ON FIBERINGS OF ALMOST COQUATERNION MANIFOLDS

BY

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After several preliminaries on almost coquaternion (metric) structures (§ 1) we introduce the notion of pseudo-coquaternion structure (§ 2). Then we give some conditions under which an almost coquaternion Riemannian manifold is the bundle space of a principal fiber bundle and investigate some properties of this fibering (§ 3).

§ 1. **Almost coquaternion (Riemannian) manifolds.** Let M be a $(4n+3)$-dimensional connected paracompact manifold.

An *almost coquaternion structure* on M is defined by three almost-co-complex structures $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, $a = 1, 2, 3$, satisfying the following conditions [11], [12, p. 13],

$$\underset{a}{\Phi} \circ \underset{b}{\Phi} - \underset{a}{\xi} \otimes \underset{b}{\eta} = - \underset{b}{\Phi} \circ \underset{a}{\Phi} + \underset{b}{\xi} \otimes \underset{a}{\eta} = \underset{c}{\Phi},$$

$$\underset{a}{\Phi}\, \underset{b}{\xi} = - \underset{b}{\Phi}\, \underset{a}{\xi} = \underset{c}{\xi},$$

$$\underset{a}{\eta} \circ \underset{b}{\Phi} = - \underset{b}{\eta} \circ \underset{a}{\Phi} = \underset{c}{\eta},$$

$$\underset{a}{\eta}(\underset{b}{\xi}) = \underset{b}{\eta}(\underset{a}{\xi}) = 0,$$

for any cyclic permutation $\{a, b, c\}$ of $\{1, 2, 3\}$.

A manifold M with an almost coquaternion structure is called an *almost coquaternion manifold*.

To an almost coquaternion structure we attached [12, p. 70] certain tensor fields of which we mention

$$\Gamma = \sum_a \underset{a}{\xi} \otimes \underset{a}{\eta},$$

$$\Lambda = \frac{1}{4}(\mathrm{id} \otimes \mathrm{id} + \sum_a \underset{a}{\Phi} \otimes \underset{a}{\Phi} + \Gamma \otimes \Gamma).$$

$$\Psi = \underset{1}{\Phi} \otimes \underset{2}{\Phi} \otimes \underset{3}{\Phi}.$$

We observe that Γ is a (1, 1) tensor field of constant rank three, for which $\Gamma^2 = \Gamma$. Therefore it gives rise to an almost product structure on M.

For each $x \in M$ let T_x denote the tangent space to M at x. We can write $T_x = V_x \oplus H_x$, where

$$V_x = \{X \mid X \in T_x,\ \Gamma X = X\},$$

$$H_x = \{X \mid X \in T_x,\ \Gamma X = 0\},$$

and we call V_x *the vertical subspace* of T_x and H_x *the horizontal subspace*.

Evidently, the vertical 3-dimensional distribution $x \xrightarrow{V} V_x$ is generated by $\underset{1}{\xi}, \underset{2}{\xi}, \underset{3}{\xi}$ and the horizontal $4n$-dimensional distribution $x \xrightarrow{H} H_x$ can be directly defined by means of 1-forms $\underset{1}{\eta}, \underset{2}{\eta}, \underset{3}{\eta}$.

Every almost coquaternion manifold has a positive definite Riemannian metric g such that [11], [12, p. 23],

$$\underset{a}{\eta}(X) = g(\underset{a}{\xi}, X),$$

$$g(\underset{a}{\Phi}X, \underset{a}{\Phi}Y) = g(X, Y) - \underset{a}{\eta}(X)\,\underset{a}{\eta}(Y),\ a = 1, 2, 3,\ \forall X, Y \in \mathfrak{X}(M).$$

g is called an *associated Riemannian metric* to the given almost coquaternion structure, $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta}, g)$, $a = 1, 2, 3$, is called an *almost coquaternion metric structure* and M is said to be an *almost coquaternion Riemannian manifold*.

An almost coquaternion metric structure can be described by means of 1-forms $\underset{a}{\eta}$ and 2-forms $\underset{a}{\Theta}(X, Y) = g(\underset{a}{\Phi}X, Y)$. We also have $g \circ \Lambda = g$.

§2. On the almost quaternion structure of $M \times R$. Let M be an almost coquaternion manifold with the structure $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, $a = 1, 2, 3$, and $M \times R$ be the product manifold of M with the real line R. If we define three tensors $\underset{a}{J}$ of type (1, 1) by

$$\underset{a}{J} = \begin{pmatrix} \underset{a}{\Phi} & \underset{a}{\xi} \\ -\underset{a}{\eta} & 0 \end{pmatrix},$$

then $\underset{a}{J}$, $a = 1, 2, 3$, is an almost quaternion structure on $M \times R$, i. e.,

$$\underset{a}{J}^2 = -\mathrm{id},\quad \underset{a}{J} \circ \underset{b}{J} = -\underset{b}{J} \circ \underset{a}{J} = \underset{c}{J}$$

hold good, where $\{a, b, c\}$ is a cyclic permutation of $\{1, 2, 3\}$.

As in [7, § 5, p. 5—1], using the Nijenhuis tensors of the almost complex structures $\underset{a}{J}$ on $M \times R$,

$$\underset{a}{N}(X, Y) = [\underset{a}{J}X, \underset{a}{J}Y] - \underset{a}{J}[\underset{a}{J}X, Y] - \underset{a}{J}[X, \underset{a}{J}Y] + \underset{a}{J}^2[X, Y],$$
$$\forall X, Y \in \mathfrak{X}(M \times R),$$

we observe that the almost coquaternion structure determines the following tensor fields on M:

$$\overset{1}{\underset{a}{N}}(X, Y) = \underset{a}{N}(X, Y;\ \underset{a}{\Phi}) + 2d\underset{a}{\eta}(X, Y)\underset{a}{\xi} \quad \text{of type } (1, 2),$$

$$\overset{2}{\underset{a}{N}}(X, Y) = -2d\underset{a}{\eta}(\underset{a}{\Phi}X, Y) - 2d\underset{a}{\eta}(X, \underset{a}{\Phi}Y) \quad \text{of type } (0, 2),$$

$$\overset{3}{\underset{a}{N}}(X) = -(L_{\underset{a}{\xi}}\underset{a}{\Phi})(X) \quad \text{of type } (1, 1),$$

$$\overset{4}{\underset{a}{N}}(X) = -(L_{\underset{a}{\xi}}\underset{a}{\eta})(X) \quad \text{of type } (0, 1),$$

$\forall X, Y \in \mathfrak{X}(M)$, where $\underset{a}{N}(X, Y;\ \underset{a}{\Phi})$ is the Nijenhuis tensor of $\underset{a}{\Phi}$.

On the other hand, using the structure tensor [1]

$$T(X, Y) = \frac{2}{3}\sum_a \underset{a}{N}(X, Y),\quad \forall X, Y \in \mathfrak{X}(M \times R),$$

of the almost quaternion structure on $M \times R$, we deduce that the almost coquaternion structure determines the following tensor fields on M

$$\overset{\alpha}{T} = \frac{2}{3}\sum_a \overset{\alpha}{\underset{a}{N}},\quad \alpha = 1, 2, 3, 4.$$

Theorem 2.1. *Any almost coquaternion structure* $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, $a = 1, 2, 3$, *on* M *satisfies the relations*

$$4\overset{1}{\underset{a}{N}}(X, Y) = \overset{1}{T}(X, Y) + \underset{a}{\Phi}(\overset{1}{T}(X,\underset{a}{\Phi}Y)) + \underset{a}{\Phi}(\overset{1}{T}(\underset{a}{\Phi}X, Y)) - \overset{1}{T}(\underset{a}{\Phi}X, \underset{a}{\Phi}Y) +$$

$$+ \underset{a}{\eta}(Y)\underset{a}{\Phi}(\overset{3}{T}(X)) + \underset{a}{\eta}(Y)\, \overset{3}{T}(\underset{a}{\Phi}X) + \underset{a}{\eta}(X)\underset{a}{\Phi}(\overset{3}{T}(Y)) - \underset{a}{\eta}(X)\overset{3}{T}(\underset{a}{\Phi}Y) +$$

$$+ \overset{2}{T}(X, \underset{a}{\Phi}Y)\underset{a}{\xi} + \overset{2}{T}(\underset{a}{\Phi}X, Y)\underset{a}{\xi} + \overset{4}{T}(X)\underset{a}{\eta}(Y)\underset{a}{\xi} - \underset{a}{\eta}(X)\overset{4}{T}(Y)\underset{a}{\xi},$$

$$4\overset{2}{\underset{a}{N}}(X, Y) = - \underset{a}{\eta}(\overset{1}{T}(X, \underset{a}{\Phi}Y)) - \underset{a}{\eta}(\overset{1}{T}(\underset{a}{\Phi}X, Y)) + \underset{a}{\eta}(\overset{3}{T}(X))\underset{a}{\eta}(Y) -$$

$$- \underset{a}{\eta}(X)\underset{a}{\eta}(\overset{3}{T}(Y)) + \overset{2}{T}(X, Y) - \overset{2}{T}(\underset{a}{\Phi}X, \underset{a}{\Phi}Y) - \overset{4}{T}(\underset{a}{\Phi}X)\underset{a}{\eta}(Y) + \underset{a}{\eta}(X)\overset{4}{T}(\underset{a}{\Phi}Y),$$

$$4\overset{3}{\underset{a}{N}}(X) = \underset{a}{\Phi}(\overset{1}{T}(X, \underset{a}{\xi})) - \overset{1}{T}(\underset{a}{\Phi}X,\underset{a}{\xi}) + \overset{3}{T}(X) + \underset{a}{\Phi}(\overset{3}{T}(\underset{a}{\Phi}X)) -$$

$$- \underset{a}{\eta}(X)\overset{3}{T}(\underset{a}{\xi}) + \overset{2}{T}(X, \underset{a}{\xi})\underset{a}{\xi} - \overset{4}{T}(\underset{a}{\Phi}X)\underset{a}{\xi},$$

$$4\overset{4}{\underset{a}{N}}(X) = \underset{a}{\eta}(\overset{1}{T}(X, \underset{a}{\xi})) + \underset{a}{\eta}(\overset{3}{T}(\underset{a}{\Phi}X)) + \overset{2}{T}(\underset{a}{\Phi}X, \underset{a}{\xi}) + \overset{4}{T}(X) - \underset{a}{\eta}(X)\overset{4}{T}(\underset{a}{\xi}),$$

$$a = 1, 2\ 3, \quad \forall X, Y \in \mathfrak{X}(M),$$

and those which are obtained by interchanging 4, $\overset{1}{T}$, $\overset{2}{T}$, $\overset{3}{T}$, $\overset{4}{T}$, *respectively with* 2, $\overset{1}{\underset{b}{N}} + \overset{1}{\underset{c}{N}}$, $\overset{2}{\underset{b}{N}} + \overset{2}{\underset{c}{N}}$, $\overset{3}{\underset{b}{N}} + \overset{3}{\underset{c}{N}}$, $\overset{4}{\underset{b}{N}} + \overset{4}{\underset{c}{N}}$, *where* $\{a, b, c\}$ *is a cyclic permutation of* $\{1, 2, 3\}$.

Proof. Like any almost quaternion structure, the almost quaternion structure $\underset{a}{J}$, $a = 1, 2, 3$, on $M \times R$ satisfies [1], [5]

$$\underset{a}{N} = \underset{a}{P} \circ T, \quad \underset{a}{N} = 2\underset{a}{P} \circ (\underset{b}{N} + \underset{c}{N}),$$

for any cyclic permutation $\{a, b, c\}$ of $\{1, 2, 3\}$, where

$$\underset{a}{P} = \frac{1}{4}(\mathrm{id} \otimes \mathrm{id} \otimes \mathrm{id} + \underset{a}{J} \otimes \mathrm{id} \otimes \underset{a}{J} + \underset{a}{J} \otimes \underset{a}{J} \otimes \mathrm{id} - \mathrm{id} \otimes \underset{a}{J} \otimes \underset{a}{J}), \quad a = 1, 2, 3,$$

are projectors and $\underset{a}{P} \circ T$ locally means $\underset{a}{P}{}^{ijk}_{rst}\, T^r_{jk}$. These relations on $M \times R$ are equivalent to the relations of Theorem 2.1 on M.

As the vanishing of $\overset{1}{\underset{a}{N}}$ (a = fixed) implies the vanishing of $\overset{2}{\underset{a}{N}}, \overset{3}{\underset{a}{N}}, \overset{4}{\underset{a}{N}}$ [7, § 5, p. 5 — 6] and the vanishing of $\overset{1}{T}$ implies the vanishing of $\overset{2}{T}$, $\overset{3}{T}$, $\overset{4}{T}$ [12, p. 38], we have.

Theorem 2.2. $\overset{1}{\underset{1}{N}}$, $\overset{1}{\underset{2}{N}}$, $\overset{1}{\underset{3}{N}}$ *vanish if any two of them vanish.* $\overset{1}{T}$ *vanishes if and only if* $\overset{1}{\underset{1}{N}}$, $\overset{1}{\underset{2}{N}}$, $\overset{1}{\underset{3}{N}}$ *simultaneously vanish.*

From the former discussions, it appears that the almost coquaternion structure $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta})$, $a = 1, 2, 3$, on M whose tensor $\overset{1}{T}$ vanishes, corresponds to the pseudo-quaternion structure $\underset{a}{J}$, $a = 1, 2, 3$, [1] on $M \times R$. It consists of three normal almost cocomplex structures. Therefore we call the almost coquaternion structure whose tensor T^1 vanishes a *pseudo-coquaternion structure* and the manifold with such a structure a *pseudo-coquaternion manifold* [12, p. 39].

§ 3. Fiberings of almost coquaternion Riemannian manifolds. First we recall some results on regular distributions [6, pp. 1—30].

Suppose that V is an integrable distribution of dimension p on a (paracompact) manifold M of dimension m and that $p < m$. We denote by M_V the set of all leaves of V. Generally M_V is not a manifold. It is a manifold only when V is a regular distribution. It can also happen that V is regular and that M_V is not a paracompact manifold.

We denote by π the canonical projection of M onto M_V defined by $x \xrightarrow{\pi}$ leaf containing x, and we quote one of Palais' results as Lemma.

Lemma 3.1. [6, pp. 21—22]. *If V is a regular distribution on M whose leaves are compact, then*

(1) *M_V is a paracompact manifold,*

(2) *$\pi : M \to M_V$ is a submersion,*

(3) *the leaves of V are the fibres of a C^∞ fibering of M with base space M_V and projection mapping π.*

Remark 3.1. As the manifold M is connected, the leaves of V and M_V are connected.

Let M be an almost coquaternion Riemannian manifold with the structure $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta}, g)$, $a = 1, 2, 3$. Joining the ideas of [11], [12, pp. 67—79] with the ideas of [10] we have.

Theorem 3.1. *If $(\underset{a}{\Phi}, \underset{a}{\xi}, \underset{a}{\eta}, g)$, $a = 1, 2, 3$, is a pseudo-coquaternion metric structure on a complete manifold M for which $\underset{a}{\Theta} = d\underset{a}{\eta}$ and the distribution V generated by $\underset{1}{\xi}$, $\underset{2}{\xi}$, $\underset{3}{\xi}$ is regular, then*

(1) *M is a principal bundle over a $4n$-dimensional Riemannian manifold M_V whose fiber is a sphere S^3 or a real projective space $RP^3 = S^3/\{\mathrm{id.}, -\mathrm{id.}\}$,*

(2) *$\Gamma = \sum_a \underset{a}{\xi} \otimes \underset{a}{\eta}$ is a connection on M,*

(3) $\eta = (\underset{1}{\eta}, \underset{2}{\eta}, \underset{3}{\eta})$ *is a connection 1-form on* M,

(4) *the metric* h *of* M_V *is related to* g *by*

$$g(X, Y) = h(\pi X, \pi Y)\circ\pi + \sum_a \underset{a}{\eta}(X)\underset{a}{\eta}(Y), \quad \forall X, Y \in \mathfrak{X}(M),$$

(5) *the tensor field*

$$\Lambda = \frac{1}{4}(\mathrm{id}\otimes\mathrm{id} + \sum_a \underset{a}{\Phi}\otimes\underset{a}{\Phi} + \Gamma\otimes\Gamma)$$

induces on M_V *a projector* $\mathfrak{P}$ *on tensors of degree 2 for which* $h\circ\mathfrak{P} = h$,

(6) *the tensor field*

$$\Psi = \underset{[1}{\Phi}\otimes\underset{2}{\Phi}\otimes\underset{3]}{\Phi}$$

induces on M_V *a tensor field* $\mathfrak{S}$ *of type* (3, 3).

Proof. (1) From $\overset{1}{T} = 0$ it follows $\overset{1}{\underset{a}{N}} = 0$, $a = 1, 2, 3$. But $\underset{a}{\Theta} = d\underset{a}{\eta}$

$\overset{1}{\underset{a}{N}} = 0$, $a = 1, 2, 3$, are equivalent to [7, § 11, p. 11—1]

$$\underset{a}{\Phi}X = \nabla_X\underset{a}{\xi}, \quad (\nabla_X\underset{a}{\Phi})(Y) = \underset{a}{\eta}(Y)X - g(X, Y)\underset{a}{\xi}$$

or

$$\underset{a}{\Theta}(X, Y) = (\nabla_X\underset{a}{\eta})(Y), \quad (\nabla_X\underset{a}{\Theta})(Y,Z) = \underset{a}{\eta}(X)g(Y,Z) - \underset{a}{\eta}(Z)g(X, Y),$$

$\forall X, Y, Z \in \mathfrak{X}(M)$, where ∇ is the Riemannian connection defined by g. Hence $\underset{1}{\xi}, \underset{2}{\xi}, \underset{3}{\xi}$ are unit Killing vector fields of the Riemannian metric g for which

$$[\underset{a}{\xi}, \underset{b}{\xi}] = -2\underset{c}{\xi},$$

where $\{a, b, c\}$ is a cyclic permutation of $\{1, 2, 3\}$, and the sectional curva tures for planes containing at least one of the vectors $\underset{1}{\xi}, \underset{2}{\xi}, \underset{3}{\xi}$ are equal to 1 at every point of M. On the other hand, from

$$(\nabla_{\underset{a}{\xi}}\underset{a}{\eta})(X) = \underset{a}{\Theta}(\underset{a}{\xi}, X) = 0,$$

it follows that all trajectories of the vector fields $\underset{1}{\xi}, \underset{2}{\xi}, \underset{3}{\xi}$ are geodesics, of M. For that reason, each leaf L of V is totally geodesic and of constant curvature 1. Obviously, by restriction, L admits a pseudo-coquaternion metric structure.

The completeness of M implies the completeness of L [3, p. 180] and hence L is a 3-dimensional spherical space form [3, p. 294] in the classification of S a s a k i [8].

The hypothesis that V is a regular distribution assures us that L can be only a sphere S^3 or a real projective space RP^3 [10]. As S^3 and RP^3 are compact Lie groups (such a group acts freely on M since $\underset{1}{\xi}, \underset{2}{\xi}, \underset{3}{\xi}$ are independent vector fields), Lemma 3.1 proves our first statement.

(2) The tensor Γ is a connection in the sense of [4, p. 46] since it satisfies the following conditions:

1°. $\Gamma(X) = \sum_a \underset{a}{\eta}(X)\underset{a}{\xi} \in V$ for every $X \in \mathfrak{X}(M)$.

2°. $\forall X \in V$, we have $X = \sum_a \underset{a}{\eta}(X)\underset{a}{\xi}$ and hence

$$\Gamma(X) = (\sum_a \underset{a}{\xi}\otimes\underset{a}{\eta})(\sum_b \underset{b}{\eta}(X)\underset{b}{\xi}) = X.$$

3°. As $\underset{a}{\eta}(X) = g(\underset{a}{\xi}, X)$, $a = 1, 2, 3$, we have

$$L_{\underset{a}{\xi}}\underset{a}{\eta} = 0, \quad L_{\underset{a}{\xi}}\underset{b}{\eta} = -L_{\underset{b}{\xi}}\underset{a}{\eta} = -2\underset{c}{\eta},$$

where $\{a, b, c\}$ is a cyclic permutation of $\{1, 2, 3\}$ and we can verify that $L_{\underset{a}{\xi}}\Gamma = 0$, $a = 1, 2, 3$, i. e., Γ and hence the distribution $x \xrightarrow{H} \mathrm{Ker}\,\Gamma_x$ are invariant under the action of the structural group.

Finally, we observe that the almost product structure on M given by $F = 2\Gamma - \mathrm{id}$ is a connection in the sense of [2].

(3) If $\{\underset{1}{\xi_0}, \underset{2}{\xi_0}, \underset{3}{\xi_0}\}$ is the Lie algebra of the structural group, then $\underset{1}{\xi}, \underset{2}{\xi}, \underset{3}{\xi}$ are considered as the fundamental vector fields corresponding to $\underset{1}{\xi_0}, \underset{2}{\xi_0}, \underset{3}{\xi_0}$, respectively. Putting

$$\eta(X) = \sum_a \underset{a}{\eta}(X)\underset{a}{\xi_0}$$

we have $\eta(\underset{a}{\xi}) = \underset{a}{\xi_0}$, $a = 1, 2, 3$.

Let t be an arbitrary element of the structural group. If $X \in H$, then $(R_t)_*X \in H$ as H is an invariant distribution. Thus, both $\eta((R_t)_*X)$ and $ad(t^{-1})\circ\eta(X)$ vanish. If $X = \underset{1}{\xi}$, then $(R_t)_*\underset{1}{\xi}$ is the fundamental vector field corresponding to $\mathrm{ad}\,(t^{-1})\underset{1}{\xi_0}$ [3, p. 51]. Thus we have

$$(R_t^*\eta)_x(\underset{1}{\xi}) = \eta_{xt}((R_t)_*\underset{1}{\xi}) = \mathrm{ad}\,(t^{-1})\underset{1}{\xi_0} = \mathrm{ad}(t^{-1})(\eta_x(\underset{1}{\xi})).$$

As every vector field $X \in \mathfrak{X}(M)$ can be decomposed into a horizontal vector field and a vertical vector field and since every vertical vector field can be expressed by means of ξ_1, ξ_2, ξ_3, we conclude from the above remarks that

$$(R_t^* \eta)(X) = \eta((R_t)_* X) = \mathrm{ad}(t^{-1}) \circ \eta(X), \quad \forall X \in \mathfrak{X}(M).$$

Hence η is a connection 1-form.

(4) As $L_{\xi_a} g = 0$, $a = 1, 2, 3$, i. e., g is an invariant Riemannian metric on M, we can define a Riemannian metric on M_V by

$$h(X, Y) = g(X^*, Y^*), \ \forall X, Y \in \mathfrak{X}(M_V),$$

where X^* denotes the lift of $X \in \mathfrak{X}(M_V)$ with respect to the connection η. h is called the *induced Riemannian metric*.

(5) From

$$(L_{X_a} \Phi)(Y) = [X, \Phi_a Y] - \Phi_a[X, Y], \ \forall X, Y \in \mathfrak{X}(M),$$

it follows

$$(L_X \Phi_a)(Y) = (\nabla_X \Phi_a)(Y) - \nabla_{\Phi_a Y} X + \Phi_a(\nabla_Y X),$$

where ∇ is the Riemannian connection defined by g. Thus we get

$$L_{\xi_a} \Phi_a = 0, \quad L_{\xi_a} \Phi_b = - L_{\xi_b} \Phi_a = - 2\Phi_c,$$

where $\{a, b, c\}$ is an even permutation of $\{1, 2, 3\}$. Consequently we have

$$L_{\xi_a} \Lambda = 0, \quad a = 1, 2, 3,$$

i. e., Λ is an invariant tensor field by the structural group. We put

$$\mathfrak{A}_p(X, Y) = d\pi((\Lambda_x)(X^*_x, Y^*_x)),$$

$\forall p \in M_V$, $x \in M$, $p = \pi(x)$, $\forall X, Y \in \mathfrak{X}(M_V)$. Obviously $\mathfrak{A}$ is well defined. We can also verify that

$$\mathfrak{A}(\mathfrak{A}(X, Y)) = \mathfrak{A}(X, Y), \ \forall X, Y \in \mathfrak{X}(M_V), \ h \circ \mathfrak{A} = h.$$

(6) From the preceding remarks it follows that

$$L_{\xi_a} \Psi = 0, \quad a = 1, 2, 3,$$

i. e., Ψ is invariant by the structural group. This means that the tensor field $\mathfrak{S}$ given by

$$\mathfrak{S}_p(X, Y, Z) = d\pi((\Psi_x)(X^*_x, Y^*_x, Z^*_x)),$$

$\forall p \in M_V$, $x \in M$, $p = \pi(x)$, $\forall X, Y, Z \in \mathfrak{X}(M_V)$, is well defined.

Remark 3.2. For other results on this fibering see [10], [11], [12, pp. 67–79].

Remark 3.3. As was remarked in [10], [11], [12, pp. 101–108] an example of our results is the Hopf fibration of S^{4n+3}. In this case the fiber is the sphere S^3 and the base space is the quaternionic projective space QP^n [9, p. 105].

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ASUPRA FIBRĂRILOR VARIETĂŢILOR APROAPE COCUATERNIONICE

Rezumat

După unele preliminarii asupra structurilor (metrice) aproape cocuaternionice (§ 1) se introduce noţiunea de structură pseudo-cocuaternionică (§ 2). Apoi se dau unele condiţii în care o varietate riemanniană aproape cocuaternionică este spaţiul total al unei fibrări principale şi se cercetează unele proprietăţi ale acestei fibrări (§ 3).