exists an order of nodes from D_2 so that D_1^j is isomorphic with D_2^j for all j, $1 \leqslant j \leqslant p$. Let A_1, A_2, A_1^j, A_2^j be the adjacency matrices of digraphs D_1, D_2, D_1^j, D_2^j respectively.

We now give a necessary and sufficient condition, so that the digraphs D_1 and D_2 are isomorphic.

Theorem 2. *The digraphs D_1 and D_2 are isomorphic if and only if there exists a permutation matrix P so that for all j, $1 \leqslant j \leqslant p$,*

$$A_1^j P = P A_2^j.$$

Proof. If D_1 and D_2 are isomorphic, then by arranging the nodes of digraph D_2 with respect to their isomorphism it follows that $A_1^j = A^j$ for all j, $1 \leqslant j \leqslant p$, and hence, we can consider the matrix

$$E = \begin{pmatrix} 1 & 0 \ldots 0 \\ \ldots & \ldots \\ 0 \ldots 0 & 1 \end{pmatrix}$$

as asked permutation matrix.

Conversely, if there exists a permutation matrix P so that $A_1^j P = P A_2^j$, it results that

$$A_1^1 P \oplus A_1^2 P \oplus \ldots \oplus A_1^p P = P A_2^1 \oplus P A_2^2 \oplus \ldots \oplus P A_2^p,$$

and hence

$$(A_1^1 \oplus A_1^2 \oplus \ldots \oplus A_1^p) P = P(A_2^1 \oplus A_2^2 \oplus \ldots \oplus A_2^p)$$

and according to main lemma $A_1 P = P A_2$ and now, according to Theorem 1, D_1 and D_2 are isomorphic.

This theorem allows us to establish if two digraphs D_1 and D_2 are isomorphic when we know that the digraphs D_1^j and D_2^j are isomorphic; it follows that the system

$$A_1^j X = X A_2^j, \qquad j = 1, 2, \ldots, p,$$

where A_1^j and A_2^j are adjacency matrices of the digraphs D_1^j and D_2^j respectively and X is a permutation matrix, is compatible if and only if the digraphs D_1 and D_2 are isomorphic.

ASUPRA IZOMORFISMULUI DIGRAFURILOR

Rezumat

În această Notă se arată că două digrafuri D_1 și D_2 sînt izomorfe dacă și numai dacă sistemul de ecuații matriciale $A_1^j X = X A_2^j$ $(1 \leqslant j \leqslant p)$ este compatibil, A_1^j și A_2^j fiind matricele de incidență ale grafurilor D_i^j — obținute din D_i prin înlăturarea muchiilor incidente cu vîrful v_j.

ON A PROBLEM OF FRANK HARARY

BY

C. SMADICI and L. SMADICI

The well known Ulam's conjecture states that two graphs are isomorphic if and only if the subgraphs obtained by removing one after another a vertex are isomorphic. P. J. Kelly [2] proved that this conjecture is true for trees. Frank Harary and Ed. Palmer [3] showed that it is true for tournements which are not strong connexe.

In this paper we deal with the following Harary's problem, [1], for undirected graphs: let $\bar{G}^j = (\bar{V}^j, \bar{U}^j)$ be p graphs, with $|\bar{V}^j| = p$, $(j = 1, \ldots, p)$, each of them having at least an isolated vertex; we wish to establish if there exists a graph $G = (V, U)$, such that for every $j = 1, \ldots, p$, the graph $G^j = (V, U^j)$, in which U^j gets from U by removing all the edges linking the vertex $v_j \in V$ with the other vertices of G, is isomorphic to the given graph $\bar{G}^j$.

Let $G = (V, U)$ be a graph without loops, with $|V| = p$, $|U| = q$, and denote by $\rho(v)$ the degree of a vertex $v \in V$. Consider the square matrix $A = (a_{ij})$ given by:

$$(1) \qquad a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in U, \\ 0 & \text{otherwise,} \end{cases} \qquad (i, j = 1, \ldots, p).$$

Obviously, we have $a_{ij} = a_{ji}$ and

$$\sum_i a_{ij} = \rho(v_j), \qquad (i, j = 1, \ldots, p).$$

Let x_i be the number of vertices having the degree i in G, $(0 \leqslant i \leqslant p - 1)$, and x_i^j be the number of vertices having the degree i in G^j. We can state the relationship between these numbers as

Theorem 1. *If $v_j \in V$ and $\rho(v_j) = k$ then :*

$$x_0^j \geqslant x_0 + 1 > 0,$$

$$x_s^j \geqslant \begin{cases} x_s - \sum_{i=0}^{s-1}(x_i^j - x_i) + 1 \geqslant 0, & \text{for } 1 \leqslant s < k, \\ x_s - \sum_{i=0}^{s-1}(x_i^j - x_i) \geqslant 0, & \text{for } k \leqslant s \leqslant p-2. \end{cases}$$

Proof. If the vertex v_j is incident to t_i^j vertices of degree i in the graph G, then from the definition of the graph G^j it follows :

$$\begin{aligned} (3) \qquad & x_0^j = x_0 + t_1^j + 1, \\ & x_i^j = x_i - t_i^j + t_{i+1}^j, \quad \text{for } 1 \leqslant i \leqslant p-2, \quad i \neq k, \\ & x_k^j = x_k - t_k^j + t_{k+1}^j - 1, \\ & x_i \geqslant t_i^j \geqslant 0, \qquad i = 1, \ldots, p-1, \quad i \neq k, \\ & x_k \geqslant t_k + 1 > 0, \end{aligned}$$

which imply

$$(4) \qquad \begin{aligned} \sum_{i=0}^{s} x_i^j &= \sum_{i=0}^{s} x_i + t_{s+1}^j + 1, \quad \text{if } 0 \leqslant s < k, \\ \sum_{i=0}^{s} x_i^j &= \sum_{i=0}^{s} x_i + t_{s+1}^j, \qquad \text{if } k \leqslant s \leqslant p-2. \end{aligned}$$

Hence

$$(5) \qquad t_{s+1}^j = \begin{cases} \sum_{i=0}^{s}(x_i^j - x_i) - 1 ; & 0 \leqslant s < k, \\ \sum_{i=0}^{s}(x_i^j - x_i) ; & k \leqslant s \leqslant p-2 \end{cases}$$

and using (3), we can write

$$x_0^j \geqslant x_0 + 1 > 0,$$

$$(6) \qquad x_s^j \geqslant \begin{cases} x_s - t_s^j = x_s - \sum_{i=0}^{s-1}(x_i^j - x_i) + 1 \geqslant 0 ; & 0 \leqslant s < k, \\ x_k - t_k^j - 1 = x_k - \sum_{i=0}^{k-1}(x_i^j - x_i) + 1 - 1 \geqslant 0 ; & s = k, \\ x_s - t_s^j = x_s - \sum_{i=0}^{s-1}(x_i^j - x_i) \geqslant 0 ; & k < s \leqslant p-2. \end{cases}$$

Now, we show that we can determine the number of vertices v in G with $\rho(v) = i$, i. e. the numbers x_i, if G^j are given.

Theorem 2. *If q is the number of edges in the graph G, than the number of vertices v in G with $\rho(v) = i$ equals the number of graphs G_j with $q - i$ edges.*

Proof. Obviously a graph G^j with $q-i$ edges gets from G by removing the i edges which are incident to a vertex v_j with $\rho(v_j) = i$, and conversely.

The preceeding theorem allows a shorter proof of the following Harary's theorem :

Theorem 3. (F. Harary). *If q_j is the number of edges in the graph G^j then :*

$$(7) \qquad q = \sum_j q_j / p - 2.$$

Proof. According to the preceeding theorem, we have

$$(8) \qquad \sum_{j=1}^{p} q_j = \sum_{i=0}^{p-1} x_i(q - i) = q\sum_{i=0}^{p-1} x_i - \sum_{i=0}^{p-1} i x_i = qp - 2q.$$

which proves the result (7).

This theorem 3 will be useful later, as well as the theorems 1 and 2 will be.

We shall now assign to any vertex v, with $\rho(v) = k$ in G, a $(p-1)$-dimensional vector $\alpha = (\alpha_i)$, where α_i equals the number of vertices of degree i, $(1 \leqslant i \leqslant p-1)$, which are incident to v.

We remark that $\sum_i \alpha_i = k$; so, a well-determined system of p such vectors can be assigned to any graph G.

Let us assume the graphs $\overline{G}^j = (\overline{V}^j, \overline{U}^j)$, $(1 \leqslant j \leqslant p)$, being given ; we shall seek now for the graph $G = (V, U)$, with $|V| = p$, solving the already stated Harary's problem. If such a graph G exists, then we know that the number of edges and the number of vertices having the degree i, for every $i = 0, \ldots, p-1$, could be determined by means of the theorems 2 and 3. In the following we shall give a constructive procedure for finding the graph G; the fact that such a graph doesn't exist will be established by the same procedure. The algorithm involves an initial phase and two main operations, which may occur in each step.

Let us denote by s^j the associate system of vectors for the graph $\overline{G}^j$, $(1 \leqslant j \leqslant p)$. These graphs are given; therefore, the system of vectors s^j, and the numbers q^j and x_i^j defined above, are known for every i and j, $(0 \leqslant i \leqslant p-1,\ 1 \leqslant j \leqslant p)$.

Initial phase. Determine $q = \sum_j q^j / p - 2$ and x_i.

According to the theorems 2 and 3, these numbers represent the number of edges and the number of vertices v with $\rho(v) = i$, respectively, in the searched graph G.

If $q^j > q$ for some j or q is not an integer number, then, obviously the graph G doesn't exist; otherwise, we pass the next operation.

Determine successively by means of the formulas (5), for every j, $(j = 1, \dots, p)$, the numbers t_i^j, $(i = 1, \dots, p-1)$.

If we find at a certain moment a negative t_i^j, we stop the procedure, because the graph G again doesn't exist; otherwise, we get successively the non negative vectors

$$\alpha^j = (t_1^j, \dots, t_{p-1}^j) \text{ for } j = 1, \dots, p,$$

i. e. the associate system of vectors

$s = (\alpha^1, \dots, \alpha^p)$ of G, and initial phase is finished.

Now, a step of the algorithm involves the following operations.

Operation (A). We construct from a graph $\overline{G}^j$ a candidate $\widetilde{G} = (\widetilde{V}, \widetilde{U})$ for G, by linking an isolated vertex v in $\overline{G}^j$ with t_i^j vertices of degree $i-1$ $(1 \leqslant i \leqslant p-1)$; then, we check if the associate system of vectors $\tilde{s}$ of this candidate coincides with the already constructed system s.

(a) When $\tilde{s}$ doesn't coincide with s, we pass to a new step, i. e. we construct a new candidate, by linking the isolate vertex v in $\overline{G}^j$ with a new group of t_i^j vertices of the degree $i-1$, check for the corresponding coincidence, a. s. o.

Of course, our procedure is stopped if all the alternatives are exhausted; in this case, the graph G doesn't exist.

(b) When $\tilde{s}$ coincides to s we pass to the second operation.

Operation (B). We construct successively the associate systems of vectors $\tilde{s}^h$, corresponding to the graphs $\widetilde{G}^h = (\widetilde{V}, \widetilde{U}^h)$, $(h = 1, \dots, p)$, where $\widetilde{U}^h$ gets from $\widetilde{U}$ by removing the incident edges of a vertex v_h; then, we check if the set of systems $\{\tilde{s}^h\}$ coincides to the set of systems $\{s^j\}$ corresponding to the given graphs $\overline{G}^j$, $(1 \leqslant j \leqslant p)$.

(B_1) When this doesn't happen, we pass to the next step; of course, our procedure is stopped if all the alternatives occuring in the operation (A) are exhausted; in this case, the graph G doesn't exist.

(B_2) When $\{\tilde{s}^h\}$ coincides with s^j the candidate G is the desired graph G.

Remarks. 1. Let us denote by $\overline{A}^j$ the incidence matrix of the given graph $\overline{G}^j$ and suppose that in this matrix $\rho(v_i) \leqslant \rho(v_{i+1})$, $(1 \leqslant i \leqslant p-1)$. We have $\rho(v_1) = 0$, so that the first row of $\overline{A}^j$ has only zero's. Now, replace these entries by the zero-one variables $\xi_1, \dots, \xi_p$.

The incidence matrix of a candidate $\widetilde{G}$ gets from this matrix when we put for the ξ_i's a solution of the system

$$\text{(9)} \qquad \begin{cases} \xi_1 = 0, \\ \displaystyle\sum_{i=k_l^1}^{k_l^2} \xi_i = t_l^j; \quad k_l^1 = 2 \text{ and} \end{cases}$$

$$k_l^1 = 1 + \sum_{k=0}^{l-2} x_k^j; \qquad (l \geqslant 2).$$

$$k_l^2 = \sum_{k=0}^{l-1} x_k^j.$$

In this way, the different candidates $\widetilde{G}$ to be checked by the steps of the algorithm correspond to the different solutions of the system (9).

2. Let us remark that the starting graph $\overline{G}^j$ can be chosen arbitrarily among the p given graphs. For instance, we can choose the graph $\overline{G}^j$ for which the number.

$$\text{(10)} \qquad N = \binom{x_0^j}{t_1^j}\binom{x_1^j}{t_2^j}\cdots\binom{x_{p-2}^j}{t_{p-1}^j}$$

is minimal.

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ASUPRA UNEI PROBLEME A LUI FRANK HARARY

Rezumat

În această Notă sînt expuse cîteva rezultate privitoare la următoarea problemă pusă de Frank Harary : fie p grafuri $\overline{G}^j = (\overline{V}^j, \overline{U}^j)$, $1 \leqslant j \leqslant p$, cu cîte p vîrfuri, fiecare avînd cel puțin un vîrf izolat. Să se determine un graf $G = (V, U)$, cu p vîrfuri, astfel încît graful G^j, obținut din G prin înlăturarea muchiilor incidente cu vîrful v_j, să fie izomorf cu graful $\overline{G}^j$, $(1 \leqslant \leqslant j \leqslant p)$.

Se arată cum se poate determina numărul de vîrfuri de ordin i, x_i, ale grafului G. Se atașează oricărui graf un sistem de vectori $(p-1)$-dimensionali și se expune un algoritm pentru construcția unui graf care are același sistem de vectori pentru grafurile G^j ca și pentru grafurile $\overline{G}^j$, date inițial.

ALGEBRA SPINORILOR PESTE UN CORP OARECARE

DE

AGNETA VESCAN

Pînă în prezent algebra spinorilor a fost definită numai peste corpul numerelor reale și complexe. În această lucrare vom extinde definiția pentru un corp de bază oarecare, comutativ, de caracteristică diferită de 2. (Vezi [1] și [2]).

Fie deci R_n un spațiu vectorial n-dimensional peste un corp comutativ K (de caracteristică $\neq 2$) și $f(x, y)$ o formă biliniară simetrică, definită pe $R_n \times R_n$. Fie e_i, $i = 1, 2, \ldots, n$, o bază ortogonală relativ la forma biliniară $f(x, y)$. Mai considerăm spațiul liniar $\times^p R_n$ al tensorilor de p ori contravarianți. Vom numi multivector un tensor de p ori contravariant antisimetric în toți indicii. Un multivector se numește simplu dacă este produsul exterior al unui număr de vectori dați într-o anumită ordine.

Se știe (vezi [4], [5]) că un spinor φ este un sistem format dintr-un scalar, un vector, un bivector ,..., un n-vector și că poate fi scris descompus după multivectorii de bază în felul următor :

$$\varphi = \varphi^0 + \Sigma\, \varphi^{i_1} e_{i_1} + \Sigma\, \varphi^{i_1 i_2} \underset{(2)}{e_{i_1 i_2}} + \cdots +$$
$$\cdots + \Sigma\, \varphi^{i_1 i_2 \ldots i_k} \underset{(k)}{e_{i_1 i_2 \ldots i_k}} + \cdots + \varphi^{12 \ldots n} \underset{(n)}{e_{12 \ldots n}},$$

sumele se extind la toate combinările numerelor $1, 2, \ldots, n$ luate cîte k, k fiind numărul de indici. Deci $i_1 < i_2 < \ldots < i_k$. În această descompunere φ^0 este scalarul, φ^{i_1} $(i_1 = 1, 2, \ldots, n)$ componentele vectorului ,..., $\varphi^{12\ldots n}$ componenta n-vectorului, formînd în același timp componentele spinorului. Observăm că spinorul poate fi definit ca suma directă dintre un scalar, un vector, un bivector ,..., un n-vector.

Această definiție poate fi păstrată și în cazul general, componentele spinorului φ fiind elemente din corpul K.