A MIXED PROBLEM FOR A POLYVIBRATING EQUATION OF MANGERON

BY

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In this paper a new mixed problem for a high order polyvibrating equation of D. Mangeron is considered.

1. Since the habilitation thesis of D. M a n g e r o n [1], a large number of contributions have been devoted to several boundary value problems, to Darboux and Goursat problems, and to certain mixed problems for polyvibrating equations of D. Mangeron. A rather complete list of references is given at the end of the paper.

In the present work we deal with the solution of the polyvibrating equation of D. Mangeron

$$\partial^n u = \lambda c(x_1, x_2, x_3)\, u + f(x_1, x_2, x_3), \tag{1}$$

subject to the conditions

$$\partial^k u \,|_{x_i=0} = 0 \;\; (k = 0, 1, \ldots, n-2), \;\; \partial^{n-1} u \,|_{x_i=\alpha_i} = 0 \;\; (i=1, 2, 3), \tag{2}$$

where $n \geqslant 2$, $\alpha_1 = \alpha_2 = 0$, $\alpha_3 = \alpha$, and

$$\partial^k = \frac{\partial^{3k}}{\partial x_1^k\, \partial x_2^k\, \partial x_3^k}, \tag{3}$$

the "k-th order P i c o n e derivative" [2], λ is a parameter, $c(x_1, x_2, x_3)$ and $f(x_1, x_2, x_3)$ are certain given functions analytic for all finite values of their arguments. We wish to find a function $u = u(x_1, x_2, x_3)$ of class C^{3n-1} in the $x_1x_2x_3$-space with a continuous $\partial^n u$, and satisfying the equation (1) and conditions (2).

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Clearly, the above formulated mixed problem becomes a Goursat problem for $\alpha = 0$. In the next section we shall establish the solution of (1)–(2).

2. We begin with the integration of Eq. (1). We then obtain

$$\partial^{n-1}u = \int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\{\lambda c(\xi_1, \xi_2, \xi_3)\, u(\xi_1, \xi_2, \xi_3) + f(\xi_1, \xi_2, \xi_3)\}\, d\xi_1\, d\xi_2\, d\xi_3 + [\varphi_{n-1}(x_1, x_2) + \psi_{n-1}(x_2, x_3) + \theta_{n-1}(x_3, x_1)],$$

$$\partial^{n-m}u = \int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\left(\prod_{i=1}^{3}\frac{(x_i-\xi_i)^{m-1}}{(m-1)!}\right)\{\lambda c(\xi_1, \xi_2, \xi_3)\, u(\xi_1, \xi_2, \xi_3) + f(\xi_1, \xi_2, \xi_3)\}\, d\xi_1\, d\xi_2\, d\xi_3$$

$$(4) \qquad + \sum_{j=1}^{m-1}\int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\left(\prod_{i=1}^{3}\frac{(x_i-\xi_i)^{m-j-1}}{(m-j-1)!}\right)[\varphi_{n-j}(\xi_1, \xi_2) + \psi_{n-j}(\xi_2, \xi_3) +$$

$$+ \theta_{n-j}(\xi_3, \xi_1)]\, d\xi_1\, d\xi_2\, d\xi_3 + [\varphi_{n-m}(x_1, x_2) + \psi_{n-m}(x_2, x_3) + \theta_{n-m}(x_3, x_1)],$$

$$u(x_1, x_2, x_3) = \int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\left(\prod_{i=1}^{3}\frac{(x_i-\xi_i)^{n-1}}{(n-1)!}\right)\{\lambda c(\xi_1, \xi_2, \xi_3)\, u(\xi_1, \xi_2, \xi_3) +$$

$$+ f(\xi_1, \xi_2, \xi_3)\}\, d\xi_1\, d\xi_2\, d\xi_3 + \sum_{j=1}^{n-1}\int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\left(\prod_{i=1}^{3}\frac{(x_i-\xi_i)^{n-j-1}}{(n-j-1)!}\right)[\varphi_{n-j}(\xi_1, \xi_2) +$$

$$\psi_{n-j}(\xi_2, \xi_3) + \theta_{n-j}(\xi_3, \xi_1)]\, d\xi_1\, d\xi_2\, d\xi_3 + [\varphi_0(x_1, x_2) + \psi_0(x_2, x_3) + \theta_0(x_3, x_1)],$$

where $\varphi_k(x_1, x_2)$, $\psi_k(x_2, x_3)$ and $\theta_k(x_3, x_1)$ are certain sufficiently smooth functions which are to be determined.

Eqs. (2) and (4) yield the following relationships

$$(5) \qquad \begin{aligned} &\varphi_k(0, x_2) + \psi_k(x_2, x_3) + \theta_k(x_3, 0) = 0, \\ &\varphi_k(x_1, 0) + \psi_k(0, x_3) + \theta_k(x_3, x_1) = 0, \qquad (k = 0, 1, \ldots, n-2). \\ &\varphi_k(x_1, x_2) + \psi_k(x_2, 0) + \theta_k(0, x_1) = 0, \end{aligned}$$

and

$$(6) \qquad \begin{aligned} &\varphi_{n-1}(0, x_2) + \psi_{n-1}(x_2, x_3) + \theta_{n-1}(x_3, 0) = 0, \\ &\varphi_{n-1}(x_1, 0) + \psi_{n-1}(0, x_3) + \theta_{n-1}(x_3, x_1) = 0, \\ &\varphi_{n-1}(x_1, x_2) + \psi_{n-1}(x_2, \alpha) + \theta_{n-1}(\alpha, x_1) \end{aligned}$$

$$= -\int_0^{x_1}\int_0^{x_2}\int_0^{\alpha}\{\lambda c(\xi_1, \xi_2, \xi_3)\, u(\xi_1, \xi_2, \xi_3) + f(\xi_1, \xi_2, \xi_3)\}\, d\xi_1\, d\xi_2\, d\xi_3.$$

Eqs. (5) imply that

$$(7) \qquad \varphi_k(x_1, x_2) + \psi_k(x_2, x_3) + \theta_k(x_3, x_1) \equiv A_k(x_1) + B_k(x_2) + C_k(x_3)$$

and

$$(8) \qquad A_k(0) + B_k(x_2) + C_k(x_3) \equiv A_k(x_1) + B_k(0) + C_k(x_3) \equiv$$
$$\equiv A_k(x_1) + B_k(x_2) + C_k(0) \equiv 0,$$

for $k = 0, 1, \ldots, n-2$. Consequently the functions $A_k(x_1)$, $B_k(x_2)$ and $C_k(x_3)$ are all constants and we have

$$(9) \qquad \varphi_k(x_1, x_2) + \psi_k(x_2, x_3) + \theta_k(x_3, x_1) \equiv 0 \qquad (k = 0, 1, \ldots, n-2).$$

Further, Eqs. (6) yield the following relationships

$$(10) \qquad \varphi_{n-1}(x_1, x_2) + \psi_{n-1}(x_2, x_3) + \theta_{n-1}(x_3, x_1) \equiv A_{n-1}(x_1) + B_{n-1}(x_2) +$$

$$+ C_{n-1}(x_3) - \int_0^{x_1}\int_0^{x_2}\int_0^{\alpha}\{\lambda c(\xi_1, \xi_2, \xi_3)\, u(\xi_1, \xi_2, \xi_3) + f(\xi_1, \xi_2, \xi_3)\}\, d\xi_1\, d\xi_2\, d\xi_3,$$

and

$$(11) \qquad \begin{aligned} &A_{n-1}(0) + B_{n-1}(x_2) + C_{n-1}(x_3) = 0, \\ &A_{n-1}(x_1) + B_{n-1}(0) + C_{n-1}(x_3) = 0, \\ &A_{n-1}(x_1) + B_{n-1}(x_2) + C_{n-1}(\alpha) \end{aligned}$$

$$= \int_0^{x_1}\int_0^{x_2}\int_0^{\alpha}\{\lambda c(\xi_1, \xi_2, \xi_3)\, u(\xi_1, \xi_2, \xi_3) + f(\xi_1, \xi_2, \xi_3)\}\, d\xi_1\, d\xi_2\, d\xi_3.$$

Obviously we have

$$(12) \qquad A_{n-1}(0) + B_{n-1}(0) + C_{n-1}(0) = 0 \text{ and } C_{n-1}(0) = C_{n-1}(\alpha).$$

by virtue of Eqs. (10) and (11). Hence, adding Eqs. (11) side by side, we find

$$A_{n-1}(x_1) + B_{n-1}(x_2) + C_{n-1}(x_3) \tag{13}$$

$$= \frac{1}{2}\int_0^{x_1}\int_0^{x_2}\int_0^{\alpha} \{\lambda c(\xi_1, \xi_2, \xi_3)\, u(\xi_1, \xi_2, \xi_3) + f(\xi_1, \xi_2, \xi_3)\}\, d\xi_1\, d\xi_2\, d\xi_3.$$

Combining Eqs. (10) and (13), we obtain

$$\varphi_{n-1}(x_1, x_2) + \psi_{n-1}(x_2, x_3) + \theta_{n-1}(x_3, x_1) \tag{14}$$

$$= -\frac{1}{2}\int_0^{x_1}\int_0^{x_2}\int_0^{\alpha} \{\lambda c(\xi_1, \xi_2, \xi_3)\, u(\xi_1, \xi_2, \xi_3) + f(\xi_1, \xi_2, \xi_3)\}\, d\xi_1\, d\xi_2\, d\xi_3.$$

Thus, by virtue of Eqs. (4), (9) and (14), we have the following Volterra type integral equation:

$$u(x_1, x_2, x_3) = \lambda(Au)(x_1, x_2, x_3) + F(x_1, x_2, x_3), \tag{15}$$

where

$$Au = \int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\left(\prod_{i=1}^{3}\frac{(x_i - \xi_i)^{n-1}}{(n-1)!}\right) c(\xi_1, \xi_2, \xi_3)\, u(\xi_1, \xi_2, \xi_3)\, d\xi_1\, d\xi_2\, d\xi_3 \tag{16}$$

$$-\frac{1}{2}\int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\left(\prod_{i=1}^{3}\frac{(x_i - \xi_i)^{n-2}}{(n-2)!}\right) d\xi_1\, d\xi_2\, d\xi_3 \int_0^{\xi_1}\int_0^{\xi_2}\int_0^{\alpha} c(\eta_1, \eta_2, \eta_3)\, u(\eta_1, \eta_2, \eta_3)\, d\eta_1\, d\eta_2\, d\eta_3,$$

and

$$F(x_1, x_2, x_3) = \int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\left(\prod_{i=1}^{3}\frac{(x_i - \xi_i)^{n-1}}{(n-1)!}\right) f(\xi_1, \xi_2, \xi_3)\, d\xi_1\, d\xi_2\, d\xi_3 \tag{17}$$

$$-\frac{1}{2}\int_0^{x_1}\int_0^{x_2}\int_0^{x_3}\left(\prod_{i=1}^{3}\frac{(x_i - \xi_i)^{n-2}}{(n-2)!}\right) d\xi_1\, d\xi_2\, d\xi_3 \int_0^{\xi_1}\int_0^{\xi_2}\int_0^{\alpha} f(\eta_1, \eta_2, \eta_3)\, d\eta_1\, d\eta_2\, d\eta_3.$$

We can easily show, by use of the method of successive approximations, that Eq. (15) admits a unique solution $U(x_1, x_2, x_3; \lambda)$, which is of class C^{3n} for all finite x_1, x_2 and x_3, and for each finite value of λ. Note that $U(x_1, x_2, x_3; \lambda)$ depends on λ analytically. It can be verified without any difficulty that the function $U(x_1, x_2, x_3; \lambda)$ is the solution of the mixed problem (1)–(2).

Let us note that for $\alpha = 0$ Eq. (15) is a Volterra integral equation of the second kind. For $\alpha \neq 0$, Eq. (15) can be considered an extension of M. Picone's integral equations given in his comprehensive Memoires [2b, 2c] of 1911.

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O PROBLEMĂ MIXTĂ PRIVITOARE LA O ECUAȚIE POLIVIBRANTĂ A LUI MANGERON

Rezumat

După apariția acum 40 de ani a primelor lucrări ale lui D. Mangeron [1], mai mulți cercetători și-au consacrat studiile lor diferitelor probleme de contur și la limită privitoare la ecuații polivibrante, introduse în literatura de specialitate de acest autor [6]–[9], [11]–[16].

În această lucrare se studiază o nouă problemă mixtă referitoare la o ecuație polivibrantă de ordin superior a lui Mangeron.

DUAL SERIES EQUATIONS ASSOCIATED WITH BESSEL AND HYPERGEOMETRIC FUNCTIONS

BY

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1. Introduction. The present paper considers the determination of coefficients A_n in the dual series equations involving Bessel and hypergeometric functions. The application of fractional integration operators due to Saxena [3] and Erdélyi [1], transforms the given dual series equations into those involving the same Bessel function. Finally the unknown coefficients A_n are determined by the application of orthogonal property of Bessel functions. The approach made for the solution in the text is believed to be new and the analysis is purely formal.

The dual series equations to be discussed here are as follows:

$$(1.1)\qquad \sum_{n=0}^{\infty} A_n\,{}_2F_1\left[\begin{matrix} -n,\ \nu+n+1\,; \\ \nu+1\,; \end{matrix}\ x^2/z^2\right] J_\nu(x) = f(x)\,:\ 0 < x \leqslant 1,$$

$$(1.2)\qquad \sum_{n=0}^{\infty} A_n\,{}_2F_1\left[\begin{matrix} -n,\ \nu-n+2\,; \\ \nu+2n+2\,; \end{matrix}\ z^2/x^2\right] J_\nu(x) = g(x)\,;\ x > 1.$$

where $J_\nu(x)$ is the ordinary Bessel function of order ν. $f(x)$ and $g(x)$ are known functions, while A_n is an unknown coefficient.

2. Fractional Integration Operators. In the present analysis the following results are used:

Saxena [3, p. 228] has introduced the following operators of fractional integration

$$(2.1)\quad \mathcal{J}[f(x)] = \mathcal{J}[\alpha,\beta,\nu,m : f(x)] = \frac{2x^{-\nu-1}}{\Gamma(1-\alpha)}\int_0^x {}_2F_1\left[\begin{matrix}\alpha,\ \beta+m\,; \\ \beta\,;\end{matrix}\ t^2/x^2\right] t^\nu f(t)\,dt$$

and