A THEOREM OF EXISTENCE, UNIQUENESS AND ANALYTICITY OF THE SOLUTION OF THE EQUATIONS OF NON-LINEAR THERMOELASTICITY

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§ 1. **Preliminaries**. For the equations of non-linear elasticity when the magnitudes of the external loads are proportional to the parameter ε Stoppelli [5, 6] established a theorem of existence and uniqueness of the solution and found when it is possible to express them by power series in ε.

In this paper following Stoppelli's method we obtain similar results for the equations of non-linear thermoelasticity.

We denote by C_0 a homogeneous, reference configuration of an elastic body and by C the configuration of equilibrium. Let ρ be the density of the body in C_0 at temperature T_0, $\sum_0$ the boundary of C_0, P_0 an arbitrary point of C_0 and P the corresponding point of C. We use a rectangular Cartesian system with the origin O belonging to C_0 and denote by x_k, y_k the coordinates of the points P_0 and P respectively.

The equations of non-linear thermoelasticity are ([1], ch. VIII)

$$
\begin{aligned}
&(y^e_{,i}\, t^{ik})_{,k} + \varepsilon\rho F^e = 0, \quad q^k_{,k} = \varepsilon W, \quad (\text{in } C_0)\\
&y^e_{,i}\, t^{ik}\, n_k = \varepsilon f^e, \quad q^k\, n_k = \varepsilon h, \quad (\text{on } \textstyle\sum_0)\\
&t^{ik} = \frac{\partial}{\partial e_{ik}} A(e_{pq}\,(u^e_{,j}),\ T), \qquad q^k = M^k\,(u^i_{,p}\,,\ T_{,e}\,,\ T)
\end{aligned}
\tag{1.1}
$$

where we used the following notations:

t^{kl} – components of second Piola-Kirchhoff stress tensor, F^l – components of body force vector F, q^k – components of heat flux vector q per unit area of $\sum_0$, W – heat source density per unit volume in C_0, n_i – components of the unit normal to the boundary $\sum_0$, f^l – components of surface force vector f per unit area of $\sum_0$, h -flux of heat across $\sum_0$, per unit time and per unit area, u^i – components of displacement vector u,

e_{ij} — components of strain tensor, A — free energy per unit volume in C_0, T — temperature of the body, ε is a real parameter; the comma denotes partial derivation with respect to spatial variables. Repeated indices are assumed over the range (1, 2, 3).

We shall prove that the system (1.1) has a unique solution $\{y(x), T(x)\}$ which satisfies the condition

$$y(0) = 0, \qquad T(0) = T_0, \tag{1.2}$$

T_0 being a given positive constant.

§ 2. **Fundamental hypotheses and auxiliary spaces of functions.** We assume that:

1) The reference configuration is a configuration of natural equilibrium, the body is isotropic in C_0 and the region occupied by the body is compact.

2) The boundary $\sum_0$ has Hölder-continuous principal curvatures with exponent λ and it is possible to cover $\sum_0$ by a finite number of regions $\sum_0^p$ each of which can be represented parametrically on a disc C^p, every point of $\sum_0$ being interior to at least one $\sum_0^p$.

3) F, W are Hölder-continuous in C_0 with exponent λ; f, h and their derivatives with respect to parameters of some parametric representation of $\sum_0$ (as in No. 2) are Hölder-continuous with exponent λ.

4) F, W, f, h satisfy the equalities

$$\begin{gathered}\int\limits_{C_0} \rho F d\tau + \int\limits_{\Sigma_0} f d\sigma = 0, \qquad \int\limits_{C_0} W d\tau - \int\limits_{\Sigma_0} h d\sigma = 0 \\ \int\limits_{C_0} p \times F \rho d\tau + \int\limits_{\Sigma_0} p \times f d\sigma = 0\ ,\end{gathered} \tag{2.1}$$

where p is the position vector in the reference configuration.

5) The free energy function A is continuous together with its derivatives up to and including the fourth order with respect to $p_{ij} = u^i_{,j}$ and T, when they vary within the ball E_1 of the 10 — dimensional Euclidian space, having its center at the point $(0, \ldots, 0, T_0)$, and radius R_1.

6) The functions M^k and their derivatives up to the third order are continuous in the ball E_2 of the 13-dimensional Euclidian space p_{ij}, $p_k = T_{,k}$, T, having its center at the point $(0, \ldots, 0, T_0)$ and radius R_2.

Let u be a vector-valued function and T a scalar-valued function defined on C_0. We denote by S'' the set of elements $z = (u, T)$ which have Hölder-continuous second derivatives with exponent λ and satisfy the condition

$$u(0) = 0, \qquad T(0) = T_0.$$

We set

$$\begin{aligned}\| z \| = & \sum_i \max_{x \in C_0} | u^i | + \sum_{i,j} \max_{x \in C_0} | u^i_{,j} | + \sum_{i,j,k} [\max_{x \in C_0} | u^i_{,jk} | + B^i_{jk}] + \\ & + \max_{x \in C_0} | T | + \sum_i \max_{x \in C_0} | T_{,i} | + \sum_{i,j} [\max_{x \in C_0} T_{,ij} | + B_{ij}],\end{aligned} \tag{2.2}$$

where B^i_{jk} and B_{ij} are the Hölder-coefficient of $u^i_{,jk}$ and $T_{,ij}$ respectively. With (2.2) as norm, S'' becomes a Banach space. We denote by S the subspace of S'' obtained by prescribing condition

$$u^k_{,r} - u^r_{,k} = 0, \qquad T = T_0, \qquad (\text{for } x_i = 0).$$

It can be shown that S is a closed subspace of S'' and hence is also a Banach space.

Let Φ, Ψ be vector-valued functions and Θ, φ scalar-valued functions. We denote by S' the class of all elements

$$Z = (\Phi, \Theta, \Psi, \varphi)$$

with following properties:

a) Φ and Θ are defined on C_0 and are Hölder-continuous with exponent λ in C_0.

b) Ψ and φ are defined on $\sum_0$; Ψ, φ and their derivatives with respect to the parameters of some parametric representation of $\sum_0$ are Hölder-continuous with exponent λ.

c) The equalities (2.1) are satisfied if we replace F, f, W, h respectively by $\Phi, \Psi, \Theta, \varphi$.

We set

$$\begin{aligned}\| Z \| = & \sum_i [\max_{x \in C_0} | \Phi_i | + \max_{x \in \Sigma_0} | \Psi_i | + D_i + C_i] + \\ & + \max_{x \in C_0} | \Theta | + \max_{x \in \Sigma_0} | \varphi | + D + C\end{aligned} \tag{2.3}$$

where D_i, D are the Hölder-coefficient of Φ_i, Θ and C_i, C are defined as in [5]. It can be verified that S' with (2.3) as norm is a Banach space.

From 5), 6) it follows the free energy function together with its derivatives up to the fourth order and the functions M^k together with their derivatives up to third order are continuous in the region $E = \{ z ; \| z - \tilde{z}_0 \| \leqslant \mathrm{R} \}$ of the space S, where $\tilde{z}_0 = (0, 0, 0, T_0)$ and $\mathrm{R} = \min (\mathrm{R}_1, \mathrm{R}_2)$.

We denote by H the largest of the maxima of the absolute values in E of the first and second derivatives of $\partial A / \partial e_{ij}$ with respect to p_{ij} and T. It is easily shown that

$$| t^{ij} (u^l_{,k}, T) - t^{ij}(v^l_{,k}, T_1) | \leqslant H(\sum_{l,k} | u^l_{,k} - v^l_{,k} | + | T - T_1 |),$$

C_0 being a natural state, it follows (if $v^i = 0$, $T_1 = T_0$) that

$$|\,t^{ij}(u^l_{,k}\,,\,T)\,| \leqslant H(\sum_{l,k}|\,u^l_{,k}\,| + |\,T - T_0\,|) \leqslant H\,\|\,z - \tilde{z}_0\|. \tag{2.4}$$

From (2.4) we have

$$\left|\sum_j \int_{C_0} (u^s_{,j}\, t^{S+1,j} - u^{S+1}_{,j}\, t^{Sj})\, d\tau\right| \leqslant H'\,\|\,z - \tilde{z}_0\,\|^2 \tag{2.5}$$

where $H' = H \operatorname{vol} C_0$.

If in u^{S+1}, t^{S+1j} an index exceeds 3 it should be diminished by 3.

§ 3. **Auxilliary theorems.** To find a theorem of existence and uniqueness we shall use the following theorems.

Theorem 1. *If $\beta = (\beta_{ij})$ is a rigid rotation, then $\{\,y(x),\ T(x)\,\}$ is a solution of problem (1.1) if and only if $\{\,\tilde{y}(x),\ T(x)\,\}$, where*

$$\tilde{y}_i = \beta_{ij}\, y_j\,, \tag{3.1}$$

is a solution of the system

$$\begin{aligned} &(\tilde{y}^l_{,t}\, t^{tS})_{,S} + \varepsilon\rho\beta^{lS} F^S = 0, \quad q^k_{,k} = \varepsilon W, \quad (\text{in } C_0),\\ &\tilde{y}^k_{,t}\, t^{tS}\, n_S = \varepsilon\beta^{kl} f^l, \quad q^k\, n_k = \varepsilon h, \quad (\text{in } \textstyle\sum_0). \end{aligned} \tag{3.2}$$

Proof. As in [3] one shows that $\{\,\tilde{y}(x),\ T(x)\,\}$ satisfies $(3.2)_1$ and $(3.2)_3$. Taking into account that:

$$q^k = J x^k_{,i}\, Q^i, \quad Q^i = y^l_{,i}\, Y_i\,(u^l_{,S}\,,\ T_{,k}\,,\ T)\ ,$$

where Q^i are the components of the heat conduction vector per unit area in the equilibrium configuration, $J = \det\,(y^l_{,i})$, Y_r are invariant under all transformations (3.1) (see [1], ch. VII), it follows that q^k are invariant under all transformations (3.1). In a same way one shows that $\{\,y(x),\ T(x)\,\}$ satisfies the equations (1.1).

We shall consider the system (3.2) and we determine the rigid rotation so as to make (3.2) have a unique solution $\{\,\tilde{y}(x),\ T(x)\,\}$ such that $\tilde{y}(0) = 0$, $T(0) = T_0$ and $\tilde{y}^i_{,j}\,(0)$ is a pure stretch, then (1.1) has a unique solution $\{\,y = x + u(x),\ T(x)\,\}$ which satisfies (1.2).

In what follows we shall write y instead of $\tilde{y}$ and we consider

$$\begin{aligned} &(y^l_{,t}\, t^{tS})_{,S} + \varepsilon\rho\beta^{lS} F^S = \Phi^l, \quad -q^k_{,k} + \varepsilon W = \Theta,\\ &-y^l_{,t}\, t^{tS}\, n_S + \varepsilon\beta^{lS} f^S = \Psi^l, \quad -q^k\, n_k + \varepsilon h = \varphi. \end{aligned} \tag{3.3}$$

We determine β in such a way that the second members of the relations (3.3) be an element of space S'. The hypotheses a), b) are obvious. The equalities $(2.1)_1$, $(2.1)_2$ are satisfied if we replace F, f, W, h respectively by Φ, Ψ, Θ, φ. The condition that β give rise to element satisfying relation $(2.1)_3$ is expressed by the equations

$$\varepsilon(A_{S+1,S+1}\,\beta_{S+2,S+1} - A_{S+2,S+2}\,\beta_{S+1,S+2}) = L_S.$$

$$\sum_{t=1}^{3}\beta_{jt}\,\beta_{kt} = \delta_{jk}\,, \tag{3.4}$$

$$\det\,(\beta_{kl}) = 1,$$

where

$$L_S = \sum_{j=1}^{3}\int_{C_0}(u^S_{,j}\, t^{S+1,j} - u^{S+1}_{,j}\, t^{Sj})\, d\tau.$$

A_{rs} is the astatic load of the F and f in the reference configuration.

In the same manner as in [5] one proves the following theorem:

Theorem 2. *If F and f do not posses an axis of equilibrium, then it is possible to find positive numbers α_1, η_1, such that for $\eta < \eta_1$ one may determine another positive number $\alpha_0 < \alpha_1$ for which the system (3.4) admits one and only one solution satisfying the relation*

$$|\,\beta_{St} - \delta_{St}\,| < \eta, \tag{3.5}$$

when

$$|\,L\,| < \alpha_0\,|\,\varepsilon\,|\,; \tag{3.6}$$

and for $L_r = 0$ this solution becomes δ_{st}.

If L satisfies (3.6), a point of the 9-dimensional space of the coordinates $\beta_{11}, \beta_{12}, \ldots, \beta_{33}$ describes a region Ω, in which the Jacobian determinant of the left-hand members of the equations (3.4) is different from zero.

From (2.5) we obtain

$$|\,L\,| = \sqrt{\sum_{S=1}^{3} L_S^2} \leqslant H'\sqrt{3}\,\|\,z - \tilde{z}_0\,\|^2\,;$$

one deduces that (3.6) is verified if

$$\|\,z - \tilde{z}_0\,\| \leqslant \alpha_0\,|\,\varepsilon\,|^{\frac{1}{2}}\ , \tag{3.7}$$

where

$$\alpha_0' = \sqrt{\frac{\alpha_0}{\sqrt{3}\,H'}}\ .$$

If $|\,\varepsilon\,| < R^2/\alpha_0')^2$ then for every z in a neighbourhood V_0 of the point z_0 defined by (3.7), there is a unique rotation such that (3.4) be satisfied. With this choice of β the right-hand side of the relation (3.4) defines a point of the space S'. In this manner (3.3) can be seen to define mapping of neighbourhood V_0 of $\tilde{z}_0$ into S'.

(3.8) $$Z = G(z).$$

It is obvious

$$G(\tilde{z}_0) = Z_0 ,$$

where

$$Z_0 = (\varepsilon F, \varepsilon W, \varepsilon f, \varepsilon h).$$

Let be

$$D(z_0, z) = \lim_{t \to 0} \frac{G(z_0 + tz) - G(z_0)}{t} = \left\{ \frac{d}{dt} G(z_0 + tz) \right\}_{t=0} ,$$

$$\omega(z_0, z) = G(z_0 + tz) - G(z_0) - D(z_0, z).$$

The transformation $G(z)$ is continuously differentiable in V_0, if for each positive number τ one may find another one δ_τ, for which the following conditions are satisfied :

1) for every $\bar{z}_0, z_0 \in V_0$ satisfying the inequality

$$\| z_0 - \bar{z}_0 \| \leqslant \delta_\tau ,$$

and for every z of S, we have

(3.9) $$\| D(z_0, z) - D(\bar{z}_0, z) \| \leqslant \tau \| z \| ;$$

2) for each z_0 of V_0, if $\| z \| < \delta_\tau$, we have

(3.10) $$\| R(z_0, z) \| \leqslant \tau \| z \| .$$

We shall use the known inverse mapping theorem of functional analysis ([4], ch. II).

Theorem 3. *If the following conditions are satisfied :*

1) $G(\tilde{z}_0) = Z_0$;

2) *$G(z)$ is continuously differentiable in the neighbourhood V_0 :*

3) *the transformation*

(3.11) $$\delta Z = D(\tilde{z}_0, \delta z)$$

is a liniar invertible transformation ; if

(3.12) $$\delta z = \Delta(\tilde{z}_0, \delta Z)$$

is the inverse transformation of (3.11), μ the norm of $\Delta(\tilde{z}_0, \delta Z)$ and

(3.13) $$\tau = \frac{k}{2\mu}, \quad \alpha = \min\left(\frac{\delta_\tau}{2}, \frac{\alpha'_0}{2}\right), \quad r = \frac{1 - 2\tau\mu}{\mu}\alpha, \quad k < 1,$$

then for every Z satisfying the inequality

(3.14) $$\| Z - Z_0 \| < r \, | \varepsilon |^{\frac{1}{2}} ,$$

there exists one and only one solution of the transformation (3.8) in the neighbourhood $\bar{V}_0$ of $\tilde{z}_0$ defined by

(3.15) $$\| z - \tilde{z}_0 \| < \alpha \, | \varepsilon |^{\frac{1}{2}} .$$

§ 4. **Theorem of existence, uniqueness and analyticity.** It is sufficient to show that $G(z)$ satisfies all the conditions of theorem 3. The first condition is evident. We prove that $G(z)$ is continuously differentiable.

We use the notations

$$z_0 = (v_1, v_2, v_3, T_1), \quad \bar{z}_0 = (v'_1, v'_2, v'_3, T_2), \quad z = (u_1, u_2, u_3, T),$$

where z_0, $\bar{z}_0 \in V_0$ and z are arbitrary points of S,

$$\mathcal{M}_k[z] = \frac{\partial}{\partial x_i}(y^k_{,j} \, t^{ij}), \quad \mathcal{N}_k[z] = y^k_{,i} \, t^{ij} \, n_j .$$

$$\mathcal{P}_k[z] = \sum_{s=1}^{3} \int_{C_0} (u^k_{,s} \, t^{k+1,s} - u^{k+1}_{,s} \, t^{ks}) \, d\tau,$$

$$\mathcal{R}[z] = \frac{\partial}{\partial x_i} M^i, \quad Q[Z] = M^k n_k .$$

If $\varphi[z]$ is an arbitrary function of z, we put

$$\delta\varphi[z_0, z] = \left\{ \frac{d}{dt} \varphi[z_0 + tz] \right\}_{t=0} .$$

If z_0 and z are fixed and t varies in a neighbourhood of $t = 0$ in such a way that $z_0 + tz$ lies in V_0, we deduce from (3.4) :

(4.1) $$\frac{d}{dt} \beta_{rs}[z_0 + tz] = \frac{1}{\varepsilon} \sum_n B_{rsn}\left(\beta_{ij}[z_0 + tz]\right) \frac{d}{dt} \mathcal{P}_n[z_0 + tz],$$

where the functions B_{rsn} and their first derivatives with respect to β_{rs} are continuous. From (4.1) we obtain

$$\delta\beta_{rs} = \frac{1}{\varepsilon} \sum_n B_{rsn}(\beta_{ij}[z_0]) \int_{C_0} [K_{lmn}(v'_{,j}, T_1) \, u^l_{,m} + K_n(v^l_{,j}, T_1) \, T] d\tau.$$

where the functions K_{lmn}, K_n and their first derivatives are continuous. We denote by H_l the greatest of the absolute values of the functions K_{lmn}, B_{rsn}, K_n and their derivatives, when p_{ij} and T vary in E and β_{rs} vary in Ω.

It can be easily shown that

$$(4.2)\qquad \begin{gathered} t^{ij}[z_0+tz] \leqslant H \| z_0 + tz - \tilde{z}_0 \|, \\ \left|\frac{d}{dt} t^{ij}[z_0+tz]\right| \leqslant H\|z\|, \left|\frac{d^2}{dt^2} t^{ij}[z_0+tz]\right| \leqslant H\|z\|^2: \end{gathered}$$

if $|\varepsilon| < 1$, it follows:

$$(4.3)\qquad \begin{gathered} \left|\frac{d}{dt}\mathcal{P}_k[z_0+tz]\right| \leqslant 2H'\|z\| \,.\, \|z_0+tz-\tilde{z}_0\|, \\ \frac{d^2}{dt^2}\left|\mathcal{P}_k[z_0+tz]\right| \leqslant H''\|z\|^2 \end{gathered}$$

where

$$H'' = (2+\alpha_1')H', \qquad \alpha_1' = \sqrt{\frac{\alpha_1}{H\sqrt{3}}}.$$

From (4.1) and (4.3), we have

$$(4.4)\qquad \begin{gathered} \left|\frac{d}{dt}\beta_{rs}[z_0+tz]\right| \leqslant \frac{6H_1H'}{\varepsilon}\|z_0+tz-\tilde{z}_0\| \,.\, \|z\|, \\ \left|\frac{d^2}{dt^2}\beta_{rs}[z_0+tz]\right| \leqslant \frac{H_2}{|\varepsilon|} \,.\, \|z\|^2, \end{gathered}$$

where

$$H_2 = 3H_1H'' + 324H_1^2H'^2\alpha_1'^2.$$

Assuming that z_0+z lies in V_0, we have:

$$\beta_{rs}[z_0+z] - \beta_{rs}[z_0] - \delta\beta_{rs}[z_0,z] = \frac{1}{2}\left\{\frac{d^2}{dt^2}\beta_{rs}[z_0+tz]\right\}_{t=\xi}, \quad 0<\xi<1.$$

Taking into account (4.4), we obtain:

$$(4.5)\qquad |\beta_{rs}[z_0+z] - \beta_{rs}[z_0] - \delta\beta_{rs}[z_0,z]| \leqslant \frac{H_2}{2|\varepsilon|} \,.\, \|z\|^2.$$

To obtain $\delta\beta_{rs}[z_0,z] - \delta\beta_{rs}[\bar{z}_0,z]$ we observe that it coincides with the variation of the function of t $\delta\beta_{rs}[\bar{z}_0 + t(z_0,\bar{z}), z]$ in the interval $[0,1]$.

In view of the fact that $|\varepsilon| < 1$, a straightforward calculation shows that

$$(4.6)\qquad |\varepsilon|\,|\delta\beta_{rs}[z_0,z] - \delta\beta_{rs}[\bar{z}_0,z]| \leqslant H_3|\varepsilon|^{-\frac{1}{2}}\|z_0-\bar{z}_0\| \,.\, \|z\|,$$

where

$$H_3 = 3H_1^2(1+54H_1H'\alpha_1')\,\mathrm{vol}\,C_0.$$

From the hypothesis imposed on the free energy function, it follows that one may determine a constant H_1 independent of ε in such a way that z_0, $\bar{z}_0 \in V_0$ imply

$$(4.7)\qquad \begin{gathered} |\delta\mathcal{M}_K[z_0,z] - \delta\mathcal{M}_K([\bar{z}_0,z]| \leqslant H_4\|z_0-\bar{z}_0\| \,.\, \|z\|, \\ |\delta\mathcal{N}_K[z_0,z] - \delta\mathcal{N}_K[\bar{z}_0,z]| \leqslant H_4\|z_0-\bar{z}_0\| \,.\, \|z\|, \\ |\mathcal{M}_K[z_0+z] - \mathcal{M}_K[z_0] - \delta\mathcal{M}_K[z_0,z]| \leqslant H_4\|z\|^2, \\ |\mathcal{N}_K[z_0+z] - \mathcal{N}_K[z_0] - \delta\mathcal{N}_K[z_0,z]| \leqslant H_4\|z\|^2. \end{gathered}$$

Using (4.6) and (4.7) one deduces that a positive number h independent of ε exists, for which

$$(4.8)\qquad \begin{gathered} |\delta\Phi_K[z_0,z] - \delta\Phi_K[\bar{z}_0,z]| \leqslant h\|z_0-\bar{z}_0\| \,.\, \|z\| \,.\, |\varepsilon|^{-\frac{1}{2}}, \\ |\delta\Psi_K[z_0,z] - \delta\Psi_K[\bar{z}_0,z]| \leqslant h\|z_0-\bar{z}_0\| \,.\, \|z\| \,.\, |\varepsilon|^{-\frac{1}{2}}, \\ |\Phi_K[z_0+z] - \Phi_K[z_0] - \delta\Phi_K[z_0,z]| \leqslant h\|z\|^2 \,.\, |\varepsilon|^{-\frac{1}{2}}, \\ |\Psi_K[z_0+z] - \Psi_K[z_0] - \delta\Psi_K[z_0,z]| \leqslant h\|z\|^2|\varepsilon|^{-\frac{1}{2}}, \end{gathered}$$

f z_0, $\bar{z}_0 \in V_0$, $z_0+z \in V_0$, $z \in S$.

Taking into account that $z_0+tz \in V_0$ and $|\varepsilon| < 1$, we obtain

$$(4.9)\qquad \begin{gathered} \left|\frac{d^2}{dt^2}\mathcal{R}[z_0+tz]\right| \leqslant N_0'\|z\|^2, \\ \left|\frac{d^2}{dt^2}Q[z_0+tz]\right| \leqslant N_0\|z\|^2, \end{gathered}$$

where

$$N_0' = N_0(2+\alpha_1');$$

N_0 is the greatest of maxima of the absolute values of the functions M^k and their derivatives of the second and third order with respect to their arguments.

The functions of t $\mathcal{R}[z_0+tz]$, $Q[z_0+tz]$ have second derivatives with respect to t and these satisfy (4.9); if $z_0+z \in V_0$ one obtains

$$(4.10)\qquad \begin{gathered} |\mathcal{R}[z_0+z] - \mathcal{R}[z_0] - \delta\mathcal{R}[z_0,z]| \leqslant \frac{N_0'}{2}\|z\|^2, \\ |Q[z_0+z] - Q[z_0] - \delta Q[z_0,z]| \leqslant \frac{N_0}{2}\|z\|^2. \end{gathered}$$

In the same way as was done for (4.6) we obtain

$$(4.11)\qquad \begin{aligned} &|\delta\mathcal{R}[z_0, z] - \delta\mathcal{R}[\bar{z}_0, z]| \leqslant N_0 \| z_0 - \bar{z}_0 \| \cdot \| z \|, \\ &|\delta Q[z_0, z] - \delta Q[\bar{z}_0, z]| \leqslant N_0 \| z_0 - \bar{z}_0 \| \cdot \| z \|. \end{aligned}$$

From (4.10) and (4.11), it follows that one may find a positive number N, independent of ε for which

$$(4.12)\qquad \begin{aligned} &|\delta\Theta[z_0, z] - \delta\Theta[\bar{z}_0, z]| \leqslant N \| z_0 - \bar{z}_0 \| \cdot \| z \|, \\ &|\delta\varphi[z_0, z] - \delta\varphi[\bar{z}_0, z]| \leqslant N \| z_0 - \bar{z}_0 \| \cdot \| z \|, \\ &|\Theta[z_0 + z] - \Theta[z_0] - \delta\Theta[z_0, z]| \leqslant N \| z \|^2, \\ &|\varphi[z_0 + z] - \varphi[z_0] - \delta\varphi[z_0, z]| \leqslant N \| z \|^2, \end{aligned}$$

if $z_0, \bar{z}_0 \in V_0$ and $z_0 + \bar{z}_0 \in V_0$ for every $z \in S$.

One may show that the Hölder-coefficients satisfy similar relations; therefore one may determine a positive number k independent of ε for which

$$\| D[z_0, z] - D[\bar{z}_0, z] \| \leqslant k |\varepsilon|^{-\frac{1}{2}} \| z_0 - \bar{z}_0 \| \cdot \| z \|,$$

$$\| \omega(z_0, z) \| \leqslant k |\varepsilon|^{-\frac{1}{2}} \| z \|^2.$$

If one assumes that

$$\delta_\tau = \frac{\tau}{k} |\varepsilon|^{\frac{1}{2}},$$

it follows that $G(z)$ is continuously differentiable.

We can show that (3.11) coincides with the fundamental linear system of thermoelasticity. One proves that the solution has Hölder-continuous second derivatives with exponent λ and the components of the displacement satisfy (2.2). Hence the transformation (3.8) has a unique solution for every Z in the neighbourhood of Z_0 defined by (3.14). We denote by $|\varepsilon| \| Z_0' \|$ the norm of the element Z_0. If ε satisfies the inequality

$$(4.13)\qquad |\varepsilon| \cdot \| Z_0 \| < r |\varepsilon|^{\frac{1}{2}},$$

then the origin of S' lies in the neighbourhood of Z_0 defined by (3.14). We denote by

$$\varepsilon_0 = \min\left\{1, \frac{R^2}{\alpha_0'^2}, \frac{r^2}{\| Z_0' \|^2}\right\},$$

If $|\varepsilon| < \varepsilon_0$ then the equation

$$(4.14)\qquad G(z) = 0$$

admits one and only one solution. Thus we proved the theorem.

Theorem 4. *If F, f, W, h have the properties 3), 4) and F, f do not posses an axis of equilibrium, then we can determine the positive numbers $\alpha', \eta', \varepsilon_0'$ in such a way that if $\eta < \eta'$ one can find other two $\alpha_0 < \alpha$; $\varepsilon_0 < \varepsilon_0'$ so that for every $|\varepsilon| < \varepsilon_0$ the system (3.1) has a unique solution*

$$\beta = (\beta_{rs}), \qquad Z = (u_1, u_2, u_3, T)$$

satisfying the conditions

$$(4.15)\qquad |\beta_{rs} - \delta_{rs}| < \eta, \qquad \| z - \tilde{z}_0 \| < \alpha |\varepsilon|^{\frac{1}{2}}.$$

If

$$|\varepsilon| < \frac{1}{9\alpha'^2},$$

one can show that the matrix $y^k_{,t}(0)$ is symmetric and positive-definite Therefore the deformation $y(x)$ defined by $y(x) = x + u(x)$ and the temperature $T(x)$ satisfy all requirements for the solution of (3.2); hence $\hat{y}_t = \beta_{jt}\, y_j$ is the solution of the original problem defined by (1.1). Thus we have the following theorem of existence and uniqueness.

Theorem 5. *If hypotheses 3), 4) are satisfied and F, f have no axis of equilibrium, then it is possible to find numbers α'', ε'' such that for $|\varepsilon| < \varepsilon''$ the system (1.1) has a unique solution $\{y(x), T(x)\}$ satisfying the conditions*

$$y(0) = 0, \quad T(0) = T_0, \quad \| z - \tilde{z}_0 \| \leqslant \alpha''.$$

In what follows we prove the analyticity of the solution. Let be

$$F(z) = z - \Delta(G(z)).$$

The equation (4.14) is equivalent to

$$z = F(z).$$

Let us consider

$$(4.16)\qquad z_0 = \tilde{z}_0, \quad z_1 = F(z_0), \ldots, z_{n+1} = F(z_n), \ldots.$$

From (3.13) and (4.13) it is easily seen that $F(z)$ satisfies the conditions of the Banach fixed-point theorem and $z_n \in \overline{V}_0$ for every n.

As in [6] one shows that sequence $\{z_n\}$ converges to an element of S. Hence a positive number ε_0 exists such that for $|\varepsilon| < \varepsilon_0$ the limit of the sequence (4.16) is the solution of the equation (4.14). We assume the free energy function and the functions M^k are analytic in the region E. In the same manner as in [6] one proves that the solution depends analytically on ε, i. e. one can determine a positive number ε_0 such that for $|\varepsilon| < \varepsilon_0$ the power series

$$\sum_{n=1}^{\infty} \frac{u_k^{(n)}}{n!} \varepsilon^n ; \qquad \sum_{k=1}^{\infty} \frac{T^{(n)}}{n!} \varepsilon^n ,$$

are uniformily convergent.

These results justify the method used in [2] for the integration of the equations of non-linear thermoelasticity.

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O TEOREMĂ DE EXISTENŢĂ, UNICITATE ŞI ANALITICITATE PENTRU SOLUŢIA ECUAŢIEI TERMOELASTICITĂŢII NELINIARE

Rezumat

În acest articol se demonstrează o teoremă de existenţă şi unicitate pentru ecuaţiile termoelasticităţii neliniare în cazul cînd mărimile forţei masice, sursei de căldură şi a datelor pe frontieră sînt proporţionale cu un parametru ε.

Dacă energia liberă şi componentele fluxului termic sînt funcţii analitice, se arată că soluţia depinde analitic de parametru.

ÜBER PERFEKTE UND KRITISCH IMPERFEKTE GRAPHEN

VON

E. OLARU

1. Einleitung. In der vorliegenden Arbeit betrachten wir endliche ungerichtete Graphen $G = (X, U)$, die keine Schlingen enthalten und in denen je zwei verschiedene Knotenpunkte durch höchstens eine Kante verbunden sind. Dabei bezeichnet X die Knotenpunktmenge und U die Kantenmenge von G.

Der Graph $G' = (X', U') = [X']_G$ heißt *Teilgraph* des Graphen G, wenn $X' \subseteq X$ und wenn zwei Knotenpunkte in G' genau dann durch eine Kante verbunden sind, wenn sie auch in G durch eine Kante verbunden sind; wir sagen auch $[X']_G$ ist der durch die Knotenpunktmenge X' aufgespannte Teilgraph von G. Ist X' eine echte Untermenge von X, so nennen wir $[X']_G$ einen echten Teilgraphen von G und wir schreiben $[X']_G \subset G$. Ein vollständiger Teilgraph von G heißt eine *Clique* in G. Mit $\omega(G)$ bezeichnen wir die *Dichte* von G, d. h. die maximale Anzahl der Knotenpunkte, deren Menge eine Clique in G aufspannt.

Eine Teilmenge $S \subseteq X$ heißt eine *Menge unabhängiger Knotenpunkte* (in Bezug auf G), wenn es gilt

$$[S]_G = (S, \varnothing).$$

Die maximale Anzahl der unabhängigen Knotenpunkte von G bezeichnen wir mit $\alpha(G)$; $\alpha(G)$ ist die *Stabilitätszahl* des Graphen G. Eine Menge S von in G unabhängigen Knotenpunkten mit genau $\alpha(G)$ Elementen wollen wir ein *Stabilitätssystem* des Graphen G nennen. Der Graph $G^* = (X^*, U^*)$ heißt einen *Kreis* $L_k (k \geq 3)$ (oder einen k-Kreis), wenn man die Knotenpunkte von G derart als Folge $x_1, x_2, \ldots, x_k$ anordnen kann, daß

$$U^* = \{(x_i, x_{i+1})\}_{i=1,2,\ldots,k} \quad (x_{k+1} = x_1)$$

gilt.

Die Anzahl k der Kanten heißt die *Länge* des Kreises, und man nennt den Kreis gerade oder ungerade, je nachdem, ob k eine gerade oder unge-