$$\bigcap_{\beta \in \Gamma} \mathcal{L}_E^p(T, \mu_\beta) \subset \mathcal{L}_E^p(T, m). \tag{13}$$

Relațiile (12) și (13) demonstrează afirmația.

Teorema 4. *Fie $\mathcal{B}$ semitributul părților boreliene relativ compacte ale lui T. Spațiul $\mathcal{E}_E(\mathcal{B})$ al funcțiilor boreliene etajate este dens în fiecare spațiu $\mathcal{L}_E^p(T, m)$, $1 \leqslant p < \infty$.*

Într-adevăr, datorită unui rezultat din [3, p. 148], care se poate adapta cu ușurință în cazul de față, $\mathcal{E}_E(\mathcal{B})$ este dens în fiecare spațiu $\mathcal{L}_E^P(T, \mu_\beta)$, încît rezultă $\mathcal{E}_E(\mathcal{B})$ dens și în $\mathcal{L}_E^P(T, m)$, pentru orice P, $1 \leqslant P < \infty$.

Spațiul $\mathcal{L}_E^1(T, m)$ construit aici coincide cu spațiul funcțiilor integrabile în raport cu o măsură regulată, cu variație finită $m : \mathcal{B} \to \mathcal{L}_\psi(E, F)$ ([2], p. 287).

După cum s-a văzut în tratarea problemei ce face obiectul acestei Note, un rol important este jucat de noțiunea de β-modul al unei măsuri majorate, care apare cu rol analog β-variației unei măsuri definite ca funcție de mulțime, în procesul de integrare în raport cu aceasta.

Menționăm că metoda de construire a spațiului $\mathcal{L}_E^p(T, m)$ apare în mod natural, dacă se ține seamă de teorema de reprezentare a operațiilor liniare ψ-majorate, $U : \mathcal{K}_E(T) \to F$ [2, p. 290].

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UNE MÉTHODE D'INTEGRATION PAR RAPPORT A UNE MESURE DÉFINIE SUR UN ESPACE LOCALEMENT COMPACT, A VALEURS DANS UN ESPACE LOCALEMENT CONVEXE

Résumé

Dans cette Note on donne une méthode de construction de l'espace des fonctions intégrables par rapport à une mesure définie sur un espace localement compact, à valeurs dans un espace localement convexe. A ce but on utilise la méthode de Bourbaki qui considère les mesures vectorielles comme opérateurs linéaires.

Par l'introduction de la notion de β-module d'une mesure majorée, on obtient une famille de mesures positives, qui permet la construction immédiate de l'espace des fonctions intégrables par rapport à cette mesure.

A SYSTEM OF VOLTERRA INTEGRAL EQUATIONS ARISING IN THE THEORY OF SUPERFLUIDITY [1])

BY

R. K. MILLER

1. Introduction. Suppose the region between two concentric infinite cylinders is filled with liquid helium. If the cylinders are rotated to set up a radial flow, this flow should eventually become steady. The purpose here is to prove this by studying the details of the flow at the boundaries. This will be accomplished by reducing the problem to a system of two nonlinear Volterra integral equations. C. C. Lin's [1] formulation of the boundary conditions for superfluid flow will be used.

If the velocity components of the flow in cylindrical coordinates are initially of the form $u_r = u_z \equiv 0$, $u_\theta = F(r)$ at $t=0$, then for $t > 0$ the (nonlinear) flow equations reduce to $u_r = u_z \equiv 0$ and $u_\theta = u(t, r)$ where $u(t, r)$ is the solution of the initial-boundary value problem

$$u_t = L(u) = \nu\,\{u_{rr} + (1/r)u_r - (1/r^2)u\}, \tag{1.1}$$

for $t > 0$ and $r_1 < r < r_2$, where $0 < r_1 < r_2 < \infty$,

$$u(0, r) = F(r),\quad r_1 < r < r_2\,. \tag{1.2}$$

and

$$u_r(t, r_j) = (-1)^{j+1} g_j(u(t, r_j) - \Omega_j(t)),\quad j = 1,2 \text{ and } t > 0. \tag{1.3}$$

where $g_j(u) = k_j u^3$, ν, $k_j > 0$ and Ω_j is the speed of the cylinder located at $r = r_j$. It is assumed that $\Omega_j(t) \to \Omega_j$ as $t \to \infty$ where $\Omega_j \geqslant 0$.

This problem will be reduced to an equivalent system of nonlinear integral equations for the boundary velocities $u_1(t) = u(t, r_1)$ and $u_2(t) = u(t, r_2)$. This system can be written in matrix-vector notation as

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$$\text{(E)} \qquad u(t) = f(t) - \int_0^t a(t-s)\, G(u(s) - \Omega(s))\, ds,$$

where

$$\text{(1.4)} \qquad u(t) = \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \ f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix}, \quad \Omega(t) = \begin{pmatrix} \Omega_1(t) \\ \Omega_2(t) \end{pmatrix}, \quad G(u) = \begin{pmatrix} g_1(u_1) \\ g_2(u_2) \end{pmatrix}$$

and $a(t)$ is a two by two matrix with entries $a_{ij}(t)$. The functions $f_j(t)$ and $a_{ij}(t)$ have the following form. Let $\{\lambda_n\}$ and $\{y_n(x)\}$ be the set of all eigenvalues and corresponding orthonormal eigenfunctions of the problem

$$\text{(1.5)} \qquad L_0 y = \frac{d}{dx}\left(x \frac{dy}{dx}\right) - \frac{1}{x} y = -\lambda x y(x),\ y'(r_1) = y'(r_2) = 0.$$

(The eigenfunctions $y_n(x)$ are Bessel functions of the form $y_n(x) = \alpha_n J_1(\sqrt{\lambda_n} x) + \beta_n Y_1(\sqrt{\lambda_n} x)$. Let $\{F_n\}$ be the set of Fourier coefficients of the initial function F, that is

$$\text{(1.6)} \qquad F(x) \sim \sum_{n=1}^{\infty} F_n y_n(x), \qquad F_n = \int_{r_1}^{r_2} F(x) y_n(x) x\, dx.$$

Then one has

$$\text{(1.7)} \qquad f_j(t) = \sum_{n=1}^{\infty} F_n y_n(r_j) \exp(-\nu\lambda_n t),\ j = 1, 2$$

and

$$\text{(1.8)} \qquad a_{ij}(t) = \nu \sum_{n=1}^{\infty} r_j y_n(r_i) y_n(r_j) \exp(-\nu\lambda_n t).$$

for $i, j = 1, 2$.

System (E) will be studied by applying some results and techniques developed by Weis [2] for Volterra integral equations arising in the theory of nonlinear heat problems. The analysis here is more complicated than that in [2] because $\lambda = 0$ is not an eigenvalue of problem (1.5). Moreover the limiting algebraic equations obtained here are more complicated. Friedman [3, pp. 206–212] has studied some problems similar to (1.1 – 3). His hypothesis are not satisfied here and his conclusions are incompatible with ours. For other related work see Mann and Wolf [4], Roberts and Mann [5], Padmavally [6], Levinson [7], Friedman [8, 9] and Miller [10, 11].

2. Equivalence of the two problems. The purpose of this section is to derive equation (E) from problem (1.1 – 3) and to prove the equivalence of the two problems under the following precise hypothesis on the functions $F(r)$, $\Omega_j(t)$ and $g_j(u)$:

(H 1) $F \in C^1[r_1, r_2]$, $F'(r_1) = F'(r_2) = 0$,

(H 2) $\Omega_j(t) \in C^1[0, \infty)$ for $j = 1, 2$,

(H 3) $g_j(u_j) \in C^1(-\infty, \infty)$ for $j = 1, 2$.

Note that the eigenvalues λ_n of problem (1.5) are all positive and the corresponding eigenfunctions $y_n(r)$, are orthogonal and complete in $L^2[r_1, r_2]$ with weight function $\rho(r) = r$. Without loss of generality it is assumed that the functions y_n are normalized so that

$$\int_{r_1}^{r_2} y_n(r) y_m(r) r\, dr = \delta_{nm}.$$

Moreover $y_1(r) \neq 0$ on $r_1 \leq r \leq r_2$, so one may assume $y_1(r) > 0$ on $[r_1, r_2]$. Let $G(r, \rho)$ be the Green's functions for (1.5), that is

$$\text{(2.1)} \qquad G(r, \rho) = \frac{r_1^2 r_2^2}{2(r_2^2 - r_1^2)} \left(\frac{\rho}{r_2^2} + \frac{1}{\rho}\right)\left(\frac{r}{r_1^2} + \frac{1}{r}\right),$$

when $r_1 \leq r \leq \rho \leq r_2$ and $G(r, \rho) = G(\rho, r)$ when $r > \rho$. Since all eigenvalues are positive, then by Mercer's theorem (see e. g. [12, p. 124] or [13]) it follows that

$$\text{(2.2)} \qquad G(r, \rho) = \sum_{n=1}^{\infty} y_n(r) y_n(\rho)/\lambda_n,$$

with convergence uniform and absolute in the region $r_1 \leq r, \rho \leq r_2$. Moreover if F satisfies (H 1) above, then its Fourier series (1.6) converges uniformly and absolutely to $F(r)$ on $r_1 \leq r \leq r_2$. These facts will be needed in the sequel.

General results on existence, uniqueness and continuation of solutions of integral equations will also be needed later. Results of this type can be found in [14, Chapter 2].

Lemma 1. *If (H 1) is true, then the functions $f(t)$ and $a(t)$ defined by (1.7) and (1.8) above satisfy the following conditions:*

(i) *$f(t)$ is continuous and bounded on $0 \leq t < \infty$.*

(ii) *$f'(t)$ exists for $t > 0$ and $f' \in L^1(0, 1) \cap C(0, \infty)$.*

(iii) *$a_{ij}(t) \in L^1(0, \infty) \cap C(0, \infty)$ for $i, j = 1, 2$,*

(iv) $\lim_{h \to 0} \int_0^h |f(t+h) - f(t)|/h\, dt = 0.$

Proof. Since the Fourier series (1.6) converges uniformly and absolutely on $r_1 \leq r \leq r_2$, then

$$|f_j(t)| \leqslant \sum_{n=1}^{\infty} \exp(-\nu\lambda_n t) \, |F_n y_n(r_j)| \leqslant \sum_{n=1}^{\infty} |F_n y_n(r_j)| < \infty,$$

for all $t \geqslant 0$. This means that the series for $f_j(t)$ converges uniformly on R^+. Thus (i) is true. To prove (ii) note that

$$|f_j'(t)| \leqslant \sum_{n=1}^{\infty} \nu\lambda_n \exp(-\nu\lambda_n t) \, |F_n y_n(r_j)| < \infty,$$

for all $t > 0$. Moreover

$$\int_0^{\infty} |f'(t)| \, dt \leqslant \sum_{n=1}^{\infty} \int_0^{\infty} \nu\lambda_n \exp(-\nu\lambda_n t) \, dt \, |F_n y_n(r_j)|$$

$$= \sum_{n=1}^{\infty} |F_n y_n(r_j)| < \infty.$$

This proves (ii).

Since the series (2.2) converges uniformly and absolutely, then there exists a uniform bound

$$K = \sup\{ |y_n(r_i) \, y_n(r_j)| / \lambda_n : i, j = 1, 2, \; n = 1, 2, 3, \ldots \}.$$

On using this in (1.8) it follows that

$$|a_{ij}(t)| \leqslant \sum_{n=1}^{\infty} (r_j \nu K \lambda_n) \exp(-\nu\lambda_n t).$$

with the last series uniformly convergent on compact subsets of $0 < t < \infty$. Thus $a_{ij} \in C(0, \infty)$. Moreover one has

$$\int_0^{\infty} |a_{ij}(t)| \, dt \leqslant \sum_{n=1}^{\infty} \left(\int_0^{\infty} \exp(-\nu\lambda_n t) dt \right) r_j \nu \, |y_n(r_j) \, y_n(r_i)| =$$

$$= r_j \sum_{n=1}^{\infty} |y_n(r_i) \, y_n(r_j)| / \lambda_n < \infty.$$

This proves (iii).

To prove (iv) fix $h > 0$ and compute

$$\int_0^{h} |f_j(t+h) - f_j(t)| / h \, dt \leqslant \sum_{n=1}^{\infty} |F_n y_n(r_j)| \, (1 - \exp(-\nu\lambda_n h)) / h$$

$$\times \int_0^{h} \exp(-\nu\lambda_n t) \, dt.$$

The series on the right is dominated uniformly for $0 < h \leqslant 1$ by the summable series $\sum_{n=1}^{\infty} |F_n y_n(r_j)| / (\nu\lambda_1)$. Moreover for each fixed $n \geqslant 1$ one has

$$F_n y_n(r_j) \, (1 - \exp(-\nu\lambda_n h)) / (h) \times \int_0^{h} \exp(-\nu\lambda_n t) dt \to 0,$$

as $h \to 0$. Therefore (iv) follows from the Lebesgue dominated convergence theorem. This completes the proof of Lemma 1.

Theorem 1. *Suppose* $0 < T \leqslant +\infty$ *and suppose* $u(t, r)$ *satisfies the following conditions:*

(i) $u(t, r)$ *exists and is continuous on* $0 \leqslant t < T$, $r_1 \leqslant r \leqslant r_2$,

(ii) u_t, u_r *and* u_{rr} *exist and are continuous on* $0 < t < T$, $r_1 < r < r_2$,

(iii) $u(t, r)$ *satisfies* (1.1), (1.2) *and* (1.3) *in the sense that*

$$\lim_{r \to r_1^+} u_r(t, r) = g_1(u(t, r_1) - \Omega_1(t)), \quad \lim_{r \to r_2^-} u_r(t, r) = g_2(u(t, r_2) - \Omega_2(t))$$

for each $t \in (0, T)$.

(iv) *Define* $A_j(t) = (-1)^{j+1} g_j(u(t, r_j) - \Omega_j(t))$ *for* $j = 1, 2$.

Assume the functions $A_j(t)$ *are in* $C[0, T) \cap C^1(0, T)$ *and are absolutely continuous in a neighborhood of* $t = 0$. *If* (H 1) *and* (H 2) *are true, then the functions* $u_j(t) = u(t, r_j)$ *satisfy equation* (E) *for all* $t \in [0, T)$.

Proof. Define $B_1(r) = (r - r_2)^2 / (2(r_1 - r_2))$, $B_2(r) = (r - r_1)^2 / (2(r_2 - r_1))$, $K(t, r) = A_1(t) B_1(r) + A_2(t) B_2(r)$ and $w(t, r) = u(t, r) - K(t, r)$. Then w will satisfy

$$w_t - L(w) = -K_t + L(K) \equiv h(t, r),$$

$$w(0, r) = F(r) - A_1(0) \, B_1(r) - A_2(0) \, B_2(r) \equiv F_1(r),$$

and

$$\frac{\partial w}{\partial r}(t, r_j) = 0 \quad \text{for } j = 1, 2.$$

Since $F_1 \in C[r_1, r_2]$, h is continuous for $0 < t < T$, $r_1 \leqslant r \leqslant r_2$ and

$$\int_0^{t} \int_{r_1}^{r_2} |h(t, r)| \, dr \, dt < \infty$$

for $0 < t < T$, then $w(t, r)$ can be found in the usual way using the Green's function (see [15, Theorem 1]). This means that

$$w(t, r) = \int_{r_1}^{r_2} \left\{ \sum_{n=1}^{\infty} \exp(-\nu\lambda_n t) y_n(r) y_n(\rho) \right\} F_1(\rho)\, d\rho +$$

$$+ \int_0^t \int_{r_1}^{r_2} \left\{ \sum_{n=1}^{\infty} \exp(-\nu\lambda_n(t-s)) y_n(r) y_n(\rho) \right\} h(s, \rho) d\rho\, ds = u(t, r) - K(t, r).$$

The definitions of F_1, h and K, integration by parts and the absolute convergence of the series (2.2) can be used to rearrange this expression into the form

$$u(t, r) = \sum_{n=1}^{\infty} F_n y_n(x) \exp(-\nu\lambda_n t) + \tag{2.3}$$
$$+ \int_0^t \left\{ \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \nu \exp(-\lambda_n \nu(t-s)) (-1)^j r_j y_n(r_j) y_n(r) A_j(s) \right\} ds.$$

Since $u(t, r)$ is continuous at $r = r_j$, thus by setting $r = r_1$ and $r = r_2$ in (2.3) one obtains (E). This proves Theorem 1.

Theorem 2. *Suppose* (H 1) – (H 3) *are true. If* $u_1(t)$ *and* $u_2(t)$ *are continuous solutions of* (E) *on* $0 \leqslant t < T$ *where* $0 < T \leqslant +\infty$, *then these two functions are the boundary values of a function* $u(t, r)$ *which satisfies conditions* (i) – (iv) *of Theorem 1.*

Proof. Since the function f defined by (1.7) satisfies conditions (i) and (ii) of Lemma 1 and (H 2) and (H 3) are true, then results in [16, Theorem 2] imply that the solution $u(t)$ of (E) is continuous, has a continuous derivative $u'(t)$ on $0 < t < T$ and u' has at worst an integrable singularity at $t = 0$. Define $A_j(t) = (-1)^{j+1} g_j(u_j(t) - \Omega_j(t))$ for $j = 1, 2$ and define $u(t, r)$ using (2.3). The steps in the proof of Theorem 1 can be reversed to see that conditions (i) and (ii) of that theorem are true for the function $u(t, r)$. Moreover one has

$$\lim_{r \to r_j} u_r(t, r_j) = A_j(t) = (-1)^{j+1} g_j(u_j(t) - \Omega_j(t)).$$

Therefore (2.3) can be rewritten as

$$u(t, r) = \sum_{n=1}^{\infty} F_n y_n(r) \exp(-\nu\lambda_n t) - \tag{2.4}$$
$$- \int_0^t \left\{ \sum_{n=1}^{\infty} \sum_{j=1}^{2} \nu \exp(-\nu\lambda_n(t-s)) r_j y_n(r_j) y_n(r) g_j(u_j(s) - \Omega_j(s)) \right\} ds.$$

By putting $r = r_i$ in (2.4) and comparing with (E) one sees that

$$u(t, r_i) = f_i(t) - \int_0^t \sum_{j=1}^{2} a_{ij}(t-s)\, g_j(u_j(s) - \Omega_j(s))\, ds = u_j(t),$$

or $u(t, r_i) \equiv u_i(t)$ for $0 \leqslant t < T$. This completes the proof of Theorem 2.

Corollary 1. *If* (H 1 – 3) *are true, then there exists at most one function* $u(t, r)$ *which satisfies conditions* (i) – (iv) *of Theorem 1.*

Proof. Under these hypotheses system (E) has at most one solution on any interval $0 \leqslant t < T \leqslant +\infty$.

The results of this section show that the problem of existence on $0 \leqslant t < \infty$ and behavior as $t \to \infty$ of the solution of (1.1 – 3) can be reduced to the same problems for the integral equation (E). These problems will be studied in the remainder of this paper.

3. Asymptotic Behavior of Solutions. Some results of Weis [2] will be needed. These results are labeled as Theorems 3, 4 and 5 below. Consider a system of equations

$$x(t) = f(t) + \int_0^t a(t-s)\, g(s, x(s))\, ds, \quad t \geqslant 0, \tag{3.1}$$

where x, f and g have values in real n-space R^n and $a(t)$ is n by n matrix valued. Let $R^+ = [0, \infty)$ and assume:

(A) the matrix a is locally of class L^1 on $0 \leqslant t < \infty$,

(G) $g : R^+ \times R^n \to R^n$ is continuous, $g(t, x)$ is locally Lipschitz continuous in x, and g has „separated structure”, that is $g(t, x) = \mathrm{col}\,(g_i(t, x_i) : i = 1, 2, \ldots, n)$.

Definition 1. *Given a matrix* $a(t)$ *satisfying* (A) *the resolvent* $r(t)$ *of* $a(t)$ *is the unique, locally integrable, matrix solution of the equation*

$$r(t) = a(t) - \int_0^t a(t-s)\, r(s)\, ds.$$

Definition 2. *Given a vector or matrix valued function* $h(t)$ *we shall say that* h *is nonnegative, monotone increasing, monotone decreasing,* $h(t) \geqslant 0$, *etc. when each component of* $h(t)$ *has the stated property.*

Theorem 3. (Weis). *In* (3.1) *let* $a(t)$ *satisfy* (A), $f(t) \equiv 0$ *and* $g(s, x) = -Nx + T(s)$. *If for each* $N > 0$ *and each continuous and nonnegative function* $T(s)$ *the solution of* (3.1) *is nonnegative, then the resolvent* r_N *of* Na *is nonnegative.*

Theorem 4. (Weis). *Assume* (A), (G) *and* $g(s, 0) \geqslant 0$ *for all* s *in* R^+. *Let* r_N *(the resolvent of* Na*) be nonnegative when* $N > 0$. *Let* $f \in C(R^+)$ *and let*

$$f(t) - \int_0^t r_N(t-s) f(s)\, ds \geqslant 0,$$

for all $t \in R^+$. Then the solution of (3.1) *is nonnegative for as long as it exists.*

The proof in [2] for the last theorem is easily modified to prove the following:

Theorem 4'. *Assume* (A) *and that $g(s, x) = \mathrm{col}\,(h_j(s)x_j + T_j(s) : j = 1, 2, \ldots, n)$ where $h_j(s)$ and $T_j(s)$ are continuous and $T_j(s) \geqslant 0$. Suppose r_N is nonnegative all $N > 0$. If f is locally L^1 on R^+ and if*

$$f(t) - \int_0^t r_N(t-s) f(s)\, ds \geqslant 0 \qquad \text{a. e.}$$

for each $N > 0$, then the solution of (3.1) *is nonnegative for as long as it exists.*

Theorem 5. (Weis). *Consider* (3.1) *and a second system*

$$y(t) = f_0(t) + \int_0^t a(t-s)\, g_0(s, y(s))\, ds. \tag{3.2}$$

Assume f and $f_0 \in C(R^+)$, a satisfies (A), *g and g_0 satisfy* (G) *and r_N is nonnegative for all $t \in R^+$ and $N > 0$. If $g_0(s, x) \geqslant g(s, x)$ for all $(s, x) \in R^+ \times R^n$ and if*

$$f_0(t) - f(t) - \int_0^t r_N(t-s)\,(f_0(s) - f(s))\, ds \geqslant 0,$$

for all $t \in R^+$ and all $N > 0$, then $y(t) \geqslant x(t)$ for as long as both solution exist.

These results will now be applied to system (E) with $R^n = R^2$.

Lemma 2. *If $a(t)$ is given by* (1.8) *and if* (H 2–3) *are true then for any $N > 0$ the resolvent r_N is nonnegative on R^+.*

Proof. Let $T : R^+ \to R^2$ be of class C^1 and nonnegative. By Theorems 1 and 2 above the system

$$x(t) = \int_0^t a(t-s)\,(-Nx(s) + T(s))\, ds,$$

is equivalent to the boundary value problem

$$v_t = \nu\left(v_{rr} + \frac{1}{r}v_r - \frac{1}{r^2}v\right), \quad v(0, r) \equiv 0,$$

$$\frac{\partial v}{\partial r}(t, r_j) = -Nv(t, r_j) + T_j(t).$$

The maximum principle for parabolic problems implies that $v(t, r) \geqslant 0$ for all $t \geqslant 0$ so that $x_j(t) = v(t, r_j) \geqslant 0$, $j = 1, 2$. Since any continuous nonnegative function $T(s)$ can be approximated by a C^1, nonnegative function, the hypothesis of Theorem 3 are true. Apply Theorem 3 to complete the proof of Lemma 2.

Corollary 2. *Under the hypotheses of Lemma 2 one has $r_N \in L^1(0, \infty)$ for any $N > 0$.*

Proof. The proof reduces to the following: If $a(t)$ and its resolvent $r(t)$ are nonnegative and if $a \in L^1(0, \infty)$, then $r \in L^1(0, \infty)$. The proof of this statement is elementary.

Lemma 3. *Suppose F satisfies* (H 1). *$F(r)$ is nonnegative on $[r_1, r_2]$ and $f(t)$ is defined by* (1.7). *Let $y(t)$ solve*

$$y(t) = f(t) - \int_0^t Na(t-s)\, y(s)\, ds,$$

hat is

$$y(t) = f(t) - \int_0^t r_N(t-s) f(s)\, ds.$$

f $N > 0$ then $y(t) \geqslant 0$ for all $t \in$ R$^+$.

Proof. By Theorems 1 and 2 above the components of $y(t)$ are the boundary values $v(t, r_j) \equiv y_j(t)$ of the problem

$$v_t = \nu\left\{v_{rr} + \frac{1}{r}v_r - \frac{1}{r^2}v\right\}, \quad v(0, r) = F(r).$$

$$\frac{\partial v}{\partial r}(t, r_j) = -Nv(t, r_j), \; j = 1, 2.$$

Therefore the result follows from the maximum principle for parabolic equations.

Consider system (E) together with (1.4); (1.6), (1.7), (1.8) and hypotheses (H 1–3). In addition assume:

(H 4) For $j = 1, 2$ the limit $\Omega_j(t) = \Omega_j > 0$ exists as $t \to \infty$. Let $\Omega_u(t)$ be a nonnegative, nonincreasing C^1 function such that $\Omega_u(t) \geqslant \Omega(t)$ for all $t \in R^+$ and $\lim \Omega_u(t) = \lim \Omega(t)$ as $t \to \infty$. Let $\Omega_L(t)$ be a nondecreasing, C^1 function such that $\Omega_L(t) \leqslant \Omega(t)$ for all $t \in R^+$ and $\lim \Omega_L(t) = \lim \Omega(t)$

as $t \to \infty$. Since the first eigenfunction $y_1(r)$ of problem (1.5) is positive on $r_1 \leqslant r \leqslant r_2$, then there is a constant $K > 0$ such that $Ky_1(r) \geqslant F(r) \geqslant -Ky_1(r)$ on $[r_1, r_2]$. In addition choose K so large that if

$$f_0(t) = K \exp(-\nu\lambda_1 t)\begin{pmatrix} y_1(r_1) \\ y_2(r_2) \end{pmatrix}, \tag{3.3}$$

then $f_0(t) + \Omega_L(t) \geqslant 0$ for all $t \in R^+$ and $\Omega_u(0) - f_0(0) \leqslant 0$. Let $x_u(t)$ and $x_L(t)$ be the solutions of the two systems

$$x_L(t) = -f_0(t) - \int_0^t a(t-s)\, G(x_L(s) - \Omega_L(s))\, ds, \tag{3.4}$$

and

$$x_u(t) = f_0(t) - \int_0^t a(t-s)\, G(x_u(s) - \Omega_u(s))\, ds. \tag{3.5}$$

In addition to (H 1–4) assume:

(H 5) For $j = 1, 2$ let $g_j(0) = 0$ and $g_j'(u) \geqslant 0$ for $-\infty < u < \infty$.

Theorem 6. *If* (H 1–5) *are true with $\Omega_1 > 0$ and $\Omega_2 > 0$, if $x_u(t)$ and $x_L(t)$ are the functions constructed above and if $u(t)$ solves* (E), *then $x_L(t)$, $u(t)$ and $x_L(t)$ exist for all $t \in R^+$ and satisfy $-f_0(t) \leqslant x_L(t) \leqslant u(t) \leqslant x_u(t) \leqslant x_u(0)$. Moreover $x_L(t)$ is monotone nondecreasing and $x_u(t)$ is monotone nonincreasing. The limits $x_L(\infty)$ and $x_u(\infty)$ are solutions of the algebraic system of equations*

$$x + \nu M_0 M_1 G(x - \Omega) = 0, \tag{3.6}$$

where

$$M_0 = \begin{pmatrix} r_1^2 + r_2^2 & 2r_1 r_2 \\ 2r_1 r_2 & r_1^2 + r_2^2 \end{pmatrix}, \quad M_1 = \begin{pmatrix} \dfrac{r_1}{r_2^2 - r_1^2} & 0 \\ 0 & \dfrac{r_2}{r_2^2 - r_1^2} \end{pmatrix}, \quad \Omega = \begin{pmatrix} \Omega_1 \\ \Omega_2 \end{pmatrix}. \tag{3.7}$$

Proof. We first show that $x_L(t) \leqslant u(t)$ for as long as both of these functions exist. Note that $-G(x - \Omega(s)) \geqslant -G(x - \Omega_L(s))$ for all $s \geqslant 0$ and all $x \in R^2$ since $g_j(u)$ is monotone nondecreasing and $\Omega(s) \geqslant \Omega_L(s)$ for all $t \geqslant 0$. Next note that $f(t)$ is obtained from the initial function $F(r)$ via the formula (1.7). Therefore (3.3) implies that $f(t) + f_0(t)$ is obtained in the same way from the nonnegative function $F(r) + Ky_1(r)$. Lemma 3 implies that for any $N > 0$

$$f(t) + f_0(t) - \int_0^t r_N(t-s)\, \{f(s) + f_0(s)\}\, ds \geqslant 0.$$

Now apply Theorem 5 to see that $u(t) \geqslant x_L(t)$.

The same type of argument can be used to see that $x_u(t) \geqslant u(t)$ for as long as both functions exists. That is note that $Ky_1(r) - F(r) \geqslant 0$ for $r_1 \leqslant r \leqslant r_2$, that $-G(x - \Omega(s)) \leqslant -G(x - \Omega_u(s))$, apply Lemma 3 and then apply Theorem 5.

In order to see that $u_L(t) \geqslant -f_0(t)$ define $U_L(t) = u_L(t) + f_0(t)$. Then one has

$$U_L(t) = \int_0^t a(t-s)\, \{-G(U_L(s) - \Omega_L(s) - f_0(s))\}\, ds.$$

By (H 5) it follows that $G(y) \leqslant 0$ if $y \leqslant 0$. Since the number K in $f_0(t)$ was chosen so large that $\Omega_L(t) + f_0(t) \geqslant 0$ for all $t \in R^+$, then $-G(-\Omega_L(s) - f_0(s)) \geqslant 0$ for all $s \geqslant 0$. By Theorem 4 $U_L(t) = u_L(t) + f_0(t) \geqslant 0$ for as long as $u_L(t)$ exists.

Theorem 1 of [16] may be used to justify diferentiating system (3.4). Since $f_0'(t) = -\lambda_1 f_0(t)$ and $x_L(0) = -f_0(0)$, then

$$x_L'(t) = \lambda_1 f_0(t) + a(t)\{-G(-\Omega_L(0) - f_0(0))\} + \tag{3.8}$$
$$+ \int_0^t a(t-s)\, H(s)\, (\Omega_L'(s) - x_L'(s))\, ds,$$

where

$$H(s) = \begin{pmatrix} g_1'(x_{L1}(s) - \Omega_{L1}(s)) & 0 \\ 0 & g'(x_{L2}(s) - \Omega_{L2}(s)) \end{pmatrix} \geqslant 0,$$

and $\Omega_L'(s) \geqslant 0$ for all $s \geqslant 0$. For any $N > 0$ it follows from Lemma 3 that

$$\lambda_1 f_0(t) - \int_0^t r_N(t-s)\, \lambda f_0(s)\, ds \geqslant 0, \quad s \geqslant 0.$$

Since $r_N(t)$ is the resolvent of $Na(t)$, then

$$a(t)\{-G(-\Omega_L(0) - f_0(0))\} - \int_0^t r_N(t-s)\, a(s)\, \{-G(-\Omega(0) - f_0(0))\}\, ds \tag{3.9}$$
$$= (Na(t) - \int_0^t r_N(t-s)\, Na(s)\, ds)\, \{-G(-\Omega_L(0) - f_0(0))\}/N$$
$$= r_N(t)\, \{-G(-\Omega_0(0) - f_0(0))\}/N.$$

Since $\Omega_0(0) + f_0(0) \geqslant 0$, $N > 0$ and $r_N(t) \geqslant 0$ (see Lemma 2 above), then the left hand side of (3.9) is nonnegative for all $t \geqslant 0$. The hypotheses of Theorem 4' are verified. It follows that the solution of (3.8) is nonnegative. This means that $x_l(t)$ is monotone nondecreasing for as long as it exists.

In order to see that $x_u(t)$ is nonincreasing use Theorem 1 of [16] to justify differentiating (3.5). One obtains

$$x_u'(t) = -\lambda_1 f_0(t) + a(t)\{-G(f_0(0) - \Omega_u(0))\} + \int_0^t a(t-s)\,H(s)\,(-x_u'(s) + \Omega_u'(s))\,ds. \tag{3.10}$$

Here $H(s)$ is the nonnegative matrix

$$H(s) = \begin{pmatrix} g_1'(x_{u1}(s) - \Omega_{u1}(s)) & 0 \\ 0 & g_2'(x_{u2}(s) - \Omega_{u2}(s)) \end{pmatrix}.$$

Since equation (3.10) is linear, one can write $x_L(t) = -z_1(t) - z_2(t)$ where

$$z_1(t) = \lambda_1 f_0(t) - \int_0^t a(t-s)\,H(s)\,z_1(s)\,ds,$$

and

$$z_2(t) = a(t)\,G(f_0(0) - \Omega_u(0)) - \int_0^t a(t-s)\,H(s)\,(z_2(s) + \Omega_u'(s))\,ds.$$

By Lemma 3 it follows immediately that $z_1(t) \geqslant 0$. Since $f_0(0) - \Omega_u(0) \geqslant 0$, then for any $N > 0$

$$a(t)\,G(f_0(0) - \Omega_u(0)) - \int_0^t r_N(t-s)a(s)\,G(f_0(0) - \Omega_u(0))ds$$

$$= r_N(t)\,G(f_0(0) - \Omega_u(0))/N \geqslant 0.$$

In addition $-\Omega_u'(s) \geqslant 0$ for all $s \geqslant 0$. Therefore Theorem 4' applies to conclude that $z_2(t) \geqslant 0$ a. e. on R^+. This shows that $x_u(t)$ is monotone nonincreasing for as long as it exists.

If $y(t)$ is any of the three functions x_l, u or y_u, then on the interval of existence of $y(t)$ one has $-f_0(0) \leqslant y(t) \leqslant x_u(0)$. This means that $y(t)$ can be continued as a solution of the appropriate integral equation to the entire interval R^+.

To complete the proof we must compute the limits $x_L(\infty)$ and $x_u(\infty)$. Since both limits are computed in the same way we consider only $x_u(t)$. Let $h(t, s) = G(x_u(t-s) - \Omega_u(t-s))$ if $0 \leqslant s \leqslant t$ and $h(t, s) = 0$ if $t < s < \infty$. Then (3.5) can be written in the form

$$x_u(t) = f_0(t) - \int_0^\infty a(s)\,h(t, s)\,ds.$$

Since $a \in L^1(R^+)$, $|h(t, s)|$ is uniformly bounded and

$$\lim_{t\to 0} h(t, s) = G(x_u(\infty) - \Omega),$$

for each $s \geqslant 0$, then by the Lebesgue Dominated Convergence theorem one has

$$x_u(\infty) = f_0(\infty) - \int_0^\infty a(s)\,G(x_u(\infty) - \Omega)\,ds.$$

The first term $f_0(\infty)$ is zero. By using (1.8) and (2.2) one can easily compute

$$\int_0^\infty a(s)\,ds = \frac{\nu}{r_2^2 - r_1^2}\begin{pmatrix} r_1(r_1^2 + r_2^2) & 2r_1 r_2^2 \\ 2r_1^2 r_2 & r_2(r_1^2 + r_2^2) \end{pmatrix}.$$

Therefore $x_L(\infty)$ is a solution of (3.6). This completes the proof of Theorem 6.

Theorem 7. *Suppose $g_1(u)$ and $g_2(u)$ satisfy* (H 3) *and* (H 5) *above, Ω_1 and Ω_2 are nonnegative real numbers and $0 < r_1 < r_2$. In addition assume*

(H 6) *For $j = 1, 2$ the limit $g_j(u)/u = +\infty$ as $|u| \to \infty$. Then the algebraic system of equations* (3.6) *has a unique solution $x = \text{col}(z_1, z_2)$. Moreover $z_1 \geqslant 0$, $z_2 \geqslant 0$, $z_1 + z_2 > 0$ and $(z_1, z_2) \neq (\Omega_1, \Omega_2)$.*

Proof. The proof is an easy consequence of the theory of the rotation of a map and the index of a zero point, see [17] especially p. 122 and Theorem 4.7 on p. 136. Let $\varphi(x) = x + \nu M_0 M_1 G(x - \Omega)$ where $\nu > 0$, (3.7) defines M_0, M_1 and Ω and $G(x) = G(x_1, x_2) = \text{col}(g_1(x_1), g_2(x_2))$. If $S = \{x \in R^2 : x_1^2 + x_2^2 = R\}$ and R is sufficiently large, it is easy to see that the rotation of φ is defined on S and equals plus one. Thus the sum of the indices of the zeros of φ inside of S is $+1$. If $x^* = \text{col}(z_1, z_2)$ is such a zero, then its index is $(-1)^\beta$ where β is the sum of the orders of all eigenvalues λ of the matrix

$$A = \nu M_0 M_1 \begin{pmatrix} g_1'(z_1 - \Omega_1) & 0 \\ 0 & g_2'(z_2 - \Omega_2) \end{pmatrix},$$

such that $\lambda > 1$. The matrix A has the form

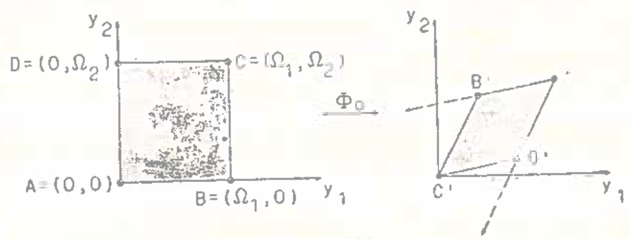
$$A = M_0(-\nu M_1 M_2) = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} -c & 0 \\ 0 & -d \end{pmatrix},$$

where $c, d \geqslant 0$, $a, b > 0$ and $a^2 - b^2 = (r_1^2 + r_2^2)^2 - (2r_1 r_2)^2 = (r_1^2 - r_2^2)^2 > > 0$. Its eigenvalues λ are solutions of the polynominal equation

$$\lambda^2 + \lambda a(c + d) + cd(a^2 - b^2) = 0.$$

Therefore Re $\lambda \leqslant 0$ for any eigenvalue. This means that $(-1)^\beta = (-1)^0 = $ index of $x^* = \mathrm{col}\,(z_1, z_2)$. Thus (3.6) has a unique solution $x^* = \mathrm{col}\,(z_1, z_2)$.

Let $\varphi_0(x) = \nu M_0 M_1 G(x - \Omega)$ so that $\varphi(x) = x + \varphi_0(x)$. Lengthly elementary arguments will establish the following statements. If x is in the second quadrant of the (x_1, x_2) – plane, then $\varphi_0(x)$ lies in the first, third or fourth quadrant. If x is in the third quadrant, then $\varphi_0(x)$ lies in the first. If x is in the fourth quadrant, then $\varphi_0(x)$ lies in the first, second or third quadrant. If $x = \mathrm{col}\,(x_1, x_2)$ with $x_1 \geqslant \Omega_1$ and $x_2 \geqslant \Omega_2$, then $\varphi_0(x)$ lies in the third quadrant. (See Figure 1 for the geometry.) These statements complete the proof of Theorem 7.



A' = Φ0(A), B' = Φ0(B), ..

Figure 1

It is very difficult to solve $\varphi(x) = 0$ exactly even when $g_j(u) = ku^3$ for $j = 1, 2$ and $k > 0$. The zeros of (3.6) can be computed numerically in this case using Newton's method. The argument in the first part of the proof of Theorem 7 above is easily modified to see that the Jacobian of $\varphi(x)$ is nonsingular if $x = \mathrm{col}\,(x_1, x_2) \geqslant 0$ and $x \neq \mathrm{col}\,(0, 0)$.

Corollary 3. *If* (H 1 – 6) *are true, then the solution of* (E) *exists for all* $t \geqslant 0$ *and has a limit* $x(\infty)$ *as* $t \to \infty$. *This limit is the unique solution of* (3.6).

Proof. It is known that $x_L(t) \leqslant u(t) \leqslant x_u(t)$ while $x_L(t)$ and $x_u(t)$ both tend to the unique solution of (3.6).

Corollary 4. *Assume* (H 1 – 6). *Then the solution of* (1.1 – 3) *exists for all* $t \geqslant 0$ *and* $u(t, r) \to v(r)$ *uniformly on* $r_1 \leqslant r \leqslant r_2$ *as* $t \to \infty$ *where* v *is the unique solution of the boundary value problem*

$$v''(r) + \frac{1}{r} v'(r) - \frac{1}{r^2} v(r) = 0, \quad r_1 \leqslant r \leqslant r_2, \tag{3.11}$$

and

$$v'(r_1) = g_1(v(r_1) - \Omega_1), \quad v'(r_2) = -g_2(v(r_2) - \Omega_2). \tag{3.12}$$

Proof. The general solution of (3.11) is $v(r) = Ar + B/r$. The function v will satisfy (3.12) if and only if A and B are chosen so that $x = \mathrm{col}(v(r_1), v(r_2))$ is a solution of (3.6). Therefore (3.12) is equivalent to

$$v(r_1) = z_1, \quad v(r_2) = z_2, \tag{3.12'}$$

where $x = \mathrm{col}\,(z_1, z_2)$ is the unique solution of (3.6). If $u(t, r)$ solves (1.1) – – (1.3), then $u(t, r_j) \to z_j$ as $t \to \infty$ (Corollary 3 above). Now apply Theorem 2 of [3, p. 158] to see that $u(t, r) \to v(r)$. This completes the proof of Corollary 3.

Corollary 5. *Assume* (H 1), (H 3), (H 5) *and* (H 7)

$$\Omega_j \in C[0, \infty) \text{ for } j = 1, 2 \text{ and } \Omega_j(\infty) \geqslant 0 \text{ exists.} \tag{47}$$

Then the conclusions of Theorem 6 and Corollary 3 remain true.

Proof. Let h be a C^1 approximation of Ω_j such that $|h(t) - \Omega_j(t)| \leqslant \varepsilon \exp(-t)$ for all $t \geqslant 0$ and let $\Omega_{j,\varepsilon}(t) = h(t) + \varepsilon$. Approximate $g_j(u)$ by a C^1 function $\gamma_j(u)$ such that $\gamma_j(u) = g_j(u)$ when $|u| \leqslant |f_0(0)| + 1$ and γ_j satisfies (H 6). With these approximate functions construct $x_L(t, \varepsilon)$, $u(t, \varepsilon)$ and $x_u(t, \varepsilon)$ as above. By Theorem 6 one has $-f_0(t) \leqslant x_L(t, \varepsilon) \leqslant u(t, \varepsilon) \leqslant x_u(t, \varepsilon) \leqslant f_0(0)$, $x_L(t, \varepsilon)$ is nondecreasing in t, and $x_u(t, \varepsilon)$ is nonincreasing in t. On taking the limit as $\varepsilon \to 0^+$ it follows that $x_L(t)$, $u(t)$ and $x_u(t)$ satisfy the same properties. Notice that x_L, u and x_u are independent of how $\gamma_j(u)$ is defined in the region $|u| \geqslant |f_0(0)| + 1$. Therefore Theorem 7 can be applied to complete the proof.

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UN SISTEM DE ECUAȚII INTEGRALE VOLTERRA CARE INTERVINE ÎN TEORIA SUPERFLUIDITĂȚII

Rezumat

O anumită problemă de curgere a fluidului de heliu lichid se reduce la ecuația căldurii cu condiții la limită neomogene. Se arată că această problemă este echivalentă cu un sistem de două ecuații integrale neliniare de tip Volterra.

Soluția acestei probleme de tip Volterra tinde asimptotic la un vector constant. Soluția corespunzătoare problemei curgerii fluidului tinde la o curgere limită independentă de timp.

INTEGRO-DIFFERENTIAL EQUATIONS IN HILBERT SPACES

BY

VIOREL BARBU

1. Introduction. In this paper we are concerned with equations of the form

$$u'(t) + \partial\varphi(u(t)) + \int_0^t a(t-s)\,Au(s)\,ds \ni f(t), \quad 0 < t < \infty. \tag{1.1}$$

Here the unknown $u(t)$ takes values in a Hilbert space H. A is a nonlinear monotone operator in H and $\partial\varphi$ is the subgradient of a convex and lower semicontinuous function $\varphi : H \to]-\infty, \infty]$. The scalar function $a(t)$ is assumed to satisfy the monotonicity condition

$$(-1)^k a^{(k)}(t) \geq 0, \quad 0 < t < \infty\,;\ k = 0, 1, 2. \tag{1.2}$$

In the scalar case, $H = R$ the equation (1.1) has been firstly considered by Levin and Nohel [11] and it was extensively studied by several authors (see [6] for a detailed information and significant results in this field).

In Section 3 we use the monotonicity property of A and to obtain —via compactness arguments—an existence result for equation (1.1). Our result includes as a special case that of Artola [2]. A related linear equation was studied by Dafermos [8]. Section 4 is concerned with asymptotic behavior as $t \to \infty$ for solutions of (1.1). The argument used there, was first applied to scalar equations by Halanay [10]. It was used recently for more general equations of the form (1.1) by Mac Camy and Wong [13].

In Section 5 we apply these results to some nonlinear boundary problems with applications in viscoelasticity theory.

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2. Preliminaries. Let H be a real Hilbert space and let X be a reflexive Banach space dense in H and with the injection map of X into