DEFORMAREA BARELOR ANIZOTROPE UŞOR CURBATE

Rezumat

Se consideră o bară mărginită de către planele $x_3 = 0$, $x_3 = h$ şi de suprafeţele (1). Se presupune că ε este un parametru mic astfel încît termenii în ε^2 pot fi neglijaţi. Domeniile $\mathcal{V}_0$ şi $\mathcal{V}_1$ mărginite de cele două plane şi de suprafeţele $\mathcal{S}_0$, $\mathcal{S}_1$ respectiv $\mathcal{S}_1$ sînt ocupate de materiale anizotrope diferite. Se studiază deformarea unei astfel de bare în cazul cînd încărcările de pe suprafaţa $\mathcal{S}_0$ sînt de forma (3). În cele din urmă problema este redusă la o problemă Almansi definită de (14) şi (15) pe care ştim să o rezolvăm [3].

ON THE QUASISTATIC TORSION OF VISCOELASTIC BEAMS

BY

GH. CIOBANU

1. Let be an isotropic homogeneous and linear viscoelastic material with the constitutive equations of the form [1] ch. 1, [6]

$$\tau_{ij}(x, t) = \lambda(t)\theta(x, t)\delta_{ij} + 2\mu(t)\varepsilon_{ij}(x, t) + $$

$$+ \int_0^t \Big[l(t, \tau)\theta(x, \tau)\delta_{ij} + 2m(t, \tau)\, \varepsilon_{ij}(x, \tau) \Big] d\tau \tag{1.1}$$

with respect to a rectangular coordinate system $Ox_1 x_2 x_3$. Here the functions $\tau_{ij}(x, t)$ and $\varepsilon_{ij}(x, t)$ are the stress tensor and the strain tensor at the point $x(x_1, x_2, x_3)$, at the moment t and $\theta(x, t) = \varepsilon_{ii}(x, t)$.

Latin subscripts have the range 1, 2, 3 and the usual convention of summation over repeated subscripts is adopted.

If $u_i(x, t)$ are the components of the displacement vector at the point x at the moment t, then

$$2\varepsilon_{ij}(x, t) = u_{i,j}(x, t) + u_{j,i}(x, t). \tag{1.2}$$

The index i after comma indicates partial differentiation with respect to x_i.

The constitutive equations (1.1) show that the viscoelastic material is assumed initially undisturbed.

If $\lambda(t)$ and $\mu(t)$ are constants then (1.1) are the constitutive equations suggested by Volterra [8]. Moreover, if $l(t, \tau) = g(t - \tau)$, $m(t, \tau) = h(t - \tau)$, (1.1) is identical in form to the accepted general linear viscoelastic constitutive equations [5], [3], [7].

We assume the elastic moduli $\lambda(t)$ and $\mu(t)$ are continuous and bounded functions on $[0, \infty)$. We suppose also there exists positive constants λ_0,

μ_0 so that $\lambda(t) \geqslant \lambda_0$, $\mu(t) \geqslant \mu_0$ for $t \geqslant 0$. We assume the relaxation moduli $l(t, \tau)$ and $m(t, \tau)$ are continuous and bounded functions on $[0, \infty)\times[0, \infty)$.

2. Let us consider an isotropic and homogeneous viscoelastic cylindrical beam of length l bounded by two cross-sections, with ends B and B'.

The rectangular cartesian system $Ox_1x_2x_3$ is chosen as follows; O is the centroid of B, Ox_1 and Ox_2 are the principal axes of inertia, Ox_3 is parallel to the generators and $x_3 = l$ is the end B'. Let the loading applied to the end B' be statically equivalent to a resultant force $\bar{R}(t)$ $(R_1(t), R_2(t), R_3(t))$ and a resultant couple $\bar{M}(t)$ $(M_1(t), M_2(t), M_3(t))$ so that

$$R_1(t) = R_2(t) = R_3(t) = 0, \qquad M_1(t) = M_2(t) = 0 \quad M_3(t) = M(t) \tag{2.1}$$

for every $t > 0$. $M(t)$ is a given continuous function of $t \geqslant 0$. This system of loading will be applied not as a concentrated system but as a distribution of tractions over the end B. We suppose the loading over the end B and the loading over the end B' are in equilibrium at every moment $t \geqslant 0$.

We assume that the body forces are absent. Therefore, the equilibrium equations at the instant $t \geqslant 0$ are

$$\tau_{ij,j}(x, t) = 0. \tag{2.2}$$

Let Σ be a cross-section in the beam and Γ its boundary with $\bar{n}(n_1, n_2, 0)$ the unit outward normal. We suppose Σ is simply-connected. On the lateral surface of the beam we impose the conditions

$$\begin{aligned} \tau_{11}(x,t)\, n_1 + \tau_{12}(x,t) n_2 &= 0, \\ \tau_{21}(x,t) m_1 + \tau_{22}(x,t) n_2 &= 0, \\ \tau_{31}(x,t)\, n_1 + \tau_{32}(x,t) n_2 &= f(x_1, x_2, t), \end{aligned} \tag{2.3}$$

where $f(x_1, x_2, t)$ is a continuous function of $(x_1, x_2) \in \Gamma$ for every $t \geqslant 0$ and continuous of $t \geqslant 0$ for every $(x_1, x_2) \in \Gamma$.

Assuming the beam is in equilibrium at any moment $t \geqslant 0$, it follows

$$\int_\Gamma f(x_1, x_2, t) ds = \int_\Gamma x_1 f(x_1, x_2, t)\, ds = \int_\Gamma x_2 f(x_1, x_2, t) ds = 0 \tag{2.4}$$

for $t \in [0, \infty)$.

The problem of quasistatic torsion consists in the solving of the equations (1.1), (1.2), (2.2) with the conditions (2.1), (2.3).

The uniqueness theorems of the solution in linear theory of viscoelasticity have been given by Volterra [8], Gurtin and Sternberg [5], Breuer and Onat [3].

By S. Venant's principle we can substitute the surface tractions at the ends by statically equivalent loads [7].

3. We seek the solution of the torsion problem in the form

$$\begin{aligned} u_1(x, t) &= -k(t)\, x_2 x_3, \\ u_2(x, t) &= k(t)\, x_1 x_3, \\ u_3(x, t) &= k(t)\, \varphi(x_1, x_2, t), \end{aligned} \tag{3.1}$$

where $\varphi(x_1, x_2, t) \in C^2(\Sigma) \cap C^1(\bar{\Sigma})$ for $t \in [0, \infty)$ and $k(t)$, the angle of twist per unit of length, is a continuous function of $t \in [0, \infty)$.

The strain-components that do not vanish are

$$\begin{aligned} 2\varepsilon_{31}(x, t) &= k(t)\, (\varphi_{,1}(x, t) - x_2), \\ 2\varepsilon_{32}(x, t) &= k(t)\, (\varphi_{,2}(x, t) + x_1). \end{aligned} \tag{3.2}$$

The stress-components are given by

$$\tau_{11}(x, t) = \tau_{22}(x, t) = \tau_{33}(x, t) = \tau_{12}(x, t) = 0,$$

$$\tau_{31}(x, t) = \mu(t)\, k(t)\, (\varphi_{,1}(x, t) - x_2) + \int_0^t m(t, \tau) k(\tau)\, (\varphi_{,1}(x, \tau) - x_2) d\tau, \tag{3.3}$$

$$\tau_{32}(x, t) = \mu(t) k(t)\, (\varphi_{,2}(x, t) + x_1) + \int_0^t m(t, \tau) k(\tau)\, (\varphi_{,2}(x, \tau) + x_1) d\tau.$$

From the equations of equilibrium (2.2) we obtain:

$$\mu(t)\Delta(k(t)\, \varphi(x, t)) + \int_0^t m(t, \tau)\Delta(k(\tau)\, \varphi(x, \tau)) d\tau = 0, \tag{3.4}$$

and by $(2.3)_3$ we have

$$\begin{aligned} &\mu(t)\left[\frac{d(k(t)\, \varphi(x, t))}{dn} - k(t)\, (x_2 n_1 - x_1 n_2)\right] + \\ &+ \int_0^t m(t, \tau)\left[\frac{d(k(\tau)\varphi(x, \tau))}{dn} - k(\tau)\, (x_2 n_1 - x_1 n_2)\right] d\tau = f(x, t). \end{aligned} \tag{3.5}$$

The conditions $(2.3)_1$ and $(2.3)_2$ are satisfied. The equations (3.4) and (3.5) are Volterra integral equations of the second kind for the determination of $\Delta(k(t)\, \varphi(x, t))$ and $(d/dn)(k(t)\, \varphi(x, t))|_\Gamma$. Therefore, we have

$$\Delta(k(t)\, \varphi(x, t)) = 0, \quad (x_1, x_2) \in \Sigma, \tag{3.6}$$

and

$$\frac{d}{dn}(k(t)\,\varphi(x,t)) = k(t)\,(x_2 n_1 - x_1 n_2) + F(x_1, x_2, t),\ (x_1, x_2) \in \Gamma \tag{3.7}$$

for $t \in [0,\infty)$.

The function $F(x_1,x_2,t)$ is the solution of the equation

$$u(t) + \int_0^t \frac{m(t,\tau)}{\mu(t)}\,u(\tau)\,d\tau = \frac{f(x,t)}{\mu(t)}. \tag{3.8}$$

$F(x_1, x_2, t)$ is a continuous function of $t \in [0,\infty)$ for $(x_1, x_2) \in \Gamma$. The problem (3.6) — (3.7) is Neumann's problem for the harmonic function $k(t)\varphi(x_1, x_2, t)$.

Integrating (3.5) over Γ we obtain

$$\int_\Gamma \frac{d}{dn}(k(t)\,\varphi(x,t))ds + \int_0^t \frac{m(t,\tau)}{\mu(t)}\left[\int_\Gamma \frac{d}{dn}(k(\tau)\,\varphi(x,\tau))ds\right]d\tau = 0, \tag{3.9}$$

because of $\int_\Gamma (x_2 n_1 - x_1 n_2)ds = 0$ and (2.4).

Volterra integral equation (3.9) has only zero-solution

$$\int_\Gamma \frac{d}{dn}(k(t)\,\varphi(x,t))ds = 0, \tag{3.10}$$

and therefore Neumann's problem (3.6), (3.7) has a unique solution apart from a constant.

From (3.10) and (3.7) we get

$$\int_\Gamma F(x_1, x_2, t)ds = 0 \tag{3.11}$$

for $t \in [0,\infty)$. The solution of the problem (3.6), (3.7) is given by

$$k(t)\,\varphi(x_1, x_2, t) = k(t)\,\varphi_1(x_1, x_2) + \varphi_2(x_1, x_2, t), \tag{3.12}$$

where $\varphi_1(x_1, x_2)$ and $\varphi_2(x_1, x_2, t)$ are the solutions of the problems

$$\begin{gathered} \Delta\varphi_1(x_1, x_2) = 0, \quad (x_1, x_2) \in \Sigma, \\ \left.\frac{d\,\varphi_1(x_1, x_2)}{dn}\right|_\Gamma = x_2 n_1 - x_1 n_2 \end{gathered} \tag{3.13}$$

and

$$\begin{gathered} \Delta\varphi_2(x_1, x_2, t) = 0, \quad (x_1, x_2) \in \Sigma, \\ \left.\frac{d\varphi_2(x_1, x_2, t)}{dn}\right|_\Gamma = F(x_1, x_2, t) \end{gathered} \tag{3.14}$$

for $t \in [0,\infty)$.

From (2.2), (2.3) and (2.4) we obtain

$$\begin{gathered} R_\alpha(t) = \int_{B'} \tau_{3\alpha}(x,t)\,d\sigma = \int_{B'} [\tau_{3\alpha}(x,t) + x_\alpha \tau_{3i,i}(x,t)]\,d\sigma = \\ = \int_{B'} [x_\alpha \tau_{3\beta}(x,t)]_{,\beta}\,d\sigma = \int_\Gamma x_\alpha \tau_{3\beta}(x,t)\,n_\beta\,ds = \int_\Gamma x_\alpha f(x_1, x_2, t)\,ds = 0,\ (\alpha = 1,2), \end{gathered} \tag{3.15}$$

$$R_3(t) = \int_{B'} \tau_{33}(x,t)\,d\sigma = 0, \quad M_1(t) = -lR_2(t) = 0, \quad M_2(t) = lR_1(t) = 0.$$

From the condition $M_3(t) = M(t)$ we obtain

$$k(t) + \int_0^t \frac{m(t,\tau)}{\mu(t)}\,k(\tau)d\tau = \frac{M(t) - M^*(t)}{D(t)}, \tag{3.16}$$

where

$$D(t) = \mu(t)\int_\Sigma (x_1\,\varphi_{1,2}(x_1, x_2) - x_2\,\varphi_{1,1}(x_1, x_2) + x_1^2 + x_2^2)d\sigma, \tag{3.17}$$

$$M^*(t) = \mu(t)\Phi(t) + \int_0^t m(t,\tau)\Phi(\tau)\,d\tau, \tag{3.18}$$

and

$$\Phi(t) = \int_\Sigma [x_1\,\varphi_{2,2}(x_1, x_2, t) - x_2\,\varphi_{2,1}(x_1, x_2, t)]d\tau. \tag{3.19}$$

$D(t)$ is called the torsional rigidity at the moment t. It is a positive function of $t \in [0,\infty)$.

Remark 1. $M^*(t)$ is a continuous function of $t \in [0,\infty)$. Indeed, if $\Phi(t)$ is continuous then $M^*(t)$ is continuous. By transforming $\Phi(t)$, and taking into account that $(x_1^2 + x_2^2)\,\varphi_2(x_1, x_2, t)$ is a single valued function for every $t \in [0,\infty)$ we get

$$\Phi(t) = \frac{1}{2}\int_\Gamma (x_1^2 + x_2^2)\,\frac{d\varphi_2(x_1, x_2, t)}{ds}\,ds. \tag{3.20}$$

Let $t_0 \geq 0$ be arbitrary. We have

$$\left| \Phi(t) - \Phi(t_0) \right| \leq \frac{1}{2} \int_{\Gamma} (x_1^2 + x_2^2) \left| \frac{d\varphi_2(x_1, x_2, t)}{ds} - \frac{d\varphi_2(x_1, x_2, t)}{ds} \right| ds.$$

Since $(d/dn)\varphi_2(x_1, x_2, t)$ is continuous of t for every $(x_1, x_2) \in \Gamma$ it follows that $(d/ds)\varphi_2(x_1, x_2, t)$ is a continuous function of t for $(x_1, x_2,) \in \Gamma$, and therefore $\lim_{t \to t_0} |\Phi(t) - \Phi(t_0)| = 0$ — i. e. $\Phi(t)$ is a continuous function of $t \in [0, \infty)$. The twist angle $k(t)$ is the solution of the Volterra integral equation (3.16). This equation has a unique continuous solution on $[0, \infty)$. Therefore, the problem of viscoelastic torsion of the beam has the following solution

$$\begin{aligned} u_1(x, t) &= -k(t)\, x_2 x_3, \quad u_2(x, t) = k(t) x_1 x_3, \\ u_3(x, t) &= k(t)\, \varphi_1(x_1, x_2) + \varphi_2(x_1, x_2, t), \end{aligned} \tag{3.21}$$

where $k(t)$ is the solution of the equation (3.16) and $\varphi_1(x_1, x_2)$, $\varphi_2(x_1, x_2, t)$ are the solutions of the problems (3.13), (3.14), respectively.

Remark 2. If we suppose that

a) $f(x_1, x_2, t)$ is continuous and bounded of $t \in [0, \infty)$ for $(x_1, x_2) \in \Gamma$,
b) $M(t)$ is continuous and bounded on $[0, \infty)$,
c) $m(t, \tau)$ is continuous on $[0, \infty) \times [0, \infty)$ and

$$\int_0^t |m(t, \tau)|\, d\tau < \mu_0, \quad t \in [0, \infty),$$

then $k(t)$ is continuous and bounded on $[0, \infty)$.

Indeed, from the condition c) we obtain

$$\int_0^t \left| \frac{m(t, \tau)}{\mu(t)} \right| d\tau < 1 \tag{3.22}$$

and taking into account a), it follows [4] (ch. 6) that the solution $F(x_1, x_2, t)$ of the equation (3.8) is continuous and bounded on $[0, \infty)$ for every $(x_1, x_2) \in \Gamma$ — i. e. $(d/dn)\varphi_2(x_1, x_2, t)|_\Gamma$ and, therefore, $(d/ds)\varphi_2(x_1, x_2, t)|_\Gamma$ are continuous and bounded on $[0, \infty)$ for $(x_1, x_2) \in \Gamma$.

By (3.20) it follows $\Phi(t)$ is continuous and bounded on $[0, \infty)$. Finally, from (3.18) we obtain that $M^*(t)$ is continuous and bounded on $[0, \infty)$.

Now, because $[M(t) - M^*(t)]/D(t)$ is continuous and bounded on $[0, \infty)$ and taking into account (3.22) it follows $k(t)$, the solution of the equation (3.16), is continuous and bounded on $[0, \infty)$ ([4]ch.6).

4. The displacement (3.21) may be decomposed in

$$\begin{aligned} u_1^{\mathrm{I}}(x, t) &= -k(t) x_2 x_3, \quad u_2^{\mathrm{I}}(x, t) = k(t) x_1 x_3, \quad u_3^{\mathrm{I}}(x, t) = \\ &= k(t)\, \varphi(x_1, x_2) \end{aligned} \tag{4.1}$$

and

$$u_1^{\mathrm{II}}(x,t) = 0, \quad u_2^{\mathrm{II}}(x,t) = 0, \quad u_3^{\mathrm{II}}(x,t) = \varphi_2(x_1, x_2, t). \tag{4.2}$$

We can easily observe that $M^*(t)$ is just the Ox_3-component of the resultant couple calculated on B', corresponding to the displacement (4.2).

If the lateral surface of the beam is free of applied forces — i. e. $f(x_1, x_2, t) \equiv 0$, it follows that $F(x_1, x_2, t) \equiv 0$ and, therefore $\varphi_2(x_1, x_2, t) = 0$, $M^*(t) = 0$.

In this case the solution of viscoelastic torsion is given by (4.1) where $\varphi_1(x_1, x_2)$ is the solution of the problem (3.13) and $k(t)$ is the solution of the equation

$$k(t) + \int_0^t \frac{m(t, \tau)}{\mu(t)} k(\tau) d\tau = \frac{M(t)}{D(t)}. \tag{4.3}$$

Remark 3. If the lateral surface of the beam is free of applied forces and if we have the conditions

a) $M(t)$ is continuous and bounded on $[0, \infty)$,
b) $m(t, \tau)$ is continuous on $[0, \infty) \times [0, \infty)$ and

$$\int_0^t |m(t, \tau)|\, d\tau < \mu_0, \quad t \in [0, \infty),$$

then the solution of viscoelastic torsion is bounded on $[0, \infty)$.

If the cylindrical lateral surfaces of the beam is free of applied forces and $\lambda(t)$, $\mu(t)$ are constants, we obtain the case treated by G. Berti [3]. In this case $u_3^{\mathrm{II}}(x, t) \equiv 0$ and $k(t)$ results as in [3].

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ASUPRA TORSIUNII CVASISTATICE A BARELOR VÎSCOELASTICE

Rezumat

Se consideră problema torsiunii pentru bare vîscoelastice, ecuaţiile constitutive pentru mediul vîscoelastic fiind date de (1.1). Soluţia este dată de (3.21).

HENRI LEBESGUE: *Oeuvres scientifiques*, vol. I. Genève, Institut de Mathématiques, Université de Genève, 1972, 340 p.

Cele cinci volume în care apare opera completă a lui Lebesgue aduc în faţa specialiştilor probleme şi rezultate în cea mai mare parte considerate şi azi clasice, idei asupra cărora marele analist a reflectat mult clarificînd concepte de bază şi motivînd metode. Scrierile lui Lebesgue prezintă interes nu numai pentru matematician, ci şi pentru cercetătorul care, cu ochi de istoric, urmăreşte să cunoască modul în care s-au născut şi au evoluat anumite concepte, cum s-au impus anumite idei şi metode, cum au fost influenţate şi cum au influenţat ele în cadrul larg şi complex pe care-l oferă cercetarea matematică şi învăţămîntul matematic. Pentru lămurirea unor asemenea aspecte sînt deosebit de utile manuscrisele inedite, notele de curs şi observaţiile în care Lebesgue însuşi se referă la formaţia sa, la influenţele pe care le-a suferit, la dificultăţile întîmpinate, la problemele cărora nu le-a putut găsi o rezolvare care să-l satisfacă. Toate astea se găsesc în opera completă, alături de lucrările publicate anterior.

Volumul pe care-l prezentăm conţine două capitole; în primul, se află: lista lucrărilor (în prezentare cronologică), unele note inedite, sumarul cursurilor ţinute la Collège de France, un articol al lui A. Denjoy cu privire la viaţa şi opera lui H. Lebesgue şi necrologul scris de P. Montel la moartea lui Lebesgue. Cap. II cuprinde lucrări care tratează probleme de teoria integrării şi derivării, precum şi unele rezultate cu privire la structura funcţiilor măsurabile.

La apariţia primului volum din opera completă a lui H. Lebesgue, se cuvine un gînd de mulţumire pentru cei cărora le datorăm conturarea şi editarea acestei lucrări de amploare: F. Châtelet, G. Choquet, P. Belgodère, P. Montel, J. Karamata şi J. Lebesgue.

Olga Costinescu

AL-FARABI: *Matematičeskie traktaty*. Alma-Ata, Izd. Kazahskoj S.S.R., 1972. 218 p.

Al-Farabi (870—950), originar din Kazahstan, este unul dintre marii gînditori şi enciclopedişti de la începutul evului mediu. El a lăsat o moştenire ştiinţifică foarte bogată, care a exercitat o influenţă deosebită asupra dezvoltării ştiinţei, atît în orient, cît şi în occident. Cu toate acestea nu toate lucrările lui Al-Farabi sînt studiate şi mai cu seamă cele matematice.

Cartea pe care o recenzăm conţine toate lucrările de matematică ale lui Al-Farabi cunoscute pînă în prezent. Ele permit să ne dăm seama de creaţia lui Al-Farabi şi să vedem în el un mare matematician al epocii sale. Cartea începe cu o prefaţă din partea colectivului de redacţie şi este urmată de o introducere,