Teoremă. *Şirul* (20) *este un şir de semne normal în sens Pascal.*

Demonstraţie. Fie X un număr natural arbitrar. Există un k aşa fel că

$$X_k \leqslant X < X_{k+1}.$$

Avem

$$\frac{N_X(\Delta)}{X} \leqslant \frac{N_{X_{k+1}}(\Delta)}{X_k} = \frac{N_{X_{k+1}}(\Delta)}{X_{k+1}} \cdot \frac{X_{k+1}}{X_k}.$$

Totodată

$$\frac{X_{k+1}}{X} = \frac{(k+1)2^k}{k2^{k-1}} = \frac{2(k+1)}{k} < 4. \tag{31}$$

Ţinînd cont de (29), (30) şi (31), obţinem

$$\overline{\lim_{k\to\infty}} \frac{N_X(\Delta)}{X} \leqslant 4 \frac{1}{2^{\delta_1+\delta_2+\ldots+\delta_r+r}},$$

şi conform primei teoreme demonstrate avem

$$\lim_{X\to\infty} \frac{N_X(\Delta)}{X} = \frac{1}{2^{\delta_1+\delta_2+\ldots+\delta_r+r}},$$

ceea ce dovedeşte normalitatea şirului (20), în sens Pascal.

Metoda folosită poate fi extinsă şi pentru p număr raţional diferit de 1/2.

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SEQUENCES OF NORMAL SIGNS IN PASCAL'S SENSE

Abstract

In this paper we give an extension of the notion of normal sequence or normal sequence in Bernoulli sense. A sequence of natural numbers which meets condition (6) is said to be a normal sequence in Pascal's sense. The criterion of Pjateckij is extended, and an example of normal sequence in Pascal's sense is built up.

ON THE STRUCTURE OF REGULAR *O*-BISIMPLE SEMIGROUPS

BY

R. J. WARNE

We extend our structure theorem for regular bisimple semigroups [3] to regular O-bisimple semigroups. We show that a regular O-bisimple semigroup S with zero, 0, is determined by $\mathcal{R} \cup 0$ and $\mathcal{L} \cup 0$ where $R(L)$ is any $\mathcal{R}$-class ($\mathcal{L}$-class) of S such that $R \cap L$ is a group. The structure of regular O-bisimple semigroups is described mod two partial semigroups with zero, 0, $R \cup 0$ and $L \cup 0$ and two functions by means of an explicit multiplication (theorem 7). We also obtain the Rees structure theorem for completely O-simple semigroups by specializing theorem 7 (theorem 10).

Unless otherwise specified, we employ the definitions and notation of [1]. In particular, $\mathcal{R}$, $\mathcal{L}$, $\mathcal{D}$ and $\mathcal{H}$ will denote Green's relations and R_a will denote the $\mathcal{R}$-class of S containing $a \in S$. If $e \in E(S)$, the collection of idempotents of S, P_e (Q_e) will denote the right unit subsemigroup (the left unit subsemigroup) of eSe.

To prove our structure theorem, we first establish six lemmas.

Lemma 1. *Let S be a regular O-bisimple semigroup with zero, 0, and let $e \in E(S)\setminus\{0\}$. Then, $S = L_eR_e \cup 0$ ($0 \notin L_eR_e$) with $ab = cd$ ($a, c \in L_e$ and $b, d \in R_e$) if and only if $c = at^{-1}$ and $d = tb$ for some $t \in H_e$ where t^{-1} is the group inverse of t in H_e.*

Proof. If $a \in S$ and $a \neq 0$, $(a, g) \in \mathcal{L}$ for some $g \in E(S)\setminus\{0\}$. Since S is O-bisimple, there exists $p \in R_e \cap L_g$. By the proof of [1, theorem 2.18], there exists $p' \in R_g \cap L_e$ such that $p'p = g$. Hence, $a = ag = (ap')p$. However, $ap' \,\mathcal{L}\, gp' \,\mathcal{L}\, p' \,\mathcal{L}\, e$. If $a \in L_e$ and $b \in R_e$, there exists $x, y \in S$ such that $xa = by = e$. Hence, if $ab = 0$, then $e = xaby = 0$ and, thus, $0 \notin L_eR_e$. Next, suppose that $ab = cd$ ($a, c \in L_e$; $b, d \in R_e$). Since $a, c \in L_e$, there exists a', $c' \in R_e$ such that $a'a = c'c = e$. Hence, $b = eb = (a'a)b = a'(cd) = (a'c)d$ and, similarly, $d = (c'a)b$. For brevity, let $s = a'c$ and $t = c'a$. Thus, $se = s = es$, $te = et = t$, $b = sd$ and $d = tb$. Since there exists x, $y \in S$ such that $bx = dy = e$, $b = stb$ and $d = tsd$ imply $st = ts = e$ and, hence, s, $t \in H_e$ with $s = t^{-1}$. Hence, $c = ce = c(dy) = (cd)y = (ab)y = at^{-1}dy = at^{-1}$.

The next lemma extends [2, corollary 1.3] to regular O-bisimple semigroups.

Lemma 2. *Let S be a regular O-bisimple semigroup with zero, 0, and let $e \in E(S) \setminus \{0\}$. Then, eSe is a regular O-bisimple semigroup with identity e.*

Proof. Let $a, b \in eSe \setminus \{0\}$. Hence, there exists $c \in S$ such that $a\mathcal{R}c$ and $c\mathcal{L}b$. Thus, $a = cx$, $c = ay$, $c = wb$ and $b = zc$ for some x, y, w, $z \in S$. Since $ec = eay = ay = c$ and, similarly, $ce = c$, $c \in eSe$. Thus, $a = c(exe)$ and $c = a(eye)$ imply $a\mathcal{R}c$ $(\in eSe)$ and, similarly, $c\mathcal{L}b (\in eSe)$. Thus, since $(eSe)^2 \subseteq eSe$ and $a \in eSe$ implies there exists $x \in S$ such that $a = a(exe)a$, eSe is a regular O-bisimple semigroup.

Choose a representative element from each $\mathcal{L}$-class of P_e (let e be the representative of the $\mathcal{L}$-class of P_e containing e). Let P'_e denote the collection of these representative elements.

Lemma 3. *Let S be a regular O-bisimple semigroup with zero, 0, and let $e \in E(S) \setminus \{0\}$. Every non-zero element of R_eL_e may be uniquely expressed in the form $x = ab$ where $a \in Q_e$ and $b \in P'_e$. Thus, if $A_e = ((x, y) \in R_e \times L_e : xy = 0)$ and $cd \neq 0 (c \in R_e,\ d \in L_e)$, $cd = \delta(c, d)\rho(c, d)$ $(\delta(c, d) \in Q_e,\ \rho(c, d) \in P'_e)$ defines mapping δ of $R_e \times L_e \setminus A_e$ into Q_e and ρ of $R_e \times L_e \setminus A_e$ into P'_e. If we define $\lambda(c, d) = \delta(c, d)$ if $cd \neq 0$ $(c \in R_e \cup 0;\ d \in L_e \cup 0)$ and $\lambda(c, d) = 0$ if $cd = 0$, λ is a mapping of $(R_e \cup 0) \times (L_e \cup 0)$ into $Q_e \cup 0$. If we define $\theta(c, d) = \rho(c, d)$ if $cd \neq 0$ and $\theta(c, d) = 0$ if $cd = 0$, θ is a mapping of $(R_e \cup 0) \times (L_e \cup 0)$ into $P'_e \cup 0$.*

Proof. Since eSe is a semigroup with identity e, P_e (Q_e) is the $\mathcal{R}(\mathcal{L})$class (L-class) of eSe containing e. Since eSe is O-bisimple by lemma 2 and $R_eL_e \subseteq eSe$, $z \in R_eL_e \setminus \{0\}$ implies $z = xy$ where $x \in Q_e$ and $y \in P_e$ by lemma 1. There exists $\bar{y} \in P'_e$ such that $y = u\bar{y}$ for some $u \in H_e$. Hence $z = (xu)\bar{y}$ with $xu \in Q_e$. Suppose that $z = cd = pq$ $(c,\ p \in Q_e,\ d,\ q \in P'_e)$. Hence, by lemma 1, there exists $t \in H_e$ such that $p = ct^{-1}$ and $q = td$. Thus, $q = d$, and, hence, $t = e$ and $p = c$. q. e. d.

Let X be a partial groupoid. An element $0 \in X$ is termed a zero of X, if $0x = 0 = x0$ for all $x \in X$. Let X^* denote the set of non-zero elements of X.

A partial groupoid R with zero, 0, containing a subsemigroup P is termed a generalized RP-system (R, P) with zero if the following conditions are valid: $b \in P$ and $c \in R^*$ imply bc is defined and $bc \in R^*$ and, if further, $a \in P$, $(ab)c = a(bc)$. $ac = bc (a, b \in P$ and $c \in R^*)$ implies $a = b$. R has a left identity 1 contained in P. A partial groupoid L with zero containing a subsemigroup Q is termed a generalized LQ-system (L, Q) with zero if the following conditions are valid: $a \in L^*$ and $b \in Q$ imply ab is defined and $ab \in L^*$ and, if further, $c \in Q$, $(ab)c = a(bc)$. $ca = cb (a, b \in Q$, $c \in L^*)$ implies $a = b$. L has a right identity 1 contained in Q.

Note. Generalized RP-systems (LQ-systems) with zero include generalized RP-(LQ-) systems introduced in [3].

Let S be a regular semigroup with zero, 0. Let $R(L)$ be an $\mathcal{R}$-class ($\mathcal{L}$-class) of S. Then, $R \cup 0$ $(L \cup 0)$ may be considered a partial groupoid with zero $(R \cup 0, .)$ $((L \cup 0, .))$ in the following manner: $a.b$ is defined in $R \cup 0$ $(L \cup 0)$ if and only if ab (product in S) $\in R \cup 0$ $(L \cup 0)$. Then, $a.b = ab$. For simplicity, we replace „." by juxtaposition.

Lemma 4. *Let S be a regular semigroup with zero, 0, and let $e \in E(S \setminus \{0\})$. $(R_e \cup 0, P_e)$ is a generalized RP-system with zero and $(L_e \cup 0, Q_e)$ is a generalized LQ-system with zero.*

Proof. Let $a \in L_e$ and $b \in Q_e$. Hence, there exists x, $y \in S$ such that $xa = yb = e$. Thus, $yxab = yeb = yb = e$ and, hence, $e \in Sab$. Since $ab = abe \in Se$, $ab \in L_e$.

Lemma 5. *Let S be a regular O-bisimple semigroup with zero, 0, and let $e \in E(S \setminus \{0\})$. If a, $c \in L_e$ and b, $d \in R_e$, $abcd = 0$ if and only if $bc = 0$. If $bc \neq 0$, $(ab)(cd) = a\lambda(b, c)\theta(b, c)d$ with $a\lambda(b, c) \in L_e$* and *$\theta(b, c)d \in R_e$.*

Proof. There exists x, $y \in S$ such that $xa = dy = e$. Hence, if $abcd = 0$, $bc = ebce = xabcdy = 0$. The remainder of the lemma is a consequence of the lemmas 3 and 4.

Lemma 6. *Let S be a regular O-bisimple semigroup with zero, 0, and let $e \in E(S) \setminus \{0\}$.* (a) *$u^{-1}\lambda(ub, cv^{-1}) = \lambda(b, c)t^{-1}$ and $\theta(ub, cv^{-1})v = t\theta(b, c)$ for $b \in R_e$, $c \in L_e$, u, $v \in H_e$ and some $t \in H_e$* (b) *$\lambda(b, c)\lambda(\theta(b, c)d, g) = \lambda(b, c\lambda(d, g))t^{-1}$ and $\theta(\theta(b, c)d, g) = t\theta(b, c\lambda(d, g))\theta(d, g)$ where b, $d \in R_e$ and c, $g \in L_e$ for some $t \in H_e$* (c) *if $b \in R_e$, there exists $b' \in L_e$ such that $\theta(b, b') = e = \lambda(b, b')$; if $b \in L_e$, there exists $b' \in R_e$ such that $\theta(b', b) = e = \lambda(b', b)$.*

Proof. (a) By lemma 5, $ubcv^{-1} = 0$ if and only if $bc = 0$. By lemma 4, $ub \in R_e$ and $cv^{-1} \in L_e$. Hence, if $bc \neq 0$, by lemma 3, $ubcv^{-1} = \lambda(ub, cv^{-1})\theta(ub, cv^{-1}) = u\lambda(b, c)\theta(b, c)v^{-1}$ and, thus, (a) follows from lemma 1. If $bc = 0$, $\lambda(ub, cv^{-1}) = \lambda(b, c) = \theta(b, c) = \theta(ub, cv^{-1}) = 0$. (b) Suppose $bcdg = 0$. If $bc \neq 0$, $\delta(b, c)\rho(b, c)dg = 0$ and, hence, $\theta(b, c)dg = 0$ with $\theta(b, c)d \in R_e$ by lemmas 3 and 4. If $bc = 0$, $\theta(b, c) = 0$. Thus, $\lambda(\theta(b, c)d, g) = \theta(\theta(b, c)d, g) = 0$. If $dg \neq 0$, $bc\,\delta(d, g)\rho(d, g) = 0$ and, hence, $bc\lambda(d, g) = 0$ with $c\lambda(d, g) \in L_e$. If $dg = 0$, $\lambda(d, g) = 0$. Hence, $\lambda(b, c\lambda(d, g)) = \theta(b, c\lambda(d, g)) = 0$. Next, suppose that $bcdg \neq 0$. Hence, $bcdg = (bc)dg = \lambda(b, c)\theta(b, c)dg = \lambda(b, c)\lambda(\theta(b, c)d, g)\theta(\theta(b, c)d, g)$ and $bcdg = bc(dg) = bc\lambda(d, g)\theta(d, g) = \lambda(b, c\lambda(d, g))\theta(b, c\lambda(d, g))\theta(d, g)$ and (b) is a consequence of lemma 1. (c) If $b \in R_e$, there exists $b' \in L_e$ such that $bb' = e$. Thus, $e = \lambda(b, b')\theta(b, b')$ and, hence, $\theta(b, b') = \lambda(b, b') = e$ by our choice of representative elements.

We are now in a position to state our structure theorem. Let (R, P) be a generalized RP-system with zero and let (L, Q) be a generalized LQ-system with zero, such that the group of units of P = group of units of $Q = U$ and such that R and L have a common zero, 0. Choose a representative element from each $\mathfrak{L}$-class of P and choose 1 to be the representative of the $\mathfrak{L}$-class conitaning 1. Denote the collection of these representative elements by P'. Let θ be a mapping of $R \times L$ into $P' \bigcup 0$ and let λ be a mapping of $R \times L$ into $Q \bigcup 0$ subject to the conditions:

I. $\lambda(ub, cv^{-1}) = u\lambda(b, c)t^{-1}$; $\theta(ub, cv^{-1}) = t\theta(b, c)v^{-1}$ where $b \in R^*$, $c \in L^*$, and $u, v \in U$ for some $t \in U$. If $(b, c) \in (R \times 0) \bigcup (0 \times L)$, $\lambda(b, c) = \theta(b, c) = 0$.

II. $\lambda(b, c)\lambda(\theta(b, c)d, g) = \lambda(b, c\lambda(d, g))z^{-1}$; $\theta(\theta(b, c)d, g) = z\theta(b, c\lambda(d, g))\theta(d, g)$ where $b, d \in R^*$, $c, g \in L^*$ for some $z \in U$.

III. If $b \in R^*$, there exists $b' \in L^*$ such that $\lambda(b, b') = \theta(b, b') = 1$; if $b \in L^*$, there exists $b' \in R^*$ such that $\lambda(b', b) = \theta(b', b) = 1$.

Define an equivalence relation η on $L \times R$ by the rule $(a, b)\eta(c, d)$ if $(a, b), (c, d) \in L^* \times R^*$ and $d = tb$ and $c = at^{-1}$ for some $t \in U$ or $(a, b), (c, d) \in (\{0\} \times R) \bigcup (L \times \{0\})$. Let $(L, Q, R, P, \lambda, \theta)_0$ denote $L \times R/\eta$ under the multiplication $(a, b)_\eta(c, d)_\eta = (a\lambda(b, c), \theta(b, c)d)_\eta$.

Theorem 7. *S is a regular O-bisimple semigroup if and only if $S \cong (L, Q, R, P, \lambda, \theta)_0$ for some collection $L, Q, R, P, \lambda, \theta$.*

Proof. We first show that $(L, Q, R, P, \lambda, \theta)_0$ is a regular O-bisimple semigroup. Clearly, $(0, 0)_\eta$ is the zero. Closure (associativity) is a consequence of I (II) and the definition of generalized RP-(LQ-) system with zero. If $(a,b), (c,d) \in L^* \times R^*$, $(a,b)_\eta (b', d)_\eta = (a, d)_\eta$, $(a, d)_\eta$, $(d', b)_\eta = (a,b)_\eta$, $(c,a')_\eta (a,d)_\eta = (c, d)_\eta$, and $(a, c')_\eta (c, d)_\eta = (a, d)_\eta$ by III, and, hence O-bisimplicity is established. Since $(b, b')_\eta (b, b')_\eta = (b, b')_\eta$ $(b \in L^*)$ by III, regularity is established [1, theorem 2.11]. Conversely, let S be a regular O-bisimple semigroup with zero, 0 and let $e \in E(S) \setminus \{0\}$. By lemma 4, $(R_e \bigcup 0, P_e)$ is a generalized RP-system with zero, 0, and $(L_e \bigcup 0, Q_e)$ is a generalized LQ-system with zero, 0. Clearly, H_e = group of units of P_e = group of units of Q_e. The mappings λ and θ are given by lemma 3 and lemmas 6 and 3 establish the validity of I—III. By lemmas 1 and 5, $(ab)\varphi = (a, b)_\eta$ $(a \in L_e, b \in R_e)$ and $0\varphi = (0, 0)_\eta$ defines an isomorphism of S onto $(L_e \bigcup 0, Q_e, R_e \bigcup 0, P_e, \lambda, \theta)_0$.

We next obtain the Rees structure theorem for completely O — simple semigroups as a corollary of Theorem 7. We first need two lemmas.

Lemma 8. *Let S be a completely O-simple semigroup and let $e \in E(S) \setminus \{0\}$. Then, $R_e \bigcup 0 = eS$ and $L_e \bigcup 0 = Se$. Each $\mathfrak{H}$-class $\mathfrak{H}$ of S is a group or $H \bigcup 0$ is a zero semigroup.*

Proof. If $a \in eS|_0$, $a = ea$ and, since S is O-simple, there exists $x \in eSe$ and $y \in Se$ such that $e = eay$. If $f = ayx$, $f^2 = ay(xay)x = ayx = f$, $ef = fe = f$, and $f \neq 0$ (if $f = 0$, $e = x(ayx)ay = 0$). Hence, $e = f = ayx \in aS$, and thus, $a \in R_e$ and $R_e \bigcup 0 = eS$. Similarly, $L_e \bigcup 0 = Se$. Since S is regular [1, theorem 2.51], $H = R_e \bigcap L_f (e, f \in E(S))$. If $a \in R_e \bigcap L_f$, $a^2 \in eS \bigcap Sf$ and, hence, $a^2 \in R_e \bigcap L_f$ and H is a group by [1, theorem 2.16] or $a^2 = 0$. If $a^2 = 0$, let $x, y \in R_e \bigcap L_f$. Hence, there exists $u, v \in S$ such that $ua = x$ and $av = y$. Thus, $xy = ua^2v = 0$.

Note. The second statement of lemma 8 also follows from [1, theorem 2.52 (1)].

Lemma 9. *Let S be a completely O-simple semigroup and let $e_1 \in E(S) \setminus \{0\}$. Then, $R_{e_1} \bigcup 0 = R = (G \times (A \bigcup B)) \bigcup 0$, where G is a group and A and B are disjoint sets under the multiplication $(g, i)(h, s) = (gh, s)$ or 0 according to whether $i \in A$ or B and $0z = z0 = 0$ for all $z \in R$. Furthermore, $L_{e_1} U0 = L = (G \times (C \bigcup D)) \bigcup 0$, where C and D are disjoint sets, under the multiplication $(g, i)(h, s) = (gh, i)$ or o according to whether $s \in C$ or D and $0y = y0 =$ for all $y \in L$. We may assume $1 \in A \bigcap C$.*

Proof. Let $\{L_j : j \in J\}$ denote the collection of non-zero $\mathfrak{L}$-classes of S and let $e_j \in E(L_j)$. Since S is O-bisimple by [1, theorem 2.51], $R_{e_1} \bigcap L_j \neq \square$ for all $j \in J$. Select $q_j \in R_{e_1} \bigcap L_j$ for each $j \in J$. If $R_{e_1} \bigcap L_j$ is a group, let q_j be its identity. Otherwise, $q_j^2 = 0$ by lemma 8. Let $A = \{j \in J | q_j^2 = q_j\}$ and $B = \{j \in J \mid q_j^2 = 0\}$. By the proof of [1, theorem 2. 18] there exists $q_j' \in R_{e_j} \bigcap L_{e_1}$ such that $q_j q_j' = e_1$ and $q_j' q_j = e_j$. If $x \in R_{e_1} \bigcap L_j$ $xq_j' \mathfrak{R} xe_j = x\mathfrak{R}e_1$ and $xq_j' \mathfrak{L} e_j q_j' = q_j' \mathfrak{L} e_1$. Thus, if we let $R_{e_1} \bigcap L_{e_1} = G$, $xq_j' = g \in G$ and, hence, $x = (xq_j')q_j = gq_j$. If $g \in G$ and $j \in J$, $gq_j \mathfrak{R} ge_1 \mathfrak{R} e_1$ and $gq_j \mathfrak{L} e_1 q_j \mathfrak{L} e_j$ and, hence, $gq_j \in R_{e_1} \bigcap L_j$. Hence, if $gq_j = hq_s$ $(g, h \in G; j, s \in J)$, $j = s$ and $g = (gq_j)q_j' = hq_j q_j' = h$. Thus, every element of R_{e_1} may be uniquely expressed in the form gq_j $(g \in G; j \in J)$. Hence, $(gq_j)\varphi = (g, j)$ and $0\varphi = 0$ defines a $1-1$ mapping of $R_{e_1} \bigcup 0$ onto R. If $j \in A$, $(gq_j)(hq_k) = (gh)q_k$ Since hRq_j, $h = q_j n$ for some $n \in S$. Hence, if $j \in B$, $(gq_j)(hq_k) = gq_j^2 nq_k = 0$. Thus, φ is an isomorphism. q. e. d.

Notation. Let G be a group with appended zero, 0: let I and J be sets with $1 \in I \bigcap J$; and let w be a function for $J \times I$ into $G \bigcup 0$ such that for each $i \in I$, there exists $j \in J$ such that $w_{ji} = w(j, i) \neq 0$ and for each $j \in J$, there exists $i \in I$ such that $w_{ji} \neq 0$. Let $M^0(G, I, J, W)$ denote $X = (G \times I \times J) \bigcup 0$ under the multiplication $(a)_{ij}(b)_{ks} = (aw_{jk}b)_{is}$ if $w_{jk} \neq 0$; $= 0$ if $w_{jk} = 0$ and $0x = x0 = 0$ for all $x \in X$.

Theorem 10 (Rees [1, chapter 3]). *S is a completely O-simple semigroup if and only if $S \cong M^0(G, I, J, W)$ for some collection G, I, J, W.*

Proof. Let S be a completely O-simple semigroup. Hence, S is a regular O-bisimple semigroup by [1, theorem 2.51]. We now employ theorem 7 and its notation. R and L are given by lemma 9 while $P = Q = U = G \times 1$. Define $f((b, j), (c, i)) = p_1(\lambda((b, j), (c, i)))$ if $\lambda((b, j), (c, i)) \neq 0$ and $f((b, j), (c, i)) = 0$ if $\lambda((b, j), (c, i)) = 0$. We note that $\lambda((b, j), (c, i)) \neq 0$ if and only if $\theta((b, j), (c, i)) \neq 0$ by I—III. Hence, utilizing I, $f((b, j), (c, i)) =$

$= bw_{ji}c$ where $w_{ji} = f((e,j), (e,i))$ (e is the identity of G). We have $((a\ i), (b, j))^{\varepsilon}\ ((c, k), (d, s))$ if and only if $ab = cd$, $i = k$, and $j = s$ and $((a, i), (b, j))_{\eta}\ ((c, k), (d, s))_{\eta} = ((abw_{jk}\ c,i), (d, s))_{\eta}$ if $w_{jk} \neq 0$; $= (0, 0)_{\eta}$ if $w_{jk} = 0$. Utilizing III, the condition on the w_{ji}'s is established. Thus, $((a, i), (b, j))_{\eta}\ \phi = (ab)_{ij}$ and $(0, 0)_{\eta}\phi = 0$ defines an isomorphism of S onto $M^0(G, I, J, W)$ where $J = A \cup B$ and $I = C \cup D$. Conversely, we employ theorem 7 to show that $M^0(G, I, J, W)$ is a completely O-simple semigroup. We may assume $w_{11} \neq 0$. Let $R = \{(a)_{1j} : j \in J,\ a \in G\} \cup 0$, $P = \{(a)_{11} : a \in G\}$, $L = \{(a)_{i1} : i \in I,\ a \in G\} \cup 0$, $Q = \{(a)_{11} : a \in G\}$. Then, (R, P) is a generalized RP system with zero, (L, Q) is a generalized LQ-system with zero, and $P = Q = U$. Let $\lambda((a)_{1j}, (b)_{i1}) = (aw_{ji}b)_{11}$ if $w_{ji} \neq 0$: $= 0$ if $w_{ji} = 0$ and let $\theta((a)_{1j}, (b)_{i1}) = (w_{11}^{-1})_{11}$ if $w_{ji} \neq 0$: $= 0$ if $w_{ji} = 0$. If $(g, h) \in (R \times 0) \cup (0 \times L)$, define $\lambda(g, h) = \theta(g, h) = 0$. It is easy to check the validity of I—III. Hence, $(L, Q, R, P, \lambda, \theta)_0$ is a regular O-bisimple semigroup and $((a)_{i1}, (b)_{1j})\,\Phi = (aw_{11})b_{ij}$ and $(0, 0)_{\eta}\,\Phi = 0$ defines an isomorphism of $(L, Q, R, P, \lambda, \theta)_0$ onto $M^0(G, I, J, W)$. The proof is completed by a straight forward calculation.

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ASUPRA STRUCTURII SEMIGRUPURILOR O-BISIMPLE REGULATE

Rezumat

În lucrare se folosesc notațiile și definițiile din [1]. În particular prin R, L, D și H se notează relațiile lui Green și prin R_a se notează o R-clasă a lui S care conține $a \in S$. Dacă $E(S)$ este mulțimea elementelor idempotente ale lui S și $e \in E(S)$, se notează prin P_e (respectiv Q_e) subsemigrupul unităților la dreapta (respectiv subsemigrupul unităților la stînga) al lui eSe.

În lucrare se extinde teorema de structură pentru semigrupuri bisimple regulate [3], la semigrupuri O-bisimple regulate. Se arată (teorema 7) că un semigrup O-bisimplu regulat S cu zero, 0, este determinat de $R \cup 0$ și $L \cup 0$, unde R (resp. L) este o R-clasă (resp. L-clasă) oarecare a lui S astfel că $R \cap L$ este un grup. Particularizînd acest rezultat se obține teorema de structură a lui Rees pentru semigrupuri O-simple complete.

ON B-MONOID

BY

KRISHNA KUMAR SRIVASTAVA

Introduction. In this paper we shall study some properties of the groupoids which satisfy one more axiom, which is a generalization of associative and commutative properties. We shall call such a system a *B-monoid* or BM.

Definition. *Let M be a set with an operation (.) satisfying the following axioms :*

(A) *for every $a, b \in M$, $a.b \in M$;*

(B) *for every $a, b, c \in M$, $a.(b.c) = c.(b.a)$.*

Then the set M is called a B-monoid.

It can be easily seen that the associative and the commutative laws together imply (B), but it neither implies nor is implied by the associative and the commutative laws individually.

With the above definition of the BM we shall now prove the following theorems.

Theorem 1. *If the identity element* (say e) *belongs to M, then the axiom* (B) *implies both the associative and commutative properties.*

The proof is obvious.

It may be noted that (B) is interesting in as much as it requires the existence of a left identity only to imply associative and commutative properties.

Corollary. *A groupoid with a left unit element satisfies* (B) *if and only if it is commutative and associative.*

Corollary. *A loop satisfies* (B) *if and only if it is an Abelian Group. For, a loop is a quasi-group with identity element.*