$(1,0)+(0,1) \in \alpha_1 M_p$ and $(1,0)-(0,1) \in \beta_1 M_p$ would imply $\alpha_1 \geqslant 2^{1/p}$ and $\beta_1 \geqslant 2^{1/p}$. We obtain the contradiction $4 < 2.2^{1/p} \leqslant \alpha_1^2 + \beta_1^2 \leqslant 2.2$. However for $1 < p < 2$, the set M_p is strictly convex.

e) The preceding results are also true in the case when X is a linear space on the field C of complex numbers. In this case we shall ask that the equality (5) to be verified for all $a_1, a_2 \in C$ and the equality (6) to be replaced by $(x, y) = \overline{(y, x)}$, where the complex inner product $(\ldots)$ is connected with the real inner product $<\ldots>$ by the equality $(x, y) = <x, y> - i <ix, y>$. Furthermore we suppose that the set M verifies the equality $\alpha M = M$ for all $\alpha \in C$ with $|\alpha| = 1$.

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ASUPRA MULȚIMILOR *H*-NETEDE DINTR-UN SPAȚIU LINIAR

Rezumat

Se dau cîteva noi proprietăți ale mulțimilor *H*-netede introduse în [3] sub denumirea de mulțimi *H*-convexe. Printre altele, se arată că pentru mulțimile absorbante, simetrice și radial mărginite, proprietatea funcționalei lui Minkowski de a fi normă hilbertiană depinde numai de *H*-netezimea frontierei radiale (teorema 2). De asemenea, se stabilesc corelații între noțiunile de *H*-netezime, convexitate și strictă convexitate.

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TRANSITIVE QUASI-UNIFORMITIES AND ZERO DIMENSIONAL BITOPOLOGICAL SPACES

BY

T. BÎRSAN

1. Introduction. The concepts of transitive quasi-uniformity and of covering quasi-uniformity was introduced by P. Fletcher [3], which initiated a systematic study of this topics.

A quasi-uniformity $\mathcal{U}$ on a set X is a *transitive quasi-uniformity* if $\mathcal{U}$ has a base (or a subbase) $\mathcal{B}$ satisfying the condition that for each $B \in \mathcal{B}$, $B \circ B = B$. In the special case of the uniformities, this concept coincides with the concept of *non-archimedean uniformity* which was considered previously by A. F. Monna [7].

Let (X, τ) be a topological space. A collection (cover) $\mathcal{C}$ of open sets is a Q-collection (Q-cover) if for each $x \in X$ $\cap \{C \in \mathcal{C} : x \in C\} \in \tau$. Let $x \in A \in \tau$. An open cover $\mathcal{C}$ of X is a cover about (x, A) if there exists $C \in \mathcal{C}$ such that $x \in C \subset A$. Every collection $\mathcal{A}$ of Q-covers of X such that, for each $A \in \tau$ and each $x \in A$, $\mathcal{A}$ contains a cover about (x, A) leads in a natural way [3, Theorem 1] to a compatible quasi-uniformity on (X, τ) which is called a *covering quasi-uniformity* for (X, τ). If $\mathcal{A}$ is the collection of all point (locally) finite open covers of X, then the corresponding covering quasi-uniformity is called the *point (locally) finite covering quasi-uniformity* for (X, τ). In [3] the equivalence of concepts of a transitive quasi-uniformity and of a covering quasi-uniformity is proved : a compatible quasi-uniformity $\mathcal{U}$ for (X, τ) is a covering quasi-uniformity if and only if $\mathcal{U}$ is a transitive quasi-uniformity.

In section 2 and 3 we give two new characterizations of transitive quasi-uniformities. In section 4 it is proved that a bitopological space (X, τ_1, τ_2) is pairwise zero dimensional if and only if it is transitive quasi-uniformizable. This latter result is an improvement of the result of I. L. Reilly [10, Theorem 4]. Some of the results of this paper generalize the corresponding results for non-archimedean uniformities [7].

2. Transitive quasi-uniformities and non-archimedean quasi-pseudometrics. A quasi-pseudo-metric p on a set X is a *non-archimedean quasi-pseudo-metric* if p satisfies the inequality

$$p(x, y) \leqslant \max\{p(x, z),\ p(z, y)\} \text{ for all } x, y, z \in X.$$

Lemma 1. *Let X be a set and $B \subset X \times X$ such that $B \supset \Delta = \{(x, x);\ x \in X\}$. Let $p_B : X \times X \to R$ be the function defined by*

$$p_B(x, y) = \begin{cases} 0 & \text{if} \quad (x, y) \in B, \\ 1 & \text{if} \quad (x, y) \notin B. \end{cases}$$

Then $B \circ B = B$ if and only if p_B is a non-archimedean quasi-pseudo-metric on X.

Proof. It is clear that $p_B(x, y) \geqslant 0$ for each $(x, y) \in X \times X$. Also, since $\Delta \subset B$, it follows that $p_B(x, x) = 0$ for each $x \in X$. It is immediate that

$$B \circ B = B \Leftrightarrow B \circ B \subset B \Leftrightarrow ((x, z) \in B,\ (z, y) \in B \Rightarrow (x, y) \in B)$$

$$\Leftrightarrow (p_B(x, z) = 0,\ p_B(z, y) = 0 \Rightarrow p_B(x, y) = 0)$$

$$\Leftrightarrow p_B(x, y) \leqslant \max\{p_B(x, z),\ p_B(z, y)\}.$$

Theorem 1. *If $\mathcal{U}$ is a transitive quasi-uniformity on X and $\mathcal{B}$ is a transitive subbase for $\mathcal{U}$, then $\mathcal{U}$ is generated by the family $\{p_B : B \in \mathcal{B}\}$ (where p_B is a non-archimedean quasi-pseudo-metric which is defined as in Lemma 1).*

Proof. It is known that a family of non-archimedean quasi-pseudo-metrics on X generates a transitive quasi-uniformity on X. Let $\mathcal{V}$ be the transitive quasi-uniformity generated by the family $\{p_B : B \in \mathcal{B}\}$. Since $B = \{(x, y) : p_B(x, y) = 0\}$ it follows that $\mathcal{B}$ is a common subbase for the quasi-uniformities $\mathcal{U}$ and $\mathcal{V}$. Hence $\mathcal{U} = \mathcal{V}$.

Theorem 2. *A quasi-uniform space $(X, \mathcal{U})$ is non-archimedean quasi-pseudo-metrizable if and only if the quasi-uniformity $\mathcal{U}$ has a countable transitive base.*

Proof. It is clear that, if p is a non-archimedean quasi-pseudo-metric compatible with $\mathcal{U}$, then the family $\{\{(x, y) : p(x, y) < 1/n\} : n \in N\}$ is a countable transitive base for $\mathcal{U}$. Conversely, let $\{B_n : n \in N\}$ be a transitive base for $\mathcal{U}$. For each $n \in N$, let $p_n = p_{B_n}$ (see Lemma 1). Now let p be defined by

$$p = \sup_{n \in N} \frac{1}{2^n} p_n .$$

The proof of the fact that p is a non-archimedean quasi-pseudo-metric on X compatible with $\mathcal{U}$ is straightforward and we omit it.

We recall that a real valued function f on the quasi-uniform space $(X, \mathcal{U})$ is *quasi-uniformly lower semi-continuous* (q.u.l.s.c.) if for each $\varepsilon > 0$ there is a $U \in \mathcal{U}$ such that $(x, y) \in U$ implies $f(y) > f(x) - \varepsilon$.

Lemma 2 [9, *Proposition 2*]. *If p is a quasi-pseudo-metric on the quasi-uniform space $(X, \mathcal{U})$, then p is q.u.l.s.c. on $(X \times X, \mathcal{U} \times \mathcal{U}^{-1})$ if and only if $\{(x, y) ; p(x, y) < \varepsilon\} \in \mathcal{U}$ for each $\varepsilon > 0$.*

The following result gives a characterization of the transitive quasi-uniformities in terms of the non-archimedean quasi-pseudo-metrics.

Theorem 3. *Every transitive quasi-uniformity $\mathcal{U}$ on X is generated by the family $\mathcal{P}_0$ of all non-archimedean quasi-pseudo-metrics which are q.u.l.s.c. on $(X \times X, \mathcal{U} \times \mathcal{U}^{-1})$ and having only two values 0 and 1.*

Proof. Let $\mathcal{V}$ be the transitive quasi-uniformity generated by the family $\mathcal{P}_0$. It is clear that the family of sets $\{\{(x, y) ;\ p(x, y) = 0\} ; p \in \mathcal{P}_0\}$ is a subbase for $\mathcal{V}$. Since, by Lemma 2, $\{(x, y) ;\ p(x, y) = 0\} \in \mathcal{U}$, for each $p \in \mathcal{P}_0$, the quasi-uniformity $\mathcal{V}$ is smaller than $\mathcal{U}$.

Conversely, if $\mathcal{B}$ is a transitive subbase for $\mathcal{U}$, then, by Lemma 1, for each $B \in \mathcal{B}$ p_B is a non-archimedean quasi-pseudo-metric on X and $B = \{(x, y) ; p_B(x, y) = 0\} \in \mathcal{U}$. Using again Lemma 2, it follows that p_B is q.u.l.s.c. on $(X \times X,\ \mathcal{U} \times \mathcal{U}^{-1})$. Hence for each $B \in \mathcal{B}$, $p_B \in \mathcal{P}_0$, so that $\mathcal{V}$ is larger than $\mathcal{U}$.

Remark. Let $X = \{x_1, x_2, \ldots, x_n\}$ be a finite space and τ a topology on X. If D_i is the smallest open set containing $x_i \in X$ and $B = \bigcup_{i=1}^{n} \{x_i\} \times D_i$, then $\{B\}$ is a base for a transitive quasi-uniformity compatible with τ. Consequently, p_B defined as in Lemma 1 is a non-archimedean quasi-pseudo-metric compatible with τ. There exists a strong interconnection between this non-archimedean quasi-pseudo-metric and the matrix characterization of topology τ given by H. Sharp [12]: if (t_{ij}) is the matrix associated to the topology τ, then $t_{ij} = 1 - p_B(x_j, x_i)$, $i, j = 1, 2, \ldots, n$.

3. Transitive quasi-uniformities and principal topologies. Let (X, τ) be a topological space. The following assertions are obviously equivalent: a) every intersection of open sets is open, b) for each $x \in X$ there exists a set $D_x \in \tau$ which is the smallest open set containing x, c) the family of the τ-closed sets is a topology on X. The topologies which satisfy one of these conditions appear frequently in the mathematical literature having different designations: they are called discrete topologies in [1], principal topologies in [13], complete topologies in [11], saturated topologies in [6], etc. We adopt the term of *principal topology* which appears in [13] related with the complementation problem of topologies.

The next lemma follows immediately.

Lemma 3. *Let τ be a principal topology on X and, for each $x \in X$, let D_x be the smallest open set which contains x. Let*

$$B_\tau = \bigcup \{\{x\} \times D_x ;\ x \in X\}.$$

Then the family $\{B_\tau\}$ reduced to a single element is a transitive base for a compatible quasi-uniformity $\mathcal{U}_\tau$ on X.

Remark. The quasi-uniformity $\mathcal{U}_\tau$ related with the principal topology τ is the largest compatible quasi-uniformity on X [2, Theorem 3.3].

Theorem 4. *Let (X, τ) be a topological space and let $\{\tau_i ;\ i \in I\}$ be a family of principal topologies on X such that $\tau = \sup\{\tau_i ;\ i \in I\}$. If, for each $i \in I$, $\mathcal{U}_{\tau_i}$ is the quasi-uniformity compatible with τ_i, constructed as in Lemma*

3. *then* $\mathcal{U}_\tau = \sup\{\mathcal{U}_{\tau_i} : i \in I\}$ *is a transitive quasi-uniformity compatible with* τ.

Proof. It is clear that $\mathcal{U}_\tau$ is a transitive quasi-uniformity and, as it is well known, the topology induced by $\mathcal{U}_\tau$ is the supremum of the family of topologies induced by $\mathcal{U}_{\tau_i}$. Therefore $\mathcal{U}_\tau$ is compatible with τ.

Definition *If* (X, τ) *is a topological space,* $\{\tau_i : i \in I\}$ *is a family of principal topologies on* X, *and* $\tau = \sup\{\tau_i : i \in I\}$, *then* $\mathcal{U}_\tau$ *is called a principal quasi-uniformity for* (X, τ).

Theorem 5. *Let* (X, τ) *be a topological space and let* $\mathcal{U}$ *be a compatible quasi-uniformity for* (X, τ). *Then* $\mathcal{U}$ *is a transitive quasi-uniformity if and only if* $\mathcal{U}$ *is a principal quasi-uniformity.*

Proof. By Theorem 4, if $\mathcal{U}$ is a principal quasi-uniformity, then $\mathcal{U}$ is a transitive quasi-uniformity. Conversely, let $\mathcal{B}$ be a transitive base for $\mathcal{U}$. Then $\mathcal{U} = \sup\{\mathcal{U}_B : B \in \mathcal{B}\}$, where $\mathcal{U}_B$ is the transitive quasi-uniformity which has the base $\{B\}$. It is clear that the topology τ_B induced by $\mathcal{U}_B$ is a principal topology, $B(x)$ being the smallest τ_B-open set which contains x. Also, since $B = \cup\{\{x\} \times B(x) : x \in X\}$, $\mathcal{U}_B$ is the quasi-uniformity compatible with τ_B constructed as in Lemma 3. Therefore $\mathcal{U}$ is a principal quasi-uniformity compatible with τ.

Remark. Let (X, τ) be a topological space. If, for each $D \in \tau$, we denote $\tau_D = \{0, D, X\}$, then $\tau = \sup\{\tau_D : D \in \tau\}$ and the corresponding principal quasi-uniformity is precisely the Pervin quasi-uniformity of τ [8, p. 16]. It is obvious that the family $\{\tau_D : D \in \tau\}$ can be replaced by the family of all finite topologies on X, included in τ.

Also, the point (locally) finite covering quasi-uniformity for (X, τ) coincides with the principal quasi-uniformity generated by the family of all principal topologies included in τ having a point (τ-locally) finite base.

If $\mathcal{B}$ is a base for a topology τ on X, then $\mathcal{B}$ is called [4] a σ*-Q-base* for τ if there exists a sequence $\{\mathcal{B}_n : n \in N\}$ of Q-collections such that $\mathcal{B} = \cup\{\mathcal{B}_n : n \in N\}$.

Lemma 4. *Let* (X, τ) *be a topological space. Then the topology* τ *has a* σ*-Q-base if and only if there exists a sequence* $\{\tau_n : n \in N\}$ *of principal topologies on* X *such that* $\tau = \sup\{\tau_n : n \in N\}$.

Proof. Let $\mathcal{B} = \cup\{\mathcal{B}_n : n \in N\}$ is a σ-Q-base for τ. Then, for each $n \in N$, $\mathcal{B}'_n = \mathcal{B}_n \cup \{X\}$ is a Q-cover of X. If $A_x^{\mathcal{B}'_n}$ denotes $\cap\{B \in \mathcal{B}'_n : x \in B\}$, the family $\{A_x^{\mathcal{B}'_n} : x \in X\}$ of τ-open sets is a base for a principal topology τ_n on X such that $A_x^{\mathcal{B}'_n}$ is the smallest τ_n-open set which contains x. Since $\mathcal{B}_n \subset \tau_n \subset \tau$, it results that $\tau = \sup\{\tau_n : n \in N\}$. Conversely, if $\tau = \sup\{\tau_n : n \in N\}$, then the corresponding principal quasi-uniformity $\mathcal{U}_\tau$ has a countable transitive subbase $\{\mathcal{B}_{\tau_n} : n \in N\}$ constructed as in Lemma 3 and therefore $\mathcal{U}_\tau$ has a countable transitive base $\{\mathcal{U}_n : n \in N\}$. It follows easily that $\mathcal{B}_n = \{U_n(x) : x \in X\}$ is a Q-cover of X (see, for example, the proof of Theorem 2.2 [4]) and $\{\mathcal{B}_n : n \in N\}$ is a σ-Q-base for τ.

Corollary 1. *Let* (X, τ) *be a topological space. Then the following conditions are equivalent:*

(i) τ *has a* σ*-Q-base;*

(ii) τ *is the supremum of countable family of principal topologies on* X;

(iii) (X, τ) *is non-archimedean quasi-pseudo-metrizable;*

(iv) (X, τ) *has a compatible transitive quasi-uniformity with a countable base.*

Proof. The result follows from Lemma 4, Theorem 2 and Theorem 5.

This Corollary has the same content as Theorem 3.2 [4]: the difference consists in the fact that we delate the condition for (X, τ) being a T_1-space.

4. Transitive quasi-uniformities and zero dimensional bitopological spaces. J. C. Kelly [5] has introduced and initiated a systematic study of bitopological spaces. Several other authors have contributed to the development of the theory of bitopological spaces. In [10] a bitopological space (X, τ, τ') is said to be *pairwise zero dimensional* if τ is zero dimensional with respect to τ' and τ' is zero dimensional with respect to τ (τ is zero dimensional with respect to τ' if τ has a base of τ'-closed sets, or, equivalently, if for each point $x \in X$ and for each τ-open set D containing x, there is a τ-open τ'-closed set G such that $x \in G \subset D$). A bitopological space (X, τ, τ') is *transitive quasi-uniformizable* if there is a pair $(\mathcal{U}, \mathcal{U}^{-1})$ of conjugate transitive quasi-uniformities on X such that τ is induced by $\mathcal{U}$ and τ' is induced by $\mathcal{U}^{-1}$. Two principal topologies on X are said to be *conjugate topologies* if either of them is the family of closed sets with respect to the other.

Lemma 5. *A bitopological space* (X, τ, τ') *is transitive quasi-uniformizable if and only if we have* $\tau = \sup\{\tau_i : i \in I\}$, $\tau' = \sup\{\tau'_i : i \in I\}$, *where for each* $i \in I$, τ_i *and* τ'_i *are principal and conjugate topologies on* X.

Proof. Let $(\mathcal{U}, \mathcal{U}^{-1})$ be a pair of conjugate transitive quasi-uniformities on X such that τ and τ' are induced by $\mathcal{U}$ and $\mathcal{U}^{-1}$, respectively. If $\mathcal{B}$ is a transitive base for $\mathcal{U}$, then $\mathcal{B}^{-1}$ is a transitive base for $\mathcal{U}^{-1}$. Obviously, $\tau = \sup\{\tau_B : B \in \mathcal{B}\}$ and $\tau' = \sup\{\tau_{B^{-1}} : B \in \mathcal{B}\}$, where τ_B denotes the topology induced on X by the quasi-uniformity $\mathcal{U}_B$, which has the base $\{B\}$. It is easy to verify that τ_B and $\tau_{B^{-1}}$ are principale and conjugate topologies on X. Conversely, it follows immediately that the principal quasi-uniformities $\mathcal{U}_\tau$ and $\mathcal{U}_{\tau'}$ corresponding to $\tau = \sup\{\tau_i : i \in I\}$ and $\tau' = \sup\{\tau'_i : i \in I\}$, respectively, are conjugate quasi-uniformities on X.

The following result states precisely the connection between the concepts of a transitive quasi-uniformity and a pairwise zero dimensionality.

Theorem 6. *Let* (X, τ, τ') *be a bitopological space. Then the following assertions are equivalent:*

(i) (X, τ, τ') *is pairwise zero dimensional;*

(ii) *for each point* $x \in X$ *and for each set* $D \in \tau$ *containing* x *there exists a function* $f : X \to \{0, 1\}$ τ*-lower semi-continuous* (τ*-l.s.c*) *and* τ'*-upper semi-continuous* (τ'*-u.s.c.*) *such that* $f(x) = 1$ *and* $f(X - D) = 0$; *also, for each point* $x \in X$ *and for each set* $D' \in \tau'$ *containing* x, *there exists a function* $f : X \to \{0, 1\}$ τ'*-l.s.c. and* τ*-u.s.c., such that* $f(x) = 1$ *and* $f(X - D') = 0$;

(iii) (X, τ, τ') *is transitive quasi-uniformizable.*

Proof. (i) $\Rightarrow$ (ii). If τ is zero dimensional with respect to τ', then for each $x \in X$ and for each $D \in \tau$ containing x, there exists a τ-open and τ'-closed set G such that $x \in G \subset D$. It is clear that the characteristic function on G is τ-l.s.c. and τ'-u.s.c., and satisfies the conditions $f(x) = 1$ and $f(X - D) = 0$. The proof of the other part of the theorem is analogous.

(ii) $\Rightarrow$ (iii). If we denote $\tau_D = \{0, D, X\}$, by using our hypotheses it results that $\tau = \sup\{\tau_D ; D\ \tau\text{-open and } \tau'\text{-closed}\}$ and $\tau' = \sup\{\tau_{X-D}; D\ \tau\text{-open and } \tau'\text{-closed}\}$. Since τ_D and τ_{X-D} are conjugate topologies, it follows by Lemma 5 that (X, τ, τ') is transitive quasi-uniformizable.

(iii) $\Rightarrow$ (i). Let (X, τ, τ') be transitive quasi-uniformizable. Since, by Lemma 5, $\tau = \sup\{\tau_i ; i \in I\}$ and $\tau' = \sup\{\tau'_i ; i \in I\}$, where for each $i \in I$ the principal topologies τ_i and τ'_i are conjugate, it follows immediately that τ has a base of τ'-closed sets and τ' has a base of τ-closed sets. Therefore (X, τ, τ') is pairwise zero dimensional.

Corollary 2. *A pairwise zero dimensional space is pairwise completely regular* [8, p. 63].

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CVASI-UNIFORMITĂŢI TRANZITIVE ŞI SPAŢII BITOPOLOGICE ZERO-DIMENSIONALE

Rezumat

În Notă sînt date mai multe caracterizări ale noţiunilor de cvasi-uniformitate tranzitivă [3] şi spaţiu bitopologic reciproc zero-dimensional [10]. Rezultatul principal stabileşte o legătură între aceste noţiuni: un spaţiu bitopologic este reciproc zero-dimensional dacă şi numai dacă el este tranzitiv cvasi-uniformizabil .

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THÉORÈMES GLOBAUX DANS LA THÉORIE DES GROUPES DE LIE-BANACH

PAR

AUREL BEJANCU

Soient B un ensemble et P une classe de bijections de B qui agit de façon simplement transitive sur B, c'est-à-dire, pour tout $x, y \in B$ il existe un élément unique $f \in P$ tel que $f(x) = y$.

Théorème 1. *B est un groupe ayant P comme ensemble des translations à gauche, si et seulement si, pour tout $f, g \in P$ on a $f \circ g \in P$.*

Démonstration. Si B est un groupe, alors il est clair que la composition de deux translations à gauche est une translation à gauche. Réciproquement, nous supposons que P vérifie la condition énoncée dans le théorème, et fixons un point $e \in B$. Pour tout $x \in B$ il existe $f_x \in P$ tel que $f_x(x) = e$. Nous définissons l'application

$$(1) \qquad \pi : B \times B \to B : \pi(x, y) = (f_x)^{-1}(y) \quad \forall (x, y) \in B \times B.$$

Il est aisé de voir que $\pi(x, e) = x$ et $\pi(x, x_*) = e$ où $x_* = f_x(e)$. On a: $f_{\pi(x,y)}(\pi(x, y)) = (f_y \circ f_x)(\pi(x, y)) = e$. En tenant compte que P agit simplement transitif sur B on déduit

$$f_{\pi(x,y)} = f_y \circ f_x \qquad \forall (x, y) \in B \times B.$$

c'est-à-dire, la loi de composition π est associative. q.e.d.

Si B est une C^∞-variété modelée par l'espace de Banach E, alors nous obtenons le

Théorème 2. *La classe de bijections P détermine sur B une structure de groupe de Lie-Banach ayant P comme ensemble des translations à gauche, si et seulement si, sont vérifiées les conditions:*

1. *Pour tout $f, g \in P$ on a $f \circ g \in P$.*
2. *L'application π est différentiable de classe C^∞.*

Maintenant nous supposons que B est une variété parallélisable et connexe. Si e est un point fixé dans B, alors le fibré vectoriel $L(B) = \bigcup_{x \in B} L(T_x B : T_e B)$ (2) possède une C^∞-section globale ω telle que $\omega(x) \in L_{is}(T_x B : T_e B)$ pour tout $x \in B$.