$= bw_{ji}c$ where $w_{ji} = f((e,j), (e,i))$ (e is the identity of G). We have $((a, i), (b, j))\eta\, ((c, k), (d, s))$ if and only if $ab = cd$, $i = k$, and $j = s$ and $((a, i), (b, j))_\eta\, ((c, k), (d, s))_\eta = ((abw_{jk}\, c, i), (d, s))_\eta$ if $w_{jk} \neq 0$; $= (0, 0)_\eta$ if $w_{jk} = 0$. Utilizing III, the condition on the w_{ji}'s is established. Thus, $((a, i), (b, j))_\eta\, \varphi = (ab)_{ij}$ and $(0, 0)_\eta \varphi = 0$ defines an isomorphism of S onto $M^0(G, I, J, W)$ where $J = A \cup B$ and $I = C \cup D$. Conversely, we employ theorem 7 to show that $M^0(G, I, J, W)$ is a completely O-simple semigroup. We may assume $w_{11} \neq 0$. Let $R = \{(a)_{1j} : j \in J,\ a \in G\} \cup 0$, $P = \{(a)_{11} : a \in G\}$, $L = \{(a)_{i1} : i \in I,\ a \in G\} \cup 0$, $Q = \{(a)_{11} : a \in G\}$. Then, (R, P) is a generalized RP system with zero, (L, Q) is a generalized LQ-system with zero, and $P = Q = U$. Let $\lambda((a)_{1j}, (b)_{i1}) = (aw_{ji}b)_{11}$ if $w_{ji} \neq 0$: $= 0$ if $w_{ji} = 0$ and let $\theta((a)_{1j}, (b)_{i1}) = (w_{11}^{-1})_{11}$ if $w_{ji} \neq 0$: $= 0$ if $w_{ji} = 0$. If $(g, h) \in (R \times 0) \cup (0 \times L)$, define $\lambda(g, h) = \theta(g, h) = 0$. It is easy to check the validity of I—III. Hence, $(L, Q, R, P, \lambda, \theta)_0$ is a regular O-bisimple semigroup and $((a)_{i1}, (b)_{1j})\,\Phi = (aw_{11})b_{ij}$ and $(0, 0)_\eta\, \Phi = 0$ defines an isomorphism of $(L, Q, R, P, \lambda, \theta)_0$ onto $M^0(G, I, J, W)$. The proof is completed by a straight forward calculation.

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ASUPRA STRUCTURII SEMIGRUPURILOR *O*-BISIMPLE REGULATE

Rezumat

În lucrare se folosesc notațiile și definițiile din [1]. În particular prin R, L, D și H se notează relațiile lui Green și prin R_a se notează o R-clasă a lui S care conține $a \in S$. Dacă $E(S)$ este mulțimea elementelor idempotente ale lui S și $e \in E(S)$, se notează prin P_e (respectiv Q_e) subsemigrupul unităților la dreapta (respectiv subsemigrupul unităților la stînga) al lui eSe.

În lucrare se extinde teorema de structură pentru semigrupuri bisimple regulate [3], la semigrupuri O-bisimple regulate. Se arată (teorema 7) că un semigrup O-bisimplu regulat S cu zero, 0, este determinat de $R \cup 0$ și $L \cup 0$, unde R (resp. L) este o R-clasă (resp. L-clasă) oarecare a lui S astfel că $R \cap L$ este un grup. Particularizînd acest rezultat se obține teorema de structură a lui Rees pentru semigrupuri O-simple complete.

ON *B*-MONOID

BY

KRISHNA KUMAR SRIVASTAVA

Introduction. In this paper we shall study some properties of the groupoids which satisfy one more axiom, which is a generalization of associative and commutative properties. We shall call such a system a *B-monoid* or BM.

Definition. *Let M be a set with an operation (.) satisfying the following axioms:*

(A) *for every* $a, b \in M$, $a.b \in M$;

(B) *for every* $a, b, c \in M$, $a.(b.c) = c.(b.a)$.

Then the set M is called a B-monoid.

It can be easily seen that the associative and the commutative laws together imply (B), but it neither implies nor is implied by the associative and the commutative laws individually.

With the above definition of the BM we shall now prove the following theorems.

Theorem 1. *If the identity element* (say e) *belongs to M, then the axiom* (B) *implies both the associative and commutative properties.*

The proof is obvious.

It may be noted that (B) is interesting in as much as it requires the existence of a left identity only to imply associative and commutative properties.

Corollary. *A groupoid with a left unit element satisfies* (B) *if and only if it is commutative and associative.*

Corollary. *A loop satisfies* (B) *if and only if it is an Abelian Group. For, a loop is a quasi-group with identity element.*

Theorem 2. *If a and b are any two elements of M and $n > 0$ is any integer then*

(1) $$(a.b)^n = a^n . b^n.$$

The proof of this theorem can be obtained by induction with regard to n.

The index laws. The index laws:

(i) $x^m . x^n \neq x^n . x^m$ and

(ii) $(x^n)^m \neq x^{nm}$

does not, in general, hold good in the case of B-monoids as is shown by the following example.

Example 1. Let $(A, +)$ be an Abelian group and for $a, b \in A$, define $a \ast b = a - b$. Then $(A, \ast)$ is a B-monoid. If we define

$$a^1 = a \text{ and } a^{n+1} = a^n \ast a.$$

then in general

$$a^n \ast a^m \neq a^m \ast a^n \text{ and } (a^n)^m \neq a^{nm}.$$

However $(a^n)^m = (a^m)^n$, a formula that holds in all B-monoids.

If the complex of all n^{th} powers of elements of a complex A in M is denoted by A^n[1], then A^n is also a B-monoid and the mappig $a \to a^n$ of M into A^n is a homomorphism, viz..

Example 2. Let $(A, \ast)$ be a B-monoid, s a sequence such that $s_n = 0$ or $s_n = 1$. For $a \in A$, define

$$a^{S(1)} = a \quad \text{and} \quad a^{S(n+1)} = \begin{cases} a \ast a^{S(n)} & \text{if } s_n = 0 \\ a^{S(n)} \ast a & \text{if } s_n = 1. \end{cases}$$

Then:

(i) $(a \ast b)^{S(n)} = a^{S(n)} \ast b^{S(n)}$ for all $a, b \in A$, $n \geqslant 1$;

(ii) $(a^{S(n)})^{S(m)} \neq (a^{S(m)})^{S(n)}$ for general m, n and s;

(iii) $a^{S(n)} = a^{T(m)}$ for all B-monoids A, and all $a \in A$, and all $n \geqslant 1$ if and only if $s_n = t_n$ for all $n \geqslant 1$.

Further, it is easy to verity that the mapping $a \to a^{S(n)}$ is a homomorphism for all s.

In the next section we shall extend the notion of ideals to B-monoids as it has been done in the case of semigroups.

If A and B are subsets of a B-monoid M, then by AB we shall mean the set of all elements ab of M with a in A and b in B.

By a left (right) ideal of a B-monoid M we mean a non empty subset of A of M such that $MA \subseteq A$ [$AM \subseteq A$]. By a two-sided ideal or simply ideal we mean a subset of M which is both a left and right ideal of M; We shall call a B-monoid M, left (right) simple if M itself is the only left (right) ideal of M. In a similar way M will be called simple if it contains no proper (two-sided) ideal.

The left ideal generated by x in M, is not in general $Mx \cup x$, and the set $Mx = \{mx : x \in M\}$ is not a left ideal, as it is so in the case of semigroups. This is shown by the example given at the end of the paper.

Examples of B-monoids. Since (B) holds if the associative and the commutative properties are both present, any system with an operation that is both associative and commutative is a B-monoid. Hence every abelian group and every commutative semigroup is a B-monoid.

Example 3. Let us consider the abelian semigroup $A = Q \times Q$, (Q the rational numbers) where $(a, b) \ast (c, d) = (ac, bd)$, $\sigma(a, b) = (b, a)$ is a homomorphism with $\sigma(\sigma(x)) = x$. Thus $(a, b)(c, d) = (a, b) \ast \sigma((c, d)) = (ad, bc)$ gives us a B-monoid $(A, .)$.

For $x \in A$, let $< x >$ be the left ideal of $(A, .)$ generated by x. Then

(i) $< (0, 0) > = \{(0, 0)\}$;

(ii) $< (0, x) > = < (y, 0) >$ for all $x, y \in Q$, with $x \neq 0 \neq y$;

(iii) in particular $< (0, 1) > = < (0, 2) >$ but $A(0, 1) \neq (0, 2)$ and $y(0, 2) \neq (0, 1)$ for all $x, y \in A$;

(iv) $A(0, 1)$ is not a left ideal of $(A, .)$, nor is $A(0, 1) \cup \{(0, 1)\}$

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ASUPRA B-MONOIZILOR

Rezumat

Definind B-monoidul ca acea mulţime M înzestrată cu o operaţie (.) satisfăcînd proprietăţile (A) şi (B), autorul arată că dacă elementul neutru $e \in M$, atunci (B) implică proprietăţile de asociativitate şi de comutativitate, iar dacă $a, b \in M$, atunci are loc (1). De asemenea se extinde noţiunea de ideal la B-monoizi, aşa cum se face în cazul semigrupurilor.

ASUPRA ÎMPĂRŢIRII CU REST ÎN INELUL CUATERNIONILOR ÎNTREGI

DE

GEORGE C. OPAIŢ

În această Notă ne propunem să studiem posibilitatea determinării cîtului şi restului împărţirii la stînga sau la dreapta a doi cuaternioni întregi şi ajungem la concluzia că acest lucru este totdeauna posibil în semigrupul multiplicativ al cuaternionilor întregi de norme impare. Acest rezultat permite definirea şi construirea celui mai mare divizor comun stîng sau drept a doi cuaternioni întregi de norme impare şi, în cele din urmă, o caracterizare completă a cuaternionilor ireductibili.

1. Împărţirea unui cuaternion întreg la un număr întreg

Definiţia 1. *Un cuaternion*

$$\alpha = a + bi + cj + dk$$

se numeşte :

— cuaternion întreg, dacă a, b, c, d *sînt numere întregi,*

— cuaternion primitiv, dacă $(a, b, c, d) = 1$ *şi*

— cuaternion primitiv în raport cu numărul întreg m, *dacă* $(a, b, c, d, m) = 1$.

Teorema 1. *Fiind dat un cuaternion întreg* $\alpha = a + bi + cj + dk$ *şi un număr întreg* m *cu* $|m| > 2$ *se poate determina un cuaternion întreg* ξ *aşa ca* $N(\alpha - m.\xi) < m^2$.

Pentru a fixa ideile putem presupune $m > 0$, căci dacă am determinat un cuaternion întreg ξ aşa ca $N(\alpha - m.\xi) < m^2$ cu $m < 0$, atunci avem şi $N(\alpha - (-m).(-\xi)) < (-m)^2$ cu $-m > 0$.

Va fi suficient să ne ocupăm numai de cuaternionii primitivi în raport cu m, deoarece dacă $(a, b, c, d, m) = h \neq 1$ şi avem un cuaternion