Cela prouve que l'opérateur G est strictement monotone.

2. Puisque $a(w, w) = (Gw, w)$, selon le théorème 3.2 il résulte

$$(Gw, w) \geqslant \beta^2 \, | w |^2_{\overset{\circ}{W}{}^2_2(\Omega)}$$

d'où

$$\frac{(Gw, w)}{| w |_{\overset{\circ}{W}{}^2_2(\Omega)}} \to \infty \text{ quand } | w |_{\overset{\circ}{W}{}^2_2(\Omega)} \to \infty, \tag{3.29}$$

c-est-à-dire l'opérateur G est coercitif.

3. Si l'on tient compte de (3.14) et (3.15) il résulte que l'opérateur G est borné. Pour montrer que l'opérateur est demicontinu, on démontre que pour chaque suite w_n convergente vers w dans $\overset{\circ}{W}{}^2_2(\Omega)$ il existe une sous-suite de Gw_n convergente faiblement à Gw.

De ce théorème et aussi du théorème de B r o w d e r [1] il résulte le

Théorème 3.4. *Si la fonction du matériel f satisfait aux conditions a), b), c), alors le problème de la déformation plastique plane pour les corps orthotropes avec consolidation admet une solution unique, faible, appartenant à l'espace $\overset{\circ}{W}{}^2_2(\Omega_2)$.*

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ASUPRA DEFORMĂRII PLASTICE PLANE A CORPURILOR ORTOTROPE CU CONSOLIDARE

Rezumat

În această lucrare se consideră problema existenței și unicității soluției problemei stării de deformare plastică plană pentru corpuri ortotrope cu consolidare. Se consideră acel tip de consolidare în care limitele de curgere cresc proporțional cu un parametru ce depinde de istoria de deformare. Teorema 3.4 arată existența și unicitatea soluției slabe a problemei puse.

ON THE EXISTENCE AND UNIQUENESS OF THE SOLUTION OF A BOUNDARY-VALUE PROBLEM IN THE THEORY OF INCOMPRESSIBLE MICROPOLAR FLUIDS

BY

VALERIU SAVA

1. Introduction. Theories of fluids with deformable microstructure were given by E r i n g e n [1] and A l l e n, D e S i l v a and K l i n e [3]. A subclass of these fluids is that of the micropolar fluids defined by E r i n g e n [2].

The purpose of the present paper is to define the weak solution of the boundary-value problem in the stationary case and homogeneous boundary conditions, for the incompressible micropolar fluids, as they were defined in [2], and to prove the existence and the uniqueness of the solution. In Sec. 2 we give the basic equations and the boundary condition for the theory of incompressible micropolar fluids. The functional spaces are introduced and the weak solution for the boundary-value problem is defined in Sec. 3. In Sec. 4, the existence and uniqueness of the solution is proved.

2. Basic Equations. Throughout this paper we employ a rectangular coordinate system x_1, x_2, x_3. All vectors are referred to this system. A comma represents partial differentiation with respect to space variable x_k, and a superposed dot indicates material differentiation.

Now we list the basic equations in the linear theory of incompressible micropolar fluids.

$$v_{i,i} = 0, \tag{2.1}$$

$$t_{ij,i} + \rho(f_j - \dot{v}_j) = 0, \tag{2.2}$$

$$m_{ij,i} + \varepsilon_{jik} t_{ik} + \rho(l_j - a^2 \dot{\varphi}_j) = 0, \tag{2.3}$$

$$t_{ij} = -\pi\delta_{ij} + \mu(v_{i,j} + v_{j,i}) + k(v_{j,i} - \varepsilon_{ijk}\,\varphi_k)\,. \tag{2.4}$$

$$m_{ij} = \alpha\varphi_{r,r}\delta_{ij} + \beta\varphi_{i,j} + \gamma\varphi_{j,i}\,. \tag{2.5}$$

In these relations we have used the following notations: t_{ij} – components of the stress tensor, m_{ij} – components of the couple stress tensor, π – undetermined pressure, v_i – components of the velocity vector, φ_i – components of the microrotation vector, f_j – components of the body force, l_j – components of the body couple, ρ – mass density, δ_{ij} – Kronecker's delta, ε_{ijk} – alternating tensor, μ – viscosity, k – vortex viscosity and α, β, γ – material coefficients.

Throughout this paper repeated indices denote summation over 1, 2, 3.

We suppose that the flow takes place in a bounded domain Ω with a smooth boundary $\partial\Omega$.

The differential equations satisfied by v_i, φ_i and π in Ω are given by (2.1) and combinations of (2.4) and (2.5) with (2.2) and (2.3), i. e.,

(2.6) $$-\pi_{,j} + (\mu + k)\, v_{j,ii} + k\varepsilon_{jik}\varphi_{k,i} + \rho(f_j - \dot{v}_j) = 0,$$

$$\gamma\varphi_{j,ii} + (\alpha + \beta)\,\varphi_{i,ji} + k\varepsilon_{jik}v_{k,i} - 2k\varphi_j + \rho(l_j - a^2\dot{\varphi}_j) = 0.$$

For the steady flows the differential equations (2.6) become

(2.7) $$-\pi_{,j} + (\mu + k)v_{j,ii} + k\varepsilon_{jik}\varphi_{k,i} + \rho(f_j - v_{j,i}v_i) = 0,$$

$$\gamma\varphi_{j,ii} + (\alpha + \beta)\,\varphi_{i,ji} + k\varepsilon_{jik}v_{k,i} - 2k\varphi_j + \rho(l_j - a^2\varphi_{j,i}v_i) = 0.$$

Here we consider the homogeneous, boundary conditions

(2.8) $$v|_{\partial\Omega} = 0, \quad \varphi|_{\partial\Omega} = 0.$$

3. Functional Spaces. Let us consider a bounded region Ω, with a smooth boundary $\partial\Omega$. We first introduce the following real functional spaces: $L_2(\Omega)$ – the Hilbert space of the vector-functions

$$u(x) = (u_1(x),\ u_2(x),\ u_3(x)),$$

with the norm induced by the inner product

$$(u,v)_{L_2(\Omega)} = \int_\Omega u_i(x)\, v_i(x)\, dx\,,$$

$W_2^1(\Omega)$ – the vector-functions space with $u_i (i = 1, 2, 3)$, square summable functions and having square summable derivatives in Ω;

$D(\Omega)$ – the set of vector-functions having continuous partial derivatives of all orders and compact support in Ω;

$\mathring{W}_2^1(\Omega)$ – the closure in $W_2^1(\Omega)$ of the set $D(\Omega)$. $\mathring{W}_2^1(\Omega)$ is a Hilbert space with the norm

(3.1) $$\|u\|_{\mathring{W}_2^1(\Omega)} = \left[\int_\Omega (u_{i,j}(x))^2\, dx\right]^{\frac{1}{2}},$$

$J(\Omega)$ – the set of functions $u(x) \in D(\Omega)$ wich are solenoidal, i. e., such that $u_{i,i} = 0$ in Ω; in $J(\Omega)$ we introduce the inner product

(3.2) $$(u,v)_{J(\Omega)} = \int_\Omega u_{i,j}(x)\, v_{i,j}(x)\, dx,$$

$\mathring{J}(\Omega)$ – the Hilbert space obtained by completion of $J(\Omega)$ in the norm

(3.3) $$\|u\|_{\mathring{J}(\Omega)} = \left[\int_\Omega (u_{i,j}(x))^2\, dx\right]^{\frac{1}{2}},$$

$\mathring{V}(\Omega)$ – the Hilbert space of all pairs (u, φ), $u \in \mathring{J}(\Omega)$, $\varphi \in \mathring{W}_2^1(\Omega)$ with the norm

(3.4) $$\|(u,\varphi)\|^2 = \|u\|^2_{\mathring{J}(\Omega)} + \|\varphi\|^2_{\mathring{W}_2^1(\Omega)}\,,$$

induced by the inner product

(3.5) $$[(u,\varphi)\,;\ (v,\psi)] = (u,v)_{J(\Omega)} + \int_\Omega \varphi_{i,j}(x)\, \psi_{i,j}(x)\, dx.$$

Now, we recall some lemmas which we shall need later: the references indicated supply the proofs.

Let $\mathcal{H}(\Omega)$ be the closure of $J(\Omega)$ in the norm of $L_2(\Omega)$.

Lemma 3.1. ([4] ch. I) *The orthogonal complement of $\mathcal{H}(\Omega)$ in $L_2(\Omega)$ is the set of all vectors of the form* grad p, *where p is singlevalued, locally square summable and has first derivatives in $L_2(\Omega)$.*

Here, as elsewhere, the derivatives are in the sense of Sobolev.

Lemma 3.2. *(Friedrichs inequality,* see for example [6] ch. IV). *If $u \in \mathring{W}_2^1(\Omega)$, then there is a positive constant* d *such that*

(3.6) $$\|u\|^2_{L_2(\Omega)} \leqslant d\, \|u\|^2_{W_2^1(\Omega)}\,.$$

Lemma 3.3. ([4] ch. V) *If $u, v \in L_4(\Omega)$ and $\varphi \in \mathring{W}_2^1(\Omega)$, then*

(3.7) $$\left|\int_\Omega u_i(x)\, v_j(x)\, \varphi_{i,j}(x)\, dx\right| \leqslant \sqrt{3}\, \|u\|_{L_4(\Omega)}\, \|v\|_{L_4(\Omega)}\, \|\varphi\|_{W_2^1(\Omega)}\,.$$

The next lemma is also proved in [4] ch. I; it is a variant of the Sobolev inequality.

Lemma 3.4. *If $u \in \mathring{W}_2^1(\Omega)$, then there is a positive constant b such that*

(3.8) $$\|u\|_{L_4(\Omega)} \leqslant b\, \|u\|_{W^1(\Omega)}\,.$$

Lemma 3.5. *If* $u \in J(\Omega)$ *and* $\varphi \in \mathring{W}_2^1(\Omega)$, *then there is a positive constant* c *such that*

$$\left|\int_{\Omega} \varphi_{(x)} \cdot \operatorname{curl} u_{(x)}\, dx\right| \leqslant c \,\|\varphi\|_{\mathring{W}_2^1(\Omega)} \|u\|_{J(\Omega)}. \tag{3.9}$$

Now, we define the weak solution of boundary-value problem (2.1) (2.7), (2.8).

Definition 3.1. *A field of pairs* $(v_{(x)}, \varphi_{(x)})$ *over* Ω *is said to be a weak solution of boundary-value problem* (2.1), (2.7), (2.8) *whenever*

i) $v(x) \in \mathring{J}(\Omega)$, $\varphi(x) \in \mathring{W}_2^1(\Omega)$ *and*
ii) *the relation*

$$\begin{aligned}\int_{\Omega} \{ (\mu + k)\, v_{k,l}\, u_{k,l} + k\varepsilon_{klm} (u_{k,l}\,\varphi_m + v_{k,l}\,\psi_m) + 2k\varphi_k\,\psi_k +\\ + \alpha\varphi_{l,l}\,\psi_{k,k} + \gamma\varphi_{k,l}\,\psi_{k,l} + \beta\varphi_{l,k}\,\psi_{k,l} - \rho v_l\, v_k\, u_{k,l} -\\ - \rho a^2\varphi_k\, v_l\,\psi_{k,l} \}\, dx = \rho \int_{\Omega} (f_k\, u_k + l_k\,\psi_k)\, dx\end{aligned} \tag{3.10}$$

holds for all $u(x) \in J(\Omega)$, $\psi(x) \in D(\Omega)$.

Evidently, if $v(x)$, $\varphi(x)$ is a classical solution of (2.1), (2.7), (2.8) corresponding to a suitable pressure $\pi(x)$, then $v(x)$, $\varphi(x)$ is a weak solution of (2.1), (2.7), (2.8). To prove that, we use lemma 3.1.

4. Existence and uniqueness of the weak solution of the boundary-value problem. We first prove some lemmas.

Let us denote the left hand side of (3.10) with $T(v, \varphi\,;\, u, \psi)$.

Lemma 4.1. *If* $v(x) \in \mathring{J}(\Omega)$, $\varphi(x) \in \mathring{W}_2^1(\Omega)$ *then* $T(v, \varphi\,;\, u, \psi)$ *is a bounded linear functional of* (u, ψ) *in* $\mathring{V}(\Omega)$.

Proof. It is easy to prove that $T(v, \varphi\,;\, u, \psi)$ is linear. Now, we consider

$$\begin{aligned}T(v, \varphi\,;\, u, \psi) = (\mu + k)\,(v, u)_{J(\Omega)} + k\int_{\Omega} \varepsilon_{kem}\,(u_{k,e}\,\varphi_m + v_{k,e}\,\psi_m)\, dx +\\ + \int_{\Omega} \{ \gamma\varphi_{k,e}\,\psi_{k,e} + \alpha\varphi_{e,e}\,\psi_{k,k} + \beta\varphi_{e,k}\,\psi_{k,e} + 2k\varphi_k\,\psi_k \}\, dx -\\ - \rho\int_{\Omega} (v_k\, v_e\, u_{k,e} + a^2\varphi_k\, v_e\,\psi_{k,e})\, dx.\end{aligned}$$

Using the lemmas 3.2 – 3.5 and Cauchy inequalities, we can write

$$\begin{aligned}|T(v, \varphi : u, \psi)| \leqslant |\mu + k|\,\|v\|_{\mathring{J}(\Omega)}\|u\|_{\mathring{W}_2^1(\Omega)} + |k|\, c\,\|\varphi\|_{\mathring{W}_2^1(\Omega)}\|u\|_{\mathring{J}(\Omega)} +\\ + |k|\, c\,\|v\|_{\mathring{J}(\Omega)}\|\psi\|_{\mathring{W}_2^1(\Omega)} + |\gamma|\,\|\varphi\|_{\mathring{W}_2^1(\Omega)}\|\psi\|_{\mathring{W}_2^1(\Omega)} +\\ + c_1\|\varphi\|_{\mathring{W}_2^1(\Omega)}\|\psi\|_{\mathring{W}_2^1(\Omega)} + \rho a^2 b^2\,\|v\|^2_{\mathring{J}(\Omega)}\|u\|_{\mathring{J}(\Omega)} +\\ + \rho a^2 b^2\,\|v\|_{\mathring{J}(\Omega)}\|\varphi\|_{\mathring{W}_2^1(\Omega)}\|\psi\|_{\mathring{W}_2^1(\Omega)} \leqslant\\ \leqslant c_2(\|v\|_{\mathring{J}(\Omega)} + \|\varphi\|_{\mathring{W}_2^1(\Omega)})\,\|(u, \varphi)\|.\end{aligned}$$

and assertion follows. Here c, c_1, c_2 are positive constants.

According to Riesz's theorem there exists $G(v, \varphi) \in \mathring{V}(\Omega)$ such that

$$T(v, \varphi : u, \psi) = [G(v, \varphi) : (u, \psi)]. \tag{4.1}$$

We observe that G is a non-linear maping from $\mathring{V}(\Omega)$ into $\mathring{V}(\Omega)$.

Lemma 4.2. *The map* $G : \mathring{V}(\Omega) \to \mathring{V}(\Omega)$ *is demi-continuous.*

Proof. Suppose $\{(v^n, \varphi^n)\} \subset \mathring{V}(\Omega)$ *is a strongly convergent sequence to* $(v, \varphi) \in \mathring{V}(\Omega)$, i. e.

$$\lim_{n\to\infty} \|(v^n, \varphi^n) - (v, \varphi)\| = 0. \tag{4.2}$$

We have to prove that

$$\lim_{n\to\infty} [G(v^n, \varphi^n) - G(v, \varphi) : (u, \psi)] = 0 \tag{4.3}$$

holds for all $(u, \psi) \in \mathring{V}(\Omega)$.

We can write

$$\begin{aligned}[G(v^n, \varphi^n) - G(v, \varphi)\,;\,(u, \psi)] = (\mu + k)\,(v^n - v, u)_{\mathring{J}(\Omega)} +\\ + k\int_{\Omega} \{ \varepsilon_{kem}(\varphi_m^n - \varphi_m)u_{k,e} + \varepsilon_{kem}\,(v_{m,e}^n - v_{m,e})\,\psi_k \}\, dx +\\ + \int_{\Omega} \{ \gamma(\varphi_{k,e}^n - \varphi_{k,e})\psi_{k,e} + \alpha(\varphi_{e,e}^n - \varphi_{e,e})\psi_{k,k} + \beta(\varphi_{e,k}^n - \varphi_{e,k})\psi_{k,e} +\\ + 2k(\varphi_k^n - \varphi_k)\,\psi_k \}\, dx -\end{aligned}$$

$$-\rho\left(\int_\Omega (v_k^n - v_k)\, v_e^n u_{k,e}\,dx + \int_\Omega v_k(v_e^n - v_e)\, u_{k,e}\,dx\right) -$$

$$-\rho a^2\left(\int_\Omega (\varphi_k^n - \varphi_k)\, v_e^n \psi_{k,e}\,dx + \int_\Omega \varphi_k(v_e^n - v_e)\psi_{k,e}\,dx\right).$$

Arguing as in the proof of lemma 4.1, the right hand side is majorized by

$$[G(v^n, \varphi^n) - G(v, \varphi); (u, \psi)] \le c_3\{\|v^n - v\|_{\mathring{J}(\Omega)} + \|\varphi^n - \varphi\|_{\mathring{W}_2^1(\Omega)} +$$
$$+ \|v^n - v\|_{\mathring{J}(\Omega)}(\|v^n\|_{\mathring{J}(\Omega)} + \|v\|_{\mathring{J}(\Omega)} + \|\varphi^n\|_{\mathring{W}_2^1(\Omega)}) + \|\varphi^n - \varphi\|_{\mathring{W}_2^1(\Omega)}\|v^n\|_{\mathring{J}(\Omega)}\} \times$$
$$\times \|(u, \psi)\|.$$

where c_3 is a constant.

Since $\{(v^n, \varphi^n)\}$ is a strong convergent sequence, it is also bounded in $\mathring{V}(\Omega)$, i. e. $\|v^n\|_{\mathring{J}(\Omega)} <$ const. and $\|\varphi^n\|_{\mathring{W}_2^1(\Omega)} <$ const. Because of (4.2), from the last inequality it follows (4.3).

Let us introduce the operator $C : \mathring{V}(\Omega) \to \mathring{V}(\Omega)$ defined by

(4.4) $$[C(v, \varphi) ; (u, \psi)] = \rho\int_\Omega \{v_e v_k u_{k,e} + a^2 v_e \varphi_k \psi_{k,e}\}\,dx.$$

Lemma 4.3. *The operator $C : \mathring{V}(\Omega) \to \mathring{V}(\Omega)$ is completely continuous on $\mathring{V}(\Omega)$.*

Proof. Consider the sequence $\{(v^n, \varphi^n)\}$ that converges weakly in $\mathring{V}(\Omega)$. From the imbeding theorem of Sobolev [8] it results that it is a strongly convergent sequence in $L_4(\Omega)\times L_4(\Omega)$. Using the definition of C we obtain by using lemma 3.3.

$$[C(v^n, \varphi^n) - C(v^m, \varphi^m) ; (u, \psi)] =$$
$$\rho\int_\Omega \{(v_k^n - v_k^m)v_e^n u_{k,e} + v_k^m(v_e^n - v_e^m)u_{k,e}\}\,dx.$$
$$+ \rho a^2\int_\Omega \{(\varphi_k^n - \varphi_k^m)v_e^n\psi_{k,e} + \varphi_k^m(v_e^n - v_e^m)\psi_{k,e}\}\,dx.$$

After some obvious calculations we get

$$|[C(v^n, \varphi^n) - C(v^m, \varphi^m) ; (u, \psi)]| \le$$
$$\le \rho a^2\{\|v^n - v^m\|_{L_4(\Omega)}(\|v^n\|_{L_4(\Omega)} + \|v^m\|_{L_4(\Omega)} + \|\varphi^m\|_{L_4(\Omega)}) +$$
$$+ \|\varphi^n - \varphi^m\|_{L_4(\Omega)}\|v^n\|_{L_4(\Omega)}\}\|(u, \psi)\|,$$

for all $(u, \psi) \in \mathring{V}(\Omega)$. Choosing $(u, \psi) = C(v^n, \varphi^n) - C(v^m, \varphi^m)$, we obtain

$$\|C(v^n, \varphi^n) - C(v^m, \varphi^m)\| \le \rho a^2\{\|v^n - v^m\|_{L_4(\Omega)}(\|v^n\|_{L_4(\Omega)} + \|v^m\|_{L_4(\Omega)} +$$
$$+ \|\varphi^m\|_{L_4(\Omega)}) + \|\varphi^n - \varphi^m\|_{L_4(\Omega)}\|v^n\|_{L_4(\Omega)}\}.$$

Since $\{(v^n, \varphi^n)\}$ is strongly convergent in $L_4(\Omega)\times L_4(\Omega)$, we have

$$\lim_{\substack{n\to\infty \\ m\to\infty}} \|C(v^n, \varphi^n) - C(v^m, \varphi^m)\| = 0.$$

Since $\mathring{V}(\Omega)$ is a complete space it follows that the sequence $\{C(v^n, \varphi^n)\}$ is strongly convergent in $\mathring{V}(\Omega)$. Therefore C carries a weakly convergent sequence $\{(v^n, \varphi^n)\} \subset \mathring{V}(\Omega)$ into the strongly convergent one $\{C(v^n, \varphi^n)\} \subset \mathring{V}(\Omega)$ i. e. C is a completely continuous operator.

Lemma 4.4. *If*

(4.5) $$\mu \ge 0, \quad k \ge 0, \quad 3\alpha + 2\gamma \ge 0, \quad -\gamma \le \beta \le \gamma,$$

then

(4.6) $$[G(v, \varphi) - G(u, \psi); (v, \varphi) - (u, \psi)] + [C(v, \varphi) - C(u, \psi);$$
$$(v, \varphi) - (u, \psi)] \ge 0,$$

holds for all $(v, \varphi), (u, \psi) \in \mathring{V}(\Omega)$.

Proof. From definitions of G and C we have

$$[G(v, \varphi) - G(u, \psi); (v, \varphi) - (u, \psi)] + [C(v, \varphi) - C(u, \psi); (v, \varphi) - (u, \psi)] =$$
$$= \int_\Omega \{\mu(v_{k,e} - u_{k,e})(v_{k,e} - u_{k,e}) + k[v_{k,e} - u_{k,e} - \varepsilon_{iek}(\varphi_i - \psi_i)][v_{k,e} - u_{k,e} -$$
$$- \varepsilon_{iek}(\varphi_i - \psi_i)]\}\,dx + \int_\Omega \{\gamma(\varphi_{k,e} - \psi_{k,e})(\varphi_{k,e} - \psi_{k,e}) + \alpha(\varphi_{k,k} - \psi_{k,k})(\varphi_{e,e} -$$
$$\psi_{e,e}) + \beta(\varphi_{e,k} - \psi_{e,k})(\varphi_{k,e} - \psi_{k,e})\}\,dx.$$

We can easily deduce that the right hand of this relation is positive for all motions, provided that (4.5) are satisfied (see also E r i n g e n [2]). The lemma is proved.

Remark 4.1. (4.5) *are the necessary and sufficient conditions that the local Clausius-Duhem inequality be satisfied.* (See E r i n g e n [2]).

Lemma 4.5. *If*

$$\mu > 0, \quad k > 0, \quad 3\alpha + 2\gamma > 0, \quad -\gamma < \beta < \gamma, \tag{4.7}$$

then

$$[G(v, \varphi) : (v, \varphi)] \geqslant c_4 \| (v, \varphi) \|^2 \tag{4.8}$$

holds for all $(v, \varphi) \in \mathring{V}(\Omega)$ *where* c_4 *is a positive constant.*

Proof. We have

$$[G(v, \varphi) : (v, \varphi)] = \int_\Omega \{ \mu v_{k,e}\, v_{k,e} + k(v_{k,e} - \varepsilon_{iek}\, \varphi_i)\, (v_{k,e} - \varepsilon_{iek}\, \varphi_i) \}\, dx +$$

$$+ \int_\Omega \{ \gamma \varphi_{k,e}\, \varphi_{k,e} + \alpha \varphi_{k,k}\, \varphi_{e,e} + \beta \varphi_{e,k}\, \varphi_{k,e} \}\, dx -$$

$$- \rho \int_\Omega v_k v_e v_{k,e}\, dx - \rho a^2 \int_\Omega \varphi_k v_e \varphi_{k,e}\, dx.$$

But

$$\int_\Omega v_e v_k v_{k,e}\, dx = 0, \qquad \int_\Omega v_e \varphi_k \varphi_{k,e}\, dx = 0,$$

by partial integrations based on the property of $v \in \mathring{J}(\Omega)$ and $\varphi \in \mathring{W}_2^1(\Omega)$. Thus

$$[G(v, \varphi) ; (v, \varphi)] = \mu \| v \|^2_{\mathring{J}(\Omega)} + k \int_\Omega (v_{k,e} - \varepsilon_{iek}\, \varphi_i)\, (v_{k,e} - \varepsilon_{iek}\, \varphi_i)\, dx +$$

$$+ \int_\Omega \{ \gamma \varphi_{k,e}\, \varphi_{k,e} + \alpha \varphi_{k,k}\, \varphi_{e,e} + \beta \varphi_{e,k}\, \varphi_{k,e} \}\, dx \geqslant$$

$$\geqslant \mu \| v \|^2_{\mathring{J}(\Omega)} + \int_\Omega \{ \gamma \varphi_{k,e}\, \varphi_{k,e} + \alpha \varphi_{k,k}\, \varphi_{e,e} + \beta \varphi_{e,k}\, \varphi_{k,e} \}\, dx \geqslant$$

$$\geqslant \mu \| v \|^2_{\mathring{J}(\Omega)} + b_1 \| \varphi \|^2_{\mathring{W}_2^1(\Omega)} \geqslant c_4 \| (v, \varphi) \|^2,$$

where

$$b_1 = \min \{ 3\alpha + \gamma + \beta, \gamma + \beta, \gamma - \beta \}, \quad c_4 = \min \{ \mu, b_1 \}.$$

The lemma is established.

We, now, state the main result.

Theorem 4.1. *Let* Ω *be a bounded domain with smooth boundary* $\partial\Omega$. *Let* $f(x)$, $l(x)$ *be such that* $\int_\Omega (f_k u_k + l_k \psi_k) dx$ *be bounded in* $\mathring{V}(\Omega)$[1]. Let μ, k, α, β, γ *be chosen so that* (4.7) *hold.*

Then there exists a weak solution of boundary-value problem (2.1), (2.7), (2.8), satisfying

$$\| (v, \varphi) \| \leqslant \frac{1}{c_4} | (f, l) |, \tag{4.9}$$

where $|(f, l)|$ *is the norm of the linear continuous functional defined by* (f, l).

Proof. From Riesz's theorem there is an $(F, L) \in \mathring{V}(\Omega)$ such that

$$\int_\Omega (f_k u_k + l_k \psi_k)\, dx = [(F, L) ; (u, \psi)] \tag{4.10}$$

for all $(u, \psi) \in \mathring{V}(\Omega)$. Let us consider $u \in J(\Omega)$, $\psi \in D(\Omega)$. From (4.1) and (4.10) it follows that (3.10) can be written in the form

$$[G(v, \varphi) ; (u, \psi)] = [(F, L) : (u, \psi)]. \tag{4.11}$$

Since u, φ are arbitrary in $J(\Omega)$ and $D(\Omega)$ respectively, and $J(\Omega)$, $D(\Omega)$ are dense in $\mathring{J}(\Omega)$ and $\mathring{W}_2^1(\Omega)$ respectively, the proof of existence of weak solution is reduced to that of the existence of the solution of the following nonlinear functional equation

$$G(v, \varphi) = (F, L) \tag{4.12}$$

in the space $\mathring{V}(\Omega)$.

The existence of the solution of (4.12) follows immediately from the following theorem due to F. B r o w d e r [5].

Theorem. *Let* X *be a reflexive separable Banach space,* X^* *its conjugate space considered as the space of bounded conjugate linear functionals on* X. *For* $w \in X^*$, $u \in X$, *denote the value of* w *at* u *by* (w, u). *Let* G *be a (not necessarily linear) operator from* X *to* X^* *satisfying the following conditions :*

(i) *There exists a completely continuous operator* C *from* X *to* X^* *such that for all* u, v *in* X

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[1]) The sufficient conditions that $\int_\Omega (f_k u_k + l_k \psi_k)\, dx$ be a bounded linear functional of (u, ψ) in $\mathring{V}(\Omega)$ are given in [4] ch. II.

$$\operatorname{Re}\{(G(u) - G(v),\ u - v) + (C(u) - C(v),\ u - v)\} \geqslant 0;$$

(ii) *There exists a function $S(r)$ of the real variable r with $S(r) \to \infty$ as $r \to \infty$ such that for all u in X*

$$\operatorname{Re}(G(u),\ u) \geqslant S(\|u\|)\,\|u\|.$$

(iii) *G is demi-continuous,* i. e. *it is continuous from the strong topology of X to the weak topology on X^*.*

Then G is onto X^* i. e. *the equation $Gu = w$ has a solution u in X for every w in X^*.*

If we consider X to be $\mathring{V}(\Omega)$, then we note that (i) is satisfied by lemmas 4.3 and 4.4, (ii) is satisfied by lemma 4.5 where $S(r) = c_4 r$, (iii) is satisfied by lemma 4.2. Thus there is a solution of (4.12) for all (F, L).

Now we consider the inner product of the two sides of (4.12) by (v, φ) and (4.9) follows by using the lemma 4.5.

Theorem 4.2. *Under the hypotheses of theorem 4.1 and*

$$\frac{2\rho\sqrt{3}b^2a^2}{c_4^2}\,|(f,\ l)| < 1, \tag{4.13}$$

there exists a unique weak solution of the boundary-value problem.

Proof. Suppose there is another weak solution $(\widetilde{v}, \widetilde{\varphi})$. Let us denote

$$w = v - \widetilde{v}, \qquad \chi = \varphi - \widetilde{\varphi}.$$

It is obvious that $(w, \chi) \in \mathring{V}(\Omega)$ and

$$\int_\Omega \{\mu w_{k,e}\, u_{k,e} + k w_{k,e}\, u_{k,e} + k\varepsilon_{iem}\,(u_{i,e}\,\chi_m + w_{i,e}\,\psi_m) + 2k\,\chi_e\psi_e\}\,dx +$$

$$+ \int_\Omega \{\gamma\chi_{k,e}\,\psi_{k,e} + \alpha\chi_{e,e}\,\psi_{k,k} + \beta\chi_{e,k}\,\psi_{k,e}\}\,dx -$$

$$- \rho\int_\Omega \{w_e v_k u_{k,e} + \widetilde{v}_e w_k u_{k,e} + a^2 w_e \varphi_k \psi_{k,e} + a^2\,\widetilde{v}_e \chi_k \psi_{k,e}\}\,dx = 0.$$

Since $J(\Omega)$ and $D(\Omega)$ are dense in $\mathring{J}(\Omega)$ and $\mathring{W}_2^1(\Omega)$ respectively, this relation holds for all $(u, \psi) \in \mathring{V}(\omega)$. If we put here $(u, \psi) = (w, \chi)$ we obtain

$$\int_\Omega \{\mu w_{k,e}\, w_{k,e} + k w_{k,e}\, w_{k,e} + 2k\varepsilon_{kem}\, w_{k,e}\,\chi_m + 2k\chi_k\chi_k\}\,dx +$$

$$+ \int_\Omega \{\gamma\chi_{k,e}\,\chi_{k,e} + \alpha\chi_{ee,}\,\chi_{k,k} + \beta\chi_{e,k}\,\chi_{k,e}\}\,dx - \tag{4.14}$$

$$- \rho\int_\Omega \{w_e v_k w_{k,e} + \widetilde{v}_e w_k w_{k,e} + a^2\widetilde{v}_e \chi_k \chi_{k,e} + a^2 w_e \varphi_k \chi_{k,e}\}\,dx = 0.$$

From (4.14) and lemma 4.5 we get

$$c_1\|(w,\ \chi)\|^2 \leqslant \rho\int_\Omega \{w_e v_k w_{k,e} + \widetilde{v}_e w_k w_{k,e} + a^2\widetilde{v}_e \chi_k \chi_{k,e} + a^2 w_e \varphi_k \chi_{k,e}\}\,dx.$$

But we have

$$\int_\Omega \widetilde{v}_e w_k w_{k,e}\,dx = 0, \qquad \int_\Omega \widetilde{v}_e \chi_k \chi_{k,e}\,dx = 0$$

as it follows by partial integration and on account of the property of $\widetilde{v}$, $w \in \mathring{J}(\Omega)$ and $\chi \in \mathring{W}_2^1(\Omega)$.

Thus we have

$$c_4\|(w,\ \chi)\|^2 \leqslant \rho\int_\Omega \{w_e v_k w_{k,e} + a^2 w_e \varphi_k \chi_{k,e}\}\,dx. \tag{4.15}$$

We, now, use lemmas 3.3, 3.4, majoration (4.9) and then by some obvious calculations to get

$$\|(w,\ \chi)\|^2 \leqslant \frac{2\rho\sqrt{3}b^2a^2}{c_4^2}\,|(f,\ l)|\,.\,\|(w,\ \chi)\|. \tag{4.16}$$

From (4.16) and (4.13) it follows $w = 0$, $\chi = 0$ and the proof is complete.

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DESPRE EXISTENȚA ȘI UNICITATEA SOLUȚIEI UNEI PROBLEME CU VALORI PE FRONTIERĂ ÎN TEORIA FLUIDELOR MICROPOLARE INCOMPRESIBILE

Rezumat

Se cercetează existența și unicitatea curgerii staționare tridimensionale a unui fluid micropolar incompresibil în cazul condițiilor omogene pe frontieră și a teoriei liniare propusă de Eringen în [2].

ON THE IRROTATIONAL MOTIONS OF MICROPOLAR FLUIDS

BY

MARIAN ARON

1. Introduction. The field equations for incompressible micropolar fluids, as given by Eringen [1] are

$$\operatorname{div} \mathbf{v} = 0, \tag{1.1}$$

$$(\lambda + 2\mu + k) \operatorname{grad} \operatorname{div} \mathbf{v} - (\mu + k) \operatorname{curl} \operatorname{curl} \mathbf{v} + k \operatorname{curl} \mathbf{\nu} - \operatorname{grad} p + \rho \mathbf{f} = \rho \left[\frac{\partial \mathbf{v}}{\partial t} - \mathbf{v} \times \operatorname{curl} \mathbf{v} + \frac{1}{2} \operatorname{grad} (\mathbf{v}^2) \right], \tag{1.2}$$

and

$$(\alpha + \beta + \gamma) \operatorname{grad} \operatorname{div} \mathbf{\nu} - \gamma \operatorname{curl} \operatorname{curl} \mathbf{\nu} + k \operatorname{curl} \mathbf{v} - 2k\mathbf{\nu} + \rho \mathbf{l} = \rho j \dot{\mathbf{\nu}}, \tag{1.3}$$

in which $\mathbf{v}$, $\mathbf{\nu}$, $\mathbf{f}$ and $\mathbf{l}$ represent respectively the velocity, micro-rotation, body force and body couple vectors. The constants ρ and j are the density and gyration parameters while α, β, γ, λ, μ, k are the viscosity coefficients of micropolar fluids. The material constants conform to the inequalities :

$$\begin{gathered} k \geqslant 0, \quad 2\mu + k \geqslant 0, \quad 3\lambda + 2\mu + k \geqslant 0, \\ 3\alpha + \beta + \gamma \geqslant 0, \quad \gamma + \beta \geqslant 0, \quad \gamma - \beta \geqslant 0. \end{gathered} \tag{1.4}$$

p is an unknown pressure and a dot denotes the material time derivative.

In the present Note, we consider steady – state isochoric motions with lamellar velocity fields and prove in section 2 that if $\dot{\mathbf{\nu}} = 0$ then, the field equations are satisfied if and only if the micror-otation velocity fields are also lamellar and the corresponding potential satisfies a Helmholtz equation. In section 3 we indicate a class of dynamically possible accelerationless motions of considered type.