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UN SISTEM DE ECUAŢII INTEGRALE VOLTERRA CARE INTERVINE ÎN TEORIA SUPERFLUIDITĂŢII

Rezumat

O anumită problemă de curgere a fluidului de heliu lichid se reduce la ecuaţia căldurii cu condiţii la limită neomogene. Se arată că această problemă este echivalentă cu un sistem de două ecuaţii integrale neliniare de tip Volterra.

Soluţia acestei probleme de tip Volterra tinde asimptotic la un vector constant. Soluţia corespunzătoare problemei curgerii fluidului tinde la o curgere limită independentă de timp.

INTEGRO-DIFFERENTIAL EQUATIONS IN HILBERT SPACES

BY

VIOREL BARBU

1. Introduction. In this paper we are concerned with equations of the form

$$u'(t) + \partial\varphi(u(t)) + \int_0^t a(t-s)\, Au(s)\, ds \ni f(t), \quad 0 < t < \infty. \tag{1.1}$$

Here the unknown $u(t)$ takes values in a Hilbert space H, A is a nonlinear monotone operator in H and $\partial\varphi$ is the subgradient of a convex and lower semicontinuous function $\varphi : H \to]-\infty, \infty]$. The scalar function $a(t)$ is assumed to satisfy the monotonicity condition

$$(-1)^k a^{(k)}(t) \geqslant 0, \quad 0 < t < \infty;\ k = 0, 1, 2. \tag{1.2}$$

In the scalar case, $H = R$ the equation (1.1) has been firstly considered by Levin and Nohel [11] and it was extensively studied by several authors (see [6] for a detailed information and significant results in this field).

In Section 3 we use the monotonicity property of A and to obtain — via compactness arguments—an existence result for equation (1.1). Our result includes as a special case that of Artola [2]. A related linear equation was studied by Dafermos [8]. Section 4 is concerned with asymptotic behavior as $t \to \infty$ for solutions of (1.1). The argument used there, was first applied to scalar equations by Halanay [10]. It was used recently for more general equations of the form (1.1) by MacCamy and Wong [13].

In Section 5 we apply these results to some nonlinear boundary problems with applications in viscoelasticity theory.

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2. Preliminaries. Let H be a real Hilbert space and let X be a reflexive Banach space dense in H and with the injection map of X into

H continuous. We identify H with its own dual and denote X' the dual space of X. Then the following relation holds

$$X \subset H \subset X'.$$

Let (x', x) be the pairing between an element x' of X' and x of X; if $x, x' \in H$ this is the ordinary inner product in H. We denote by $\|\ \|_X$ and $\|\ \|_{X'}$ the norm of X and X' respectively. The norm of H will be denoted by $|\ |$.

Let A be a nonlinear (multivalued) operator from X into X'. We shall identify A with a subset of product space $X \times X'$ and we shall use the following notations:

(2.1) $\quad Ax = \{y \in X';\ [x, y] \in A\},\ D(A) = \{x \in X;\ Ax \neq \emptyset\},$

(2.2) $\quad R(A) = \cup\{Ax;\ x \in D(A)\},\ A^{-1} = \{[y, x];\quad [x, y] \in A\}.$

A subset $A \subset X \times X'$ is said to be monotone if

(2.3) $\qquad (y_1 - y_2, x_1 - x_2) \geqslant 0$ for $[x_i, y_i] \in A,\ i = 1, 2.$

A is said to be maximal monotone in $X \times X'$ if it is monotone and admits no proper monotone extensions in $X \times X'$.

Let φ be a lower semicontinuous proper convex function from X into $[-\infty, \infty]$. We set

$$D(\varphi) = \{x \in X;\quad \varphi(x) < +\infty\}$$

and denote $\partial\varphi \subset X \times X'$ the subdifferential (subgradient) mapping of φ i. e.,

(2.4) $\quad \partial\varphi(x) = \{y \in X';\ \varphi(x) - \varphi(u) \leqslant (y, x - u),\ \forall\ u \in X\}.$

We recall (see e. g. [14]) that $\partial\varphi$ is maximal monotone in $X \times X'$ and $D(\partial\varphi)$ is a dense subset of $D(\varphi)$.

A monotone subset $A \subset X \times X'$ is said to be cyclically monotone if it satisfies

(2.5) $\quad (x_1 - x_0, y_0) + \ldots + (x_n - x_{n-1}, y_{n-1}) + (x_0 - x_n, y_n) \leqslant 0$

whenever $y_i \in Ax_i$ for $i = 0, 1, \ldots, n$ (n arbitrary). According to a well known result due to R. T. Rockafellar [14], a maximal monotone subset A of $X \times X'$ is the subdifferential $\partial\varphi$ of a lower semicontinuous proper convex function $\varphi : X \to]-\infty, \infty]$ iff A is cyclically monotone. In this case φ is determined by A uniquely up to an additive constant.

Suppose now that A is a maximal monotone subset of $H \times H$. Then $R(I + \lambda A) = H$ for every $\lambda > 0$ and $J_\lambda = (I + \lambda A)^{-1}$ is well defined on all of H. For every $\lambda > 0$ denote by A_λ the Yosida approximation of A i. e.

$$A_\lambda = \lambda^{-1}(I - J_\lambda),$$

In the following Lemma we collect for later use some elementary properties of J_λ and A_λ (for the proof see [7]).

Lemma 2.1. *Let A be a maximal monotone subset of $H \times H$. Then*

(i) *J_λ is a contraction defined on all of H for every $\lambda > 0$.*

(ii) *$\lim_{\lambda \to 0} J_\lambda x = x$ in H for every $x \in \overline{D(A)}$.*

(iii) *A_λ is monotone (everywhere defined) Lipschitz continuous operator (with Lipschitz constant $1/\lambda$). Moreover, $|A_\lambda u| \leqslant |A^0 u|$ and $\lim_{\lambda \to 0} A_\lambda u = A^0 u$ for every $u \in D(A)$. Here $A^0 u$ denotes the unique element of least norm in Au.*

Now, let $A = \partial\varphi$, where φ is a lover semicontinuous proper convex function from H into $]-\infty, \infty]$.

Lemma 2.2. *The function φ_λ defined by*

(2.6) $$\varphi_\lambda(u) = \inf_{v \in H}\left\{\frac{|u - v|^2}{2\lambda} + \varphi(v)\right\},\quad u \in H;\ \lambda > 0$$

is convex, Fréchet differentiable and $\partial\varphi_\lambda = A_\lambda$.

In addition,

(2.7) $$\lim_{\lambda \to 0} \varphi_\lambda(u) = \varphi(u),\ \forall\ u \in H$$

and

(2.8) $$\varphi(J_\lambda u) \leqslant \varphi_\lambda(u) \leqslant \varphi(u),\quad \forall\ u \in H.$$

For proof see H. Brézis [4].

If $1 \leqslant p \leqslant \infty$, then $L^p(0, T;\ X)$ denotes the space of strongly measurable functions $u :]0, T[\to X$ which are L^p-Bochner integrable over $]0, T[$, and the usual modification in case $p = \infty$. By $L^p_{loc}(0, \infty;\ X)$ we denote the corresponding local spaces.

Throughout this paper, $u' = du/dt$ will denote the derivative of u in the sense of vectorial distributions on real axis. It is well known that if u and u' belong to $L^1(0, T;\ X)$ then $u : [0, T] \to X$ is absolutely continuous and its ordinary derivative coincides a. e. on $]0, T[$ with u'.

Lemma 2.3. *Let $\varphi : H \to]-\infty, \infty]$ be a lower semicontinuous proper convex function and let $u \in L^2(0, T; H)$ be such that $u' \in L^2(0, T; H)$. Assume that there is $g \in L^2(0, T; H)$ such that $g(t) \in \partial\varphi(u(t))$ a. e. $t \in\ \in]0, T[$.*

Then the function $t \to \varphi(u(t))$ is absolutely continuous on $[0, T]$ and the following equality

(2.9) $$(\varphi(u(t)))' = (h(t), u'(t))$$

holds for almost all $t \in]0, T[$ and for every $h(t) \in \partial\varphi(u(t))$.

The proof of Lemma 2.3 is given in [4].

3. The main existence result. We are given a pair of real reflexive Banach spaces V, W and a real Hilbert space H which satisfy

$$V \subset W \subset H$$

with each injection mapping continuous and densely defined. Notations for normed and inner product will be the same as in § 2.

Secondly, we have a lower semicontinuous proper convex function $\varphi : H \to]-\infty, \infty]$ and a cyclically monotone operator $A : W \to W'$ which are assumed to satisfy

(i) $D(A) = W$ and A is single valued, hemicontinuous and coercive from W into W', i. e.

(3.1) $$\lim_{t \to 0} (A(tu + v), w) = (Av, w) \text{ for } u, v, w \in W;$$

(3.2) $$\lim_{\|u\|_W \to \infty} (Au, u)/\|u\|_W = +\infty;$$

(ii) $D(\varphi) \subset V$ and

(3.3) $$\lim_{|u|_V \to \infty} (\varphi(u) + |u|^2) = +\infty;$$

(iii) the injection map of V into W is compact:

(iv) for every $u \in D(\varphi)$ and for each $\varepsilon > 0$, the solution u_ε of the equation $u_\varepsilon + \varepsilon A u_\varepsilon = u$, lies in $D(\varphi)$ and satisfies

(3.4) $$\varphi(u_\varepsilon) \leqslant \varphi(u);$$

(v) $a(t)$ is a continuous function on $[0, \infty[$ such that $a \in C^2(]0, \infty[; R)$ and

(3.5) $$(-1)^k a^{(k)}(t) \geqslant 0 \text{ for } t > 0 \text{ and } k = 0, 1, 2.$$

Theorem 1. *Suppose that Hypotheses* (i) $\sim$ (v) *are satisfied. Let* u_0, f *be given such that*

(3.6) $$u_0 \in D(\varphi),\ f, f' \in L^1_{\mathrm{loc}}(0, \infty; H).$$

Then there exists at least one function $u : [0, \infty[\to H$ *which satisfies*

(3.7) $$u \in C(0, \infty; W) \cap L^\infty_{\mathrm{loc}}(0, \infty; V),$$

(3.8) $$u' \in L^2_{\mathrm{loc}}(0, \infty; H).$$

(3.9) $$u'(t) + \partial\varphi(u(t)) + \int_0^t a(t-s)\, Au(s)\, ds \ni f(t), \text{ a. e. } t > 0.$$

(3.10) $$u(0) = u_0.$$

Proof. According to a well known result due to F. Browder (see e. g. [3]), assumption (i) implies that A is maximal monotone in $W \times W'$ and $R(A) = W'$. In particular, it follows that the operator A_H defined by

$$A_H u = Au \text{ if } u \in D(A_H) = \{u \in W,\ Au \in H\}$$

is maximal monotone in $H \times H$.

On the other hand, since A is cyclically maximal monotone in $W \times W'$, there exists a unique lower-semicontinuous convex function $\psi : W \to [-\infty, +\infty]$, $\psi \not\equiv +\infty$ such that $A = \partial\psi$ and

$$\inf\{\psi(u);\ u \in W\} = 0.$$

The coercivity condition (3.2) clearly implies that $A^{-1} : W' \to W$ is bounded on every bounded subset of W'. We deduce (see for instance [5], Th. 7) that

(3.11) $$\lim_{\|u\|_W \to \infty} \psi(u)/\|u\|_W = \infty.$$

Let $\psi_H : H \to]-\infty, \infty[$ be the convex function defined by

$$\psi_H(u) = \begin{cases} \psi(u) & \text{if } u \in W \\ +\infty & \text{if } u \in H \setminus W. \end{cases}$$

Since the space W is reflexive, it follows by (3.11) that $\{u;\ \psi_H(u) \leqslant \lambda\}$ is a closed subset of H for each real λ, i. e. ψ_H is lower semicontinuous in H. Thus $\partial\psi_H$ is maximal monotone subset of $H \times H$ and this clearly implies that $A_H = \partial\psi_H$.

Let $T \in]0, \infty[$ be such that $a(T) > 0$. We consider the approximating equations

(3.12) $$u'_\lambda(t) + \partial\varphi(u_\lambda(t)) + \int_0^t a(t-s)\, A_\lambda u_\lambda(s)\, ds \ni f(t),\ 0 < t < T$$

with the initial condition

(3.13) $$u_\lambda(0) = u_0,$$

where $A_\lambda = \lambda^{-1}(I - (I + \lambda A_H)^{-1})$ and $\lambda > 0$. Hypothesis (v) implies that $a' \in L^1(0, \infty)$ and by Lemma 2.1 A_λ is Lipschitz continuous in H. Thus by using the same argument as in [4], it follows that problem (3.12) $\sim$ (3.13) has a unique solution $u_\lambda \in C(0, T; H)$ which satisfies

$$u'_\lambda \in L^2(0, T; H),\ u_\lambda(t) \in D(\partial\varphi) \text{ for } 0 \leqslant t < T.$$

In order to obtain the convergence of u_λ as $\lambda \to 0$ some apriori estimates are necessary.

Lemma 3.1. *There exists a constant* $C > 0$ *such that for all* $\lambda > 0$ *and* $t \in [0, T]$.

(3.14) $$\int_0^t |u'_\lambda(s)|^2\, ds + \left|\int_0^t A_\lambda u_\lambda(s)\, ds\right|^2 + \varphi(u_\lambda(t)) \leqslant C.$$

Proof. Let $\psi_\lambda : H \to]-\infty, \infty]$ be the function defined by

$$\psi_\lambda(u) = \inf_{v \in H} \left\{ \frac{|u - v|^2}{2\lambda} + \psi(v) \right\}, \quad u \in H\,;\ \lambda > 0$$

and let $E_\lambda(t)$ be „the energy function"

$$E_\lambda(t) = \psi_\lambda\,(u_\lambda(t)) + \frac{1}{2}\,a(t) \left| \int_0^t A_\lambda u_\lambda(s)\,ds \right|^2 -$$

$$-\frac{1}{2}\int_0^t a'(t-s) \left| \int_0^s A_\lambda u_\lambda(\tau)\,d\tau \right|^2 ds, \quad 0 \leqslant t \leqslant T.$$

By Lemma 2.2 and Lemma 2.3 it follows that

$$(u'(t),\ A_\lambda u_\lambda(t)) = (\psi_\lambda(u_\lambda(t)))' \text{ a.e. } 0 < t < T.$$

Thus a simple calculation involving (3.13) leads to

$$(3.15)\quad E'_\lambda(t) = \frac{1}{2}\,a'(t) \left| \int_0^t A_\lambda u_\lambda(s)\,ds \right|^2 - \frac{1}{2}\int_0^t a''\,(t-s) \left| \int_s^t A_\lambda u_\lambda(\tau)\,d\tau \right|^2 ds \,+$$

$$+\,(f(t) - y_\lambda(t),\ A_\lambda u_\lambda(t)), \text{ a. e. } 0 < t < T,$$

where

$$y_\lambda(t) = f(t) - u'_\lambda(t) - \int_0^t a(t-s)\,A_\lambda u_\lambda(s)\,ds.$$

The existence of the integrals occuring in (3.15) is easily justified by the hypothesis. In fact it follows readily from (3.5) that $\lim_{t \to 0} ta'(t) = 0$ and $ta''(t) \in L^1(0, \infty)$.

From (3.5) and (3.15) one observes that

$$E'_\lambda(t) \leqslant (f(t) - y_\lambda(t),\ A_\lambda u_\lambda(t)) \text{ a. e. } 0 < t < T.$$

Since $y_\lambda(t) \in \partial\varphi(u_\lambda(t))$ for all $t \in]0, T]$, it follows easily from (iv) that

$$(y_\lambda\,(t),\ A_\lambda u'_\lambda(t))) \geqslant 0 \text{ for } 0 \leqslant t \leqslant T.$$

Hence

$$(3.16)\qquad E_\lambda(t) \leqslant \psi_\lambda(u_0) + \int_0^t (f(s),\ A_\lambda u_\lambda(s))\,ds, \quad 0 \leqslant t \leqslant T.$$

Set

$$F_\lambda(t) = \int_0^t A_\lambda u_\lambda(s)\,ds, \quad 0 \leqslant t \leqslant T.$$

By integrating by parts in (3.16) we get

$$\psi_\lambda\,(u_\lambda\,(t)) + \frac{1}{2}\,a(t)\,|\,F_\lambda(t)\,|^2 \leqslant \psi_\lambda^{(u_0)} + (f(t),\ F_\lambda(t)) - \int_0^t (f'(s),\ F_\lambda(s))\,ds$$

and

$$\psi_\lambda\,(u_\lambda(t)) + \frac{1}{4}\,a\,(t)\,|\,F_\lambda(t)\,|^2 \leqslant \psi_\lambda(u_0) + \frac{|\,f(t)\,|^2}{a(t)} + \int_0^t |\,f'(s)\,|\,|\,F_\lambda(s)\,|\,ds.$$

Since $\psi_\lambda(u_0) \leqslant \psi(u_0)$ and $a(t) > a(T) > 0$ the above integral inequality yields

$$(3.17)\qquad \psi_\lambda(u_\lambda(t)) + |\,F_\lambda(t)\,|^2 \leqslant C \quad \text{for } 0 \leqslant t \leqslant T,$$

where $C > 0$ is independent of λ.

On the other hand, the equation (3.12) may be written as

$$(3.18)\qquad u'_\lambda(t) + y_\lambda(t) + a(0)\,F_\lambda(t) + \int_0^t a'(t-s)\,F_\lambda(s)\,ds = f(t).$$

If we multiply (3.18) by $y_\lambda^{(t)}$ and integrate over $]0, t[$ we obtain by Lemma 2.2.

$$\varphi(u_\lambda(t)) + \int_0^t |\,y_\lambda(s)\,|^2\,ds \leqslant \varphi(u_0) + \int_0^t |\,y_\lambda(s)\,|\,(a(0)\,|\,F_\lambda(s)\,| \,+$$

$$+\,|\,f(s)\,|\,)ds + \int_0^t\!\!\int_0^s |\,a'(s-\tau)\,|\,|\,y_\lambda(s)\,|\,|\,F_\lambda(\tau)\,|\,d\tau\,ds.$$

This, together (3.12) and (3.17) implies (3.14) and thus completes the proof of Lemma 3.1.

Proof of Theorem 1 (continued). Set $v_\lambda(t) = (I + \lambda A_H)^{-1}\,u_\lambda(t)$. Hypothesis (iv) together with Lemma 3.1 imply

$$\varphi(v_\lambda(t)) \leqslant \varphi(u_\lambda(t)) \leqslant C,\ 0 \leqslant t \leqslant T\,;\ \lambda > 0.$$

Again by Lemma 3.1 and Hypothesis (ii) we conclude that

$$(3.19)\qquad \|\,v_\lambda(t)\,\|_V + \|\,u_\lambda(t)\,\|_V \leqslant C_1$$

for all $\lambda > 0$ and $t \in [0, T]$. Here $C_1 > 0$ is independent of λ. Since $(I + \lambda A_H)^{-1}$ is a contraction on H it follows that $|v'_\lambda(t)| \leqslant |u'_\lambda(t)|$ a.e. $t \in]0, T[$. Then

$$(3.20) \qquad \int_0^T |v'_\lambda(t)|^2\, dt \leqslant \int_0^T |u'_\lambda(t)|^2\, dt \leqslant C, \ \lambda > 0$$

by Lemma 3.1.

Since the injection map of V into W is compact by hypothesis, it follows by (3.19) and (3.20) (see [1]) that $\{u_\lambda ;\ \lambda > 0\}$ and $\{v_\lambda ;\ \lambda > 0\}$ are compact subsets of $L^2(0, T : W)$. Hence there exist sequences $\{u_{\lambda_n}\}$ and $\{v_{\lambda_n}\}$ such that

$$u_{\lambda_n} \to u, \quad v_{\lambda_n} \to v \text{ as } \lambda_n \to 0$$

strongly in $L^2(0, T : W)$.

Since A is maximal monotone in $W \times W'$ and defined on all of H, a well known result due to R. T. R o c k a f e l l a r (see e. g. [14]) implies that A is locally bounded on W. Since A is single valued, we deduce that it is demi-continuous i. e. continuous from the strong topology of W into weak topology of W'. This the inequality (3.19) together with Hypothesis (iii) imply that the sets $\{Av_\lambda(t) ;\ 0 \leqslant t \leqslant T ;\ \lambda > 0\}$ and $\{Au_\lambda(t) ;\ 0 \leqslant t \leqslant T ;\ \lambda > 0\}$ are weakly compact in W'. Hence $\{Av_\lambda ;\ \lambda > 0\}$ and $\{Au_\lambda ;\ \lambda > 0\}$ are bounded subsets of $L^\infty(0, T : W')$.

On the other hand, the equality

$$v_\lambda(t) + \lambda A v_\lambda(t) = u_\lambda(t)$$

implies

$$|v_\lambda(t) - u_\lambda(t)|^2 \leqslant -\lambda(Au_\lambda(t), v_\lambda(t) - u_\lambda(t))$$

because A is monotone.

Hence

$$|v_\lambda(t) - u_\lambda(t)|^2 \leqslant \lambda\, \|Au_\lambda(t)\|_{W'} \|u_\lambda(t) - v_\lambda(t)\|_W .$$

This obviously implies $u(t) = v(t)$ a. e. $t \in]0, T[$. Since $\{u'_\lambda\}$ remains bounded in $L^2(0, T ; H)$ and $\{Av_\lambda\}$ is a bounded subset of $L^\infty(0, T ; W')$, there exists $\lambda_n \to 0$ as $n \to \infty$ such that

$$u_{\lambda_n} \to u \text{ strongly in } L^2(0, T ; H),$$

$$u'_{\lambda_n} \to u' \text{ weakly in } L^2(0, T ; H),$$

$$Av_{\lambda_n} \to w \text{ weakly-star in } L^\infty(0, T ; W').$$

But A is demicontinuous from W into W', so we conclude that $w(t) = Au(t)$ a. e. $t \in]0, T[$.

Since $\int_0^t a(t-s) A_{\lambda_n} u_{\lambda_n}(s)\, ds$ remains bounded in $L^2(0, T : H)$ by Lemma 3.1, we may assume that

$$\int_0^t a(t-s) A_{\lambda_n} u_{\lambda_n}(s)\, ds \to \int_0^t a(t-s) Au(s) ds$$

weakly in $L^2(0, T : H)$. In equation (3.12) we put $\lambda = \lambda_n$ and let $n \to \infty$. Since $\partial\varphi$ is maximal monotone it follows by a standard device that $u(t) \in D(\partial\varphi)$ a. e. $t \in]0, T[$ and

$$(3.21) \quad u'(t) + \partial\varphi(u(t)) + \int_0^t a(t-s) Au(s)\, ds \ni f(t) \text{ a. e. } t \in]0, T[.$$

We have

$$u_{\lambda_n}(0) \to u(0) \text{ weakly in } H$$

because $u'_{\lambda_n} \to u'$ weakly in $L^2(0, T ; H)$.

Hence

$$(3.22) \qquad u(0) = u_0.$$

Moreover, we have

$$(3.23) \qquad u' \in L^2(0, T ; H),\ Au \in L^\infty(0, T ; W').$$

Since $F_\lambda(t) = \int_0^t A_\lambda u_\lambda(s)\, ds$ remains bounded in H and $F_{\lambda_n}(t) \to \int_0^t Au(s)\, ds$ weakly in W' for each $t \in]0, T]$, we deduce that

$$(3.24) \qquad \int_0^t Au(s)\, ds \in H \quad \text{for } 0 \leqslant t \leqslant T.$$

Finally, in (3.14) we put $\lambda = \lambda_n$ and let $n \to \infty$, thereby obtaining:

$$(3.25) \qquad \varphi(u(t)) \leqslant C \quad \text{for } 0 \leqslant t \leqslant T.$$

By hypothesis this implies $u \in L^\infty(0, T; V)$ and

$$u \in C(0, T; W).$$

because by (3.23) u is continuous H-valued function and the injection map of V into W is compact. Thus to complete the proof of Theorem 1, we need to show that $u(t)$ can be continued on $]T, \infty[$.

We consider the equation

$$(3.26)\quad v'(t)+\partial\varphi(v(t))+\int_0^t a(t-s)\, Av(s)\, ds \ni f(t+T) - \int_0^T a(T+t-s)\, Au(s)\, ds$$

with initial value condition

$$(3.27)\qquad v(0) = u(T).$$

But $u(T) \in D(\varphi)$ by (3.25) and from (3.24) it follows that the function

$$t \to \int_0^T a(T+t-s)\, Au(s)\, ds$$ together its derivative belong to $L^1\,(0, T : H)$.

Thus by the first part of this proof it follows that the problem (3.26) ~ ~ (3.27) has at least a solution $v(t) : [0,T\,[\to H$ satisfying

$$v \in C(0,T;\, W) \cap L^\infty(0,T;\, V),\ Av \in L^\infty\ (0,T;\, W')$$

and

$$v' \in L^2(0,T;\, H).$$

It now follows readily that $\tilde{u} : [0,2T[\ \to H$ defined by

$$\tilde{u}(t) = \begin{cases} u(t) & \text{for}\quad 0 \leqslant t \leqslant T \\ v(t-T) & \text{for}\quad T < t < 2T \end{cases}$$

satisfies the equation (3.9). It may be similarly shown that u can be continued outside $[0,2T[$ q.e.d.

Remark 3.1. *If the operator $A : W \to W'$ is assumed to be bounded on every bounded subset of W, then Theorem 1 remains valid if the assumption* (ii) *is replaced by the condition*

(ii)′ *$D(\partial\varphi) \subset V$ and there are positive constants α, β such that*

$$(3.28)\qquad |(\partial\varphi)^0 u|^2 + \beta\|u\|_W^2 \geqslant \alpha\|u\|_V^2,\ \forall u \in D(\partial\varphi).$$

To prove this it suffices to observe in the proof of Theorem 1 that $\{(\partial\varphi)^0(u_\lambda);\ \lambda > 0\}$ is a bounded subset of $L^2(0, T : H)$ by Lemma 3.1, and to use the condition (3.28) instead of (3.3).

4. The asymptotic behavior of the solutions. We are interested here in the asymptotic stability of solutions of homogeneous equation

$$(4.1)\qquad u'(t) + \partial\varphi(u(t)) + \int_0^t a(t-s)\, Au(s)\, ds \ni 0,\ \text{a.e. } t > 0.$$

In order to obtain a stability result for this equation, additional assumptions are required. Specifically we suppose:

(vi) $a(t) \not\equiv a(0)$ and one of the following two conditions is satisfied

$$(4.2)\qquad \begin{aligned} &a(\infty) \neq 0, \\ &a \in L^1(0,\infty) \cup L^2(0,\infty). \end{aligned}$$

We further assume that

(vii) A^{-1} is single valued mapping.

Theorem 2. *Assume that* (i) ~ (vii) *hold. If $u \in C(0,\infty; H)$ is any solution of* (4.1) *satisfying*

$$(4.3)\qquad u(0) \in D(\varphi) \quad \text{and} \quad u', Au \in L^2_{\text{loc}}\ (0,\infty; H)$$

then

$$\lim_{t\to\infty} u(t) = A^{-1}\, 0$$

in the strong topology of W.

Proof. Let $\psi : W \to\ ]-\infty, +\infty]$ be the convex and lower semicontinuous function uniquely determined by

$$A = \partial\psi \text{ and } \inf_{u\in W}\ \psi(u) = 0.$$

Let u be any function satisfying the conditions of the theorem. We multiply the equation

$$(4.5)\qquad u'(t) + \partial\varphi(u(t)) + \int_0^t a(t-s)\, Au(s)\, ds \ni 0,\ \text{a. e. } t > 0$$

by $Au(t)$ and we integrate over $]0, t[$. Use of Lemma 2.3 and Assumption (iv) leads to

$$\psi(u(t)) + \int_0^t ds \int_0^s a(s-\tau)\,(Au(s),\ Au(\tau))\, d\tau \leqslant \psi(u(0)),\quad t > 0$$

On the other hand, from (v) and $a(t) \not\equiv a(0)$ it follows the existence of positive constants α and ε (see [13]) such that

$$\int_0^t ds \int_0^s a(s-\tau)\,(Au(s),\ Au(\tau))\, d\tau \geqslant \alpha \int_0^t ds \int_0^s e^{-\varepsilon(s-\tau)}\,(Au(s),\ Au(\tau))\, d\tau.$$

Therefore,

$$\psi(u(t)) + \alpha\int_0^t ds\int_0^s e^{-\varepsilon(s-\tau)}(Au(s),\ Au(\tau))d\tau \leqslant \psi(u(0)),\ t > 0.$$

We set

$$y_\varepsilon(t) = \int_0^t e^{-\varepsilon(t-s)} Au(s)\,ds.$$

Then

$$\psi(u(t)) + \int_0^t (y_\varepsilon(s),\ Au(s))\,ds \leqslant \psi(u(0)) \tag{4.6}$$

and

$$y_\varepsilon'(t) + \varepsilon y_\varepsilon(t) = Au(t) \text{ for } 0 \leqslant t < \infty. \tag{4.7}$$

Hence

$$\frac{1}{2}\,|\,y_\varepsilon(t)\,|^2 + \varepsilon\int_0^t |\,y_\varepsilon(s)\,|^2\,ds \leqslant \psi(u(0)) \text{ for } 0 \leqslant t < \infty. \tag{4.8}$$

We define

$$g(t) = \int_0^t a(t-s)\,Au(s)\,ds.$$

We note that

$$g(t) = a(0)\,y_\varepsilon(t) + \int_0^t (\varepsilon a(t-s) + a'(t-s))\,y_\varepsilon(s)\,ds.$$

If the second alternative of Hypothesis (vi) holds (i. e. $a \in L^1(0,\ \infty) \cup L^2(0,\ \infty)$) then the inequality (4.8) obviously implies that $g(t)$ is bounded on $[0,\ \infty[$. Now suppose that $a(\infty) > 0$. We then define „the energy function“,

$$E(t) = \psi(u(t)) + \frac{1}{2}\,a(t)\,|\,F(t)\,|^2 - \frac{1}{2}\int_0^t a'(t-s)\,|\,F(t) - F(s)\,|^2\,ds,$$

where

$$F(t) = \int_0^t Au(s)\,ds.$$

The assumption (v) implies just as in section 3 that $E'(t) \leqslant 0$ a. e. $t > 0$.

Therefore,

$$\psi(u(t)) + \frac{1}{2}\,a(t)\,|\,F(t)\,|^2 \leqslant \psi(u_0) \text{ for } t \geqslant 0. \tag{4.9}$$

Since $g(t)$ can be written as

$$g(t) = a(0)\,F(t) + \int_0^t a'(t-s)\,F(s)\,ds,$$

we conclude that

$$|\,g(t)\,| \leqslant M \text{ for } 0 \leqslant t < \infty, \tag{4.10}$$

where M is a positive constant.

Next, we take inner product of equation (4.5) with $u'(t)$ to obtain

$$|\,u'(t)\,|^2 + (\varphi(u(t)))' \leqslant |\,g(t)\,|\,|\,u'(t)\,|,\ \text{a. e. } t > 0. \tag{4.11}$$

Let u_0 be arbitrary but fixed in $D(\varphi)$. We have,

$$\varphi(u(t)) \leqslant \varphi(u_0) - (u'(t) + g(t),\ u(t) - u_0),\ \text{a. e. } t > 0. \tag{4.12}$$

But $|\,u(t)\,|$ and $|\,g(t)\,|$ are both bounded by the estimates (4.5) and (4.10). Thus the previous estimates imply that

$$(\varphi(u(t)))^2 + (\varphi(u(t)))' \leqslant C_1,\ \text{a. e. } t > 0$$

because φ is bounded from below by an affine function. Hence

$$|\,\varphi(u(t))\,| \leqslant C \text{ for } 0 \leqslant t < \infty \tag{4.13}$$

where C is a nonnegative suitable chosen constant.

Let w be arbitrary in W. Since $u \in C(0,\ \infty\,;\ H)$ and since the imbedding of V in W is compact, it follows immediately that $u \in C(0,\ \infty\,:\ W)$. We have already observed that A is demicontinuous from W into W'. Since $\|\,u(t)\,\|_W$ is bounded this implies that the function $t \to (Au(t),\ w)$ is uniformly continuous on the halfaxis $t > 0$. Then (4.7) and (4.8) imply just as in [10], that $(y(t),\ w) \to 0$ and $(y'(t),\ w) \to 0$ as $t \to \infty$. Hence

$$Au(t) \to 0 \text{ weakly in } W' \text{ as } t \to \infty. \tag{4.14}$$

But $\|\,u(t)\,\|_V$ is bounded by (4.13). Thus there exist $u_\infty \in W$ and $t_n \to \infty$ such that $u(t_n) \to u_\infty$ strongly in W. Since A is demicontinuous, (4.14) implies that $Au_\infty = 0$. Consequently, we conclude from (vii) that

$$\lim_{t\to\infty} u(t) = A^{-1}0 \text{ strongly in } W.$$

This completes the proof of the theorem.

5. **Examples.** Throughout this section, Ω will denote a bounded open subset of R^n with smooth boundary Γ. By $H^k(\Omega)$ and $H_0^k(\Omega)$ we shall denote usual Sobolev spaces.

Example 1. Let g be a real continuous function defined on $R =]-\infty, \infty[$ satisfying

(5.1) $$|g(r)| \leqslant C_1(|r|^{p-1} + 1), \quad r \in R.$$

and

(5.2) $$rg(r) \geqslant C_2(|r|^p - 1), \quad r \in R,$$

where $C_1, C_2 > 0$ and p is such that

(5.3) $$2 \leqslant p \leqslant 2n/(n-2) \qquad \text{if } n \geqslant 3,$$
$$p \text{ arbitrary in } [2, \infty[\text{ if } 1 \leqslant n \leqslant 3.$$

Besides g, we are given a maximal monotone subset β in $R\times R$ such that

(5.4) $$r'g(r) > 0 \quad \text{for any } [r, r'] \in \beta.$$

Since β is maximal monotone in $R\times R$, it is cyclically maximal monotone, so that there exists a lower semicontinuous convex function $j : R \to]-\infty, \infty]$ such that $j \not\equiv +\infty$ and

$$\partial j = \beta.$$

Finally, we are given a continuous function $a : [0, \infty[\to R$ which satisfies

(5.5) $$(-1)^k a^{(k)}(t) \geqslant 0 \text{ for } t > 0, \ k = 0, 1, 2.$$

Our first application is concerned with existence and asymptotic behavior for solutions of the boundary value problem

(5.6) $$\frac{\partial u}{\partial t} - \Delta u - \int_0^t a(t-s)\, g(u(s))\, ds = f(t) \text{ a. e. on }]0, \infty[\times\Omega,$$

(5.7) $$-\frac{\partial u}{\partial n} \in \beta(u) \qquad \text{a. e. on }]0, \infty[\times\Gamma,$$

(5.8) $$u(0, x) = u_0(x) \qquad \text{a. e. on } \Omega,$$

where g, β and a are assumed to satisfy the above conditions.

Corollary 1. *Let f, u_0 be given satisfying*

(5.9) $$f, f' \in L^1_{\mathrm{loc}}(0, \infty; L^2(\Omega)),$$

(5.10) $$u_0 \in H^1(\Omega) \text{ and } j(u_0) \in L^1(\Gamma).$$

Then the problem (5.6) $\simeq$ (5.7) admits at least one solution $u(t, x)$ satisfying

(5.11) $$u \in C(0, \infty; L^p(\Omega)) \cap L^\infty_{\mathrm{loc}}(0, \infty; H^1(\Omega))$$

(5.12) $$u(t) \in H^2(\Omega) \text{ a. e. } t > 0$$

(5.13) $$\frac{\partial u}{\partial t} \in L^2_{\mathrm{loc}}(0, \infty; L^2(\Omega)).$$

Suppose further, that $a(t)$ satisfies the additional hypothesis (vi) *and g is strictly monotone. If $n \leqslant 3$ and $f \equiv 0$ then for each u_0 satisfying* (5.10) *there exists a solution u of* (5.6), (5.8) *which satisfies* (5.11), (5.12), (5.13) *and also*

(5.14) $$\lim_{t\to\infty} u(t) = g^{-1}(0) \text{ strongly in } L^p(\Omega).$$

Proof. Let A denote the monotone operator from $L^p(\Omega)$ into $L^{p'}(\Omega)$ $(1/p + 1/p' = 1)$ given by

$$(Au)(x) = g(u(x)) \text{ a. e. } x \in \Omega, \ u \in L^p(\Omega),$$

and let $\varphi : L^2(\Omega) \to]-\infty, \infty]$ be the convex function defined by

$$\varphi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_\Omega |\operatorname{grad} u|^2\, dx + \int_\Gamma j(u)\, d\sigma & \text{if } u \in H^1(\Omega),\ j(u) \in L^1(\Gamma) \\ +\infty & \text{otherwise.} \end{cases}$$

It is proved in [5] that φ is lower semicontinuous in $L^2(\Omega)$ and its sub differential $\partial\varphi \subset L^2(\Omega)\times L^2(\Omega)$ is given by

$$\partial\varphi(u) = -\Delta u,$$

with

$$D(\partial\varphi) = \left\{u \in H^2(\Omega);\ -\frac{\partial u}{\partial n} \in \beta(u) \text{ a. e. on } \Gamma\right\},$$

where $\partial/\partial n$ denotes the outward normal derivative. We have assumed the boundary Γ of Ω to be sufficiently smooth to define the „trace" $\partial u/\partial n|_\Gamma$ for $u \in H^2(\Omega)$ (see e. g. [12]). We take $V = H^1(\Omega)$, $W = L^p(\Omega)$ and $H = L^2(\Omega)$. If A and φ are defined as above then the assumptions (i) and (ii) follows easily from the definitions and from the estimates (5.1) and

(5.2). Hypothesis (iii) follows from (5.3) by using Sobolev's Imbedding Theorem and (iv) is clearly implied by (5.4). Hypothesis (vi) follows immediately from the corresponding property of g.

Theorem 1 and Theorem 2 can therefore be applied to the present situation. To complete the proof of Corolary 1, we need only to verify that under our assumptions, every solution u of the problem (5.6) $\sim$ $\sim$ (5.8) satisfying (5.11) $\sim$ (5.13), satisfies also $Au \in L^2_{\text{loc}}(0, \infty; L^2(\Omega))$ i. e.,

$$g(u(t, x)) \in L^2_{\text{loc}}(0, \infty; L^2(\Omega)). \tag{5.15}$$

Indeed, if $n \leqslant 3$ then $H^2(\Omega) \subset C(\bar{\Omega})$ so that (5.12) implies that $u(t) \in C(\bar{\Omega})$ a. e. $t > 0$. In particular it follows that $g(u(t)) \in C(\bar{\Omega}) \subset L^2(\Omega)$ a. e. $t > 0$.

Finally, (5.15) follows from continuity of g and from (5.11). Thus the proof is complete.

Example 2. We take $V = H^1_0(\Omega)$, $W = L^p(\Omega)$ and $H = L^2(\Omega)$. Let $\varphi : H \to]-\infty, \infty]$ be given by:

$$\varphi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_{\Omega} |\operatorname{grad} u|^2 \, dx & \text{if } u \in H^1_0(\Omega) \text{ and } u \geqslant 0 \text{ a. e. on } \Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

It follows easily that φ is lower semicontinuous in H and its subdifferential $\partial\varphi$ is given by

$$\partial\varphi(u) = \{v \in L^2(\Omega) ; v = -\Delta u \text{ a. e. on } [u > 0] \text{ and}$$
$$v \leqslant -\Delta u \text{ a. e. on } [u = 0]\}, \ u \in D(\partial\varphi),$$

where $D(\partial\varphi) = \{u \in H^1_0(\Omega) \cap H^2(\Omega) ; u \geqslant 0 \text{ a. e. on } \Omega\}$.

The operator $A : L^p(\Omega) \to (L^p(\Omega))'$ is defined as in the proof of Theorem 3 where g is assumed to satisfy (5.1), (5.2), (5.3) and the additional hypothesis:

$$g(0) = 0 \quad \text{and} \quad rg(r) > 0 \text{ for all } r \in R. \tag{5.16}$$

Thus applying Theorem 1 and Theorem 2 we obtain

Corollary 2. *Let (5.1), (5.2), (5.3), (5.5) and (5.16) be satisfied. Let u_0, f be given, satisfying*

$$u_0 \in H^1_0(\Omega), \quad u \geqslant 0 \text{ a. e. on } \Omega, \tag{5.17}$$

$$f, f' \in L^1_{\text{loc}}(0, \infty; L^2(\Omega)). \tag{5.18}$$

Then there exists at least one function $u(t, x)$ which satisfies

$$u \in C(0, \infty; L^p(\Omega)) \cap L^\infty_{\text{loc}}(0, \infty; H^1_0(\Omega)) \tag{5.19}$$

$$u(t) \in H^1_0(\Omega) \cap H^2(\Omega) \text{ a. e. } t > 0. \tag{5.20}$$

$$u(t, x) \geqslant 0 \text{ a. e. on }]0, \infty[\times \Omega. \tag{5.21}$$

$$\begin{aligned} &\frac{\partial u}{\partial t} - \Delta u + \int_0^t a(t-s)\, g(u(s))\, ds = f \text{ a. e. on } [u > 0], \\ &\frac{\partial u}{\partial t} - \Delta u + \int_0^t a(t-s)\, g(u(s))\, ds \leqslant f \text{ a. e. on } [u = 0]. \end{aligned} \tag{5.22}$$

$$u(0, x) = u_0(x) \qquad \text{a. e. on } \Omega. \tag{5.23}$$

Suppose further, that $a(t)$ satisfies (vi), *$f \equiv 0$ and $n \leqslant 3$. Then*

$$\lim_{t \to \infty} u(t) = 0 \quad \textit{strongly in } L^p(\Omega). \tag{5.24}$$

Example 3. Consider the mixed boundary – initial value problem

$$u_t(t, x) = u_{xx}(t, x) + \int_0^t a(t-s)(\sigma(u_x))_x \, ds + f(t, x), \quad (t, x) \in]0, \infty[\times]0, 1[\tag{5.25}$$

$$u(t, 0) = u(t, 1) = 0, \qquad 0 < t < \infty. \tag{5.26}$$

$$u(0, x) = u_0(x), \qquad 0 < x < 1. \tag{5.27}$$

We have used here the standard notations: $u_t = \partial u/\partial t$ and $u_x = \partial u/\partial x$.

We suppose that $\sigma(r)$ is a continuously differentiable function on R, satisfying

$$\sigma'(r) \geqslant 0, \quad r \in R, \tag{5.28}$$

and

$$|\sigma(r)| \leqslant m_1 |r| + m_2, \quad r \in R, \tag{5.29}$$

where m_1, m_2 are positive constants.

Define the operator A from $H^1_0(0, 1)$ into its dual space by

$$(Au, v) = \int_0^1 \sigma(u_x) v_x \, dx, \quad u, v \in H^1_0(0, 1).$$

It is easy to see that under our assumptions on σ the operator A is cyclically monotone, and demicontinuous from $H^1_0(0, 1)$ into $(H^1_0(0, 1))'$. If we impose the further condition

$$r\sigma(r) \geqslant m(|r|^2 + 1), \qquad m > 0, \tag{5.30}$$

then A is coercive on $H_0^1(0, 1)$.

Let $\varphi : L^2(0, 1) \to]-\infty, \infty]$ be the convex function defined by

$$\varphi(u) = \begin{cases} \dfrac{1}{2}\displaystyle\int_0^1 |u_x|^2 \, dx & \text{if} \quad u \in H_0^1(0,1), \\ +\infty & \text{otherwise}. \end{cases}$$

We have

$$\partial\varphi(u) = -u_{xx}$$

for all $u \in H_0^1(0,1) \cap H^2(0,1) = D(\partial\varphi)$. It follows easily that the conditions (i), (ii)′, (iii) and (iv) in § 3 are satisfied with $V = H_0^1(0.1) \cap H^2(\Omega)$, $W = H_0^1(0, 1)$, $H = L^2(0, 1)$ and A, φ defined as before. If we apply Theorem 1 (see Remark 3.1), we have the following result:

Corollary 3. *Suppose that $a(t)$ satisfies the assumption* (v) *in* 3. *Furthermore, we assume that condition* (5.28), (5.29) *and* (5.30) *hold for* σ. *Let* f, u_0 *be given satisfying*

$$u_0 \in H_0^1(0, 1), \tag{5.31}$$

$$f, f' \in L^1_{\text{loc}}(0, \infty; L^2(0, 1)). \tag{5.32}$$

Then the problem (5.25) ~ (5.27) *admits at least one solution* $u(t, x)$ *satisfying*

$$u \in C(0, \infty; H_0^1(0, 1)) \cap L^\infty_{\text{loc}}(0, \infty; H_0^1(0,1) \cap H^2(0, 1)), \tag{5.33}$$

$$u_t \in L^2_{\text{loc}}(0, \infty; L^2(0, 1)). \tag{5.34}$$

Remark 5.1. Equations of the form (5.25) are of physical interest and were considered by many authors. For a different approach to the study of related problem, we refer to J. M. Greenberg, R. C. Mac Camy and V. J. Mizel [9].

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Note added in proof. After this paper vas submitted for publication the author received from Professor R. C. MacCamy a preprint entitled „Asymptotic stability for a class of functional differential equations" where similar results are proved by different techniques.

ECUAȚII INTEGRO-DIFERENȚIALE ÎN SPAȚII HILBERT

Rezumat

Se studiază existența și comportarea asimptotică a soluțiilor unei ecuații integro-diferențiale abstracte de tip Levin-Nohel.