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REMARKS ABOUT DIFFERENTIAL OPERATORS ON FOLIATE MANIFOLDS

BY

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The study of differential operators on vector bundles is an important subject for many theories in analysis, differential geometry and topology and, in view of this fact, it is worth to consider the special case of a foliate manifold. In this case, it is of course interesting to look for operators depending only on the leaves.

These are the questions to be investigated in the present Note, whose starting points are the paper of Molino [9], where the special case of a connection is investigated, and of Kamber-Tondeur [6], where a general theory of invariant differential operators is constructed. We also recall that all the mentioned authors started from an older paper of Atiyah [1], which investigated such problems in the complex analytic case.

The main results which we shall expose here are: a description of the analytical configurations involved by the theory of differential operators on foliate manifolds, the calculation of the obstruction to the existence of foliate (projectable [9]) differential operators of a given symbol and the proof of the existence of such operators when corresponding connections exist.

1. Foliate vector bundles

1.1. Let V^{n+m} be a C^∞ differentiable paracompact manifold, endowed with a foliation $\mathcal{F}$ of codimension n. The pair $(V, \mathcal{F})$ is called a *foliate manifold* and it is characterized by the existence of an atlas $\mathcal{A} = \{U_0, (x_0^a, x_0^u)\}$ ($\cup U_0 = V$; $a = 1,\dots, n$; $u = n+1,\dots, n+m$) such that on an intersection of coordinate neighbourhoods $U_0 \cap U_{0'}$, one has

$$\frac{\partial x_{0'}^a}{\partial x_0^u} = 0. \tag{1.1}$$

The maximal connected integral manifolds of the Pfaff system $dx^a = 0$ are called the *leaves* of $\mathcal{F}$, the subsets of an U_0 given by $x^a =$ const. are the *slices* and, generally, the elements which depend only on the leaves are called *foliate*. Particularly, we can speak about foliate differentiable functions $f: V \to R$, which are characterized by

$$\frac{\partial f}{\partial x^u} = 0 \tag{1.2}$$

and define a subsheaf $\Phi(V)$ of the sheaf $F(V)$ of germs of real differentiable functions on V. Generally, the foliate elements are induced by corresponding elements of submanifolds of the neighbourhoods U_β transverse to the slices of $\mathcal{F}$, which are called *local transverse manifolds* of $\mathcal{F}$. This explains why in [9] the term „foliate" is replaced by „projectable".

For a survey paper regarding foliate manifolds we send to Haefliger [4] and for the notation and terminology to be used to our previous paper [13]. All the manifolds, functions and bundles of this Note are of class C^∞. Besides the already introduced indices: $a, b, \ldots = 1, \ldots, n$; $u, v, \ldots = n+1, \ldots, n+m$, we also consider $i, j, \ldots = 1, \ldots, n+m$ and $\alpha, \beta, \ldots = 1, \ldots, r$.

1.2. Now, consider a differentiable vector bundle $p: E \to V$ of rank r (i.e. with fibre R^r). It will be called a *foliate vector bundle* (f.v.b.), if it has a local trivialization whose transition functions are matrices of foliate functions and in this case only such trivializations will be used. Hence, E is an f.v.b. if and only if it induces locally constant bundles on the leaves of $\mathcal{F}$.

Such bundles were considered by Molino [9] and in our papers [13, 14]. We remark that they are different from Milnor's foliate vector bundles [8]. If E is an f.v.b. in our sense, the manifold E has the local coordinates (x^a, x^u, s^α) where s^α are fibre coordinates for E and one sees that $x^a =$ const, $s^\alpha =$ const, are the slices of a foliation $\mathcal{E}$ on E, whose leaves are covering spaces for the leaves of $\mathcal{F}$. Now, the pair $(E, \mathcal{E})$ induces on the leaves of $\mathcal{F}$ vector bundles which are foliate in Milnor's sense.

Passing by, let us still remark that Hermann's definition of the linear holonomy group of a leaf [5] can be carried over to any f.v.b. on V and this gives linear representations of the fundamental groups of the leaves as transformation groups on the corresponding local fibres of E.

In order to give the most important example of an f.v.b., we consider first the subbundle $T(\mathcal{F})$ of the tangent bundle $T(V)$ defined by the tangent spaces to the leaves of $\mathcal{F}$ and called the *structural bundle* of $\mathcal{F}$ [13]. It is to be remarked that a vector field X on V is a section of $T(\mathcal{F})$ (it is a *structural field* or a field tangent to $\mathcal{F}$) if and only if $Xf = 0$ for any foliate function f on V.

Now, let

$$Tr(\mathcal{F}) = T(V)/T(\mathcal{F}) \tag{1.3}$$

be the corresponding quotient bundle. Using the atlas $\mathcal{A}$, we see that $Tr(\mathcal{F})$ has a local trivialization with transition functions $\partial x^a_{(1)}/\partial x^b_{(0)}$, whence it follows that $Tr(\mathcal{F})$ is an f.v.b. This is the required example and $Tr(\mathcal{F})$ is called the *transversal bundle of* $\mathcal{F}$. We shall denote by π the canonical projection

$$\pi: T(V) \to Tr(\mathcal{F}). \tag{1.4}$$

Let $g: E \to E'$ be a vector bundle homomorphism of the f.v.b. E and E'. We shall say that g is *foliate* if the local coordinate expressions of g are such. Now, it is easy to see that the f.v.b. on V and their foliate homomorphisms make up an additive category.

The following remarks are easily verified by considering the transition functions of the corresponding bundles. If E and E' are f.v.b., the vector bundles E^*, $L(E, E')$, $E \otimes E'$, $\wedge^p E$ are f.v.b. too. If $g: E \to E'$ is a foliate homomorphism of constant rank, $\ker g$ and $\operatorname{im} g$ are f.v.b. and the canonical isomorphism $E/\ker g \approx \operatorname{im} g$ is foliate. Hence, if the short exact sequence

$$0 \to E' \to E \to E'' \to 0 \tag{1.5}$$

is made up by f.v.b. and foliate homomorphisms then $E'' \approx E/E'$. It is known that such an exact sequence splits but, generally, the splitting is not foliate.

1.3. A cross-section s of an f.v.b. E is called *foliate* if it is constant on the leaves of $\mathcal{F}$. Hence, s is foliate if and only if one has locally $s = s^\alpha \sigma_\alpha$, where σ_α are basic cross-sections for E (with respect to the trivialization having foliate transition functions!) and $\partial s^\alpha/\partial x^u = 0$.

It follows that for the f.v.b. E we can associate not only the sheaf $\underline{E}$ of germs of differentiable sections, which is an $F(V)$-module, but also the subsheaf $\underline{\underline{E}} \subset \underline{E}$ of germs of foliate sections, which is a locally free $\Phi(V)$-module of finite rank. Moreover, it is important to remark that one can prove, just like for the differentiable or analytic case [3, VIII C] that any locally free $\Phi(V)$-module of finite rank is the sheaf of germs of foliate sections of an f.v.b. Also, the foliate homomorphisms $g: E \to E'$ can be identified with the homomorphisms of $\Phi(V)$-modules $g: \underline{\underline{E}} \to \underline{\underline{E}}'$ since, obviously, the homomorphism g is foliate if and only if it sends foliate sections to foliate sections.

We hawe another characterization of the sheaf $\underline{\underline{E}}$ too. Let X be a structural vector field on V. Then, the formula

$$Xs = (Xs^\alpha)\sigma_\alpha \tag{1.6}$$

obviously defines for every f.v.b. E an additive operator $X: \underline{E} \to \underline{E}$ and it follows that $\underline{\underline{E}} = \cap \ker X$, for all the structural fields X.

The sheaf $\underline{\underline{Tr}}(\mathcal{F})$ plays an important role for the deformations of $\mathcal{F}$ [13]. More general, the vector fields Y on V such that πY is a foliate cross-section of $Tr(\mathcal{F})$ are called *foliate vector fields*. They are characterized by the fact that for every foliate function f on V the function Yf is foliate too. The foliate vector fields define on V infinitesimal transformations which send leaves to leaves [13]. Using the characterization given in 1.2 for structural fields and the previous characterization of foliate fields, we see that if X is structural, Y foliate and f a foliate function, then $[X, Y]f = 0$, whence

Proposition 1.1. *The bracket between a structural field and a foliate field is a structural field.*

Other important cases are those of the bundles $L^p_{\text{alt}}\,(Tr, E)$ and $L^p_{\text{sym}}\,(Tr, E)$ whose foliate sections are respectively the *E-valued foliate p-forms* [12, 13, 14] *and E-valued foliate symmetric covariant tensor fields*. These names are explained by the fact that the projection (1.4) induces monomorphisms

$$
(1.7)\qquad
\begin{aligned}
\pi^*_F &= \pi^* : L^p_{\text{alt}}\,(Tr, E) \to L^p_{\text{alt}}\,(T, E),\\
\pi^*_F &= \pi^* : L^p_{\text{sym}}\,(Tr, E) \to L^p_{\text{sym}}\,(T, E).
\end{aligned}
$$

1.4. Consider again an f.v.b. E on V and denote by

$$
(1.8)\qquad J^k E = \bigcup_{x \in V} J^k E_x
$$

the k-jet bundle of E [11]. $J^k E_x$ is the space of the k-jets of local cross-sections in E around x and we recall that, if s_α are the local fibre coordinates in E, a k-jet has the local coordinates

$$
(1.9)\qquad \frac{\partial^{|i|} s^\alpha}{\partial x^{i_1} \ldots \partial x^{i_h}}\ (|i| = i_1 + \ldots i_h : h = 0, \ldots, k).
$$

Now, for every $x \in V$, we can consider the subspace $J^k_f E_x \subset J^k E_x$ made up of jets of foliate sections, whose coordinates expressed by the help of the atlas $\mathcal{A}$ are obviously

$$
(1.10)\qquad \frac{\partial^{|a|} s^\alpha}{\partial x^{a_1} \ldots \partial x^{a_h}}\ (|a| = a_1 + \ldots + a_h : h = 0, \ldots, k)
$$

and it is easy to see that

$$
(1.11)\qquad J^k_f E = \bigcup_{x \in V} J^k_f E_x
$$

is a subbundle of $J^k E$. Moreover, $J^k_f E$ is an f.v.b. and the canonical jet map [11]

$$
(1.12)\qquad j^k : \underline{E} \to \underline{J^k E}
$$

induces a jet map

$$
(1.13)\qquad j^k_f : \underline{\underline{E}} \to \underline{\underline{J^k_f E}}\,.
$$

The quotient bundle

$$
(1.14)\qquad Q^k E = J^k E \,/\, J^k_f E
$$

will also be important. We call it the *quotient k-jet bundle* and, using again the atlas $\mathcal{A}$, it has the local coordinates (1.9), where at least one of the indices i is an index u.

The mentioned bundles are connected by the exact sequence

$$
(J^k E)\qquad 0 \to J^k_f E \xrightarrow{i^k} J^k E \xrightarrow{p^k} Q^k E \to 0,
$$

where i^k is the inclusion map and p^k the canonical projection. According to a well known theorem of Whitney, this sequence splits and we can consider a left inverse a^k of i^k and a corresponding right inverse q^k of p^k such that the following relations hold

$$
(1.15)\qquad a^k i^k = \text{id},\ p^k q^k = \text{id},\ i^k a^k + q^k p^k = \text{id},\ p^k i^k = 0,\ a^k q^k = 0.
$$

It is known [11] that one has a natural epimorphism of jet bundles

$$
(1.16)\qquad \rho^{k-1} : J^k E \to J^{k-1} E,
$$

which associates to the k-jet of a section the $(k-1)$-jet of the same section and whose kernel is $L^k_{\text{sym}}\,(T, E)$. In just the same manner one gets an epimorphism

$$
(1.17)\qquad \rho^{k-1}_f : J^k_f E \to J^{k-1}_f E,
$$

whose kernel is $L^k_{\text{sym}}\,(Tr, E)$ and one can obviously construct the following diagram

$$
(\Delta^k E)\qquad
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & L^k_{\text{sym}}\,(Tr, E) & \xrightarrow{\varepsilon_f} & J^k_f E & \xrightarrow{\rho^{k-1}_f} & J^{k-1}_f E & \to & 0\\
 & & \downarrow{\scriptstyle \pi^*} & & \downarrow{\scriptstyle i^k} & & \downarrow{\scriptstyle i^{k-1}} & & \\
0 & \to & L^k_{\text{sym}}\,(T, E) & \xrightarrow{\varepsilon} & J^k E & \xrightarrow{\rho^{k-1}} & J^{k-1} E & \to & 0\\
 & & \downarrow{\scriptstyle \chi} & & \downarrow{\scriptstyle p^k} & & \downarrow{\scriptstyle p^{k-1}} & & \\
0 & \to & L^k_{\text{sym}}\,(T, E)/L^k_{\text{sym}}\,(Tr, E) & \xrightarrow{\bar{\varepsilon}} & Q^k E & \xrightarrow{\bar{\rho}^{k-1}} & Q^{k-1} E & \to & 0\\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

which has commutative squares, exact columns and exact first and second rows. Then, it follows from the "9-lemma" that the third row is exact too. Hence, we have

Proposition 1.2. *For every f.v.b. E on V and every $k = 1, 2, \ldots$, there is an associated diagram $(\Delta^k E)$ with exact rows and columns and commutative squares.*

Since the first row of $(\Delta^k E)$ is foliate, we also have

Proposition 1.3. *For every E and k as above, there is the following exact sequence of $\Phi\,(V)$-modules*

$$(S^kE) \qquad 0 \to L^k_{\mathrm{sym}}(Tr, E) \xrightarrow{\varepsilon_f} J^k_f E \xrightarrow{\rho^{k-1}_f} J^{k-1}_f E \to 0.$$

The exact sequence $(S^k E)$ will play an important role in section 3.

1.5. Let us consider the second row of the diagram $(\Delta^k E)$ for a vector bundle E. This is an exact sequence which, according to Whitney's theorem again, splits and a splitting of it is called a k^{th} order connection on E [7]. A connection is defined either by ist *retraction* $\tilde{\omega}$ which is a left inverse of ε or by the corresponding *lift* ω, which is a right inverse of ρ^{k-1}, in such a manner that the following relations hold

$$(1.18) \qquad \tilde{\omega}\varepsilon = \mathrm{id}, \; \rho^{k-1}\omega = id, \; \varepsilon\tilde{\omega} + \omega\rho^{k-1} = \mathrm{id}, \; \rho^{k-1}\varepsilon = 0, \; \tilde{\omega}\omega = 0.$$

This implies the following exact sequence

$$(1.19) \qquad 0 \to J^{k-1}E \xrightarrow{\omega} J^k E \xrightarrow{\tilde{\omega}} L^k_{\mathrm{sym}}(T, E) \to 0.$$

Now, suppose that E is an f.v.b. endowed with a k^{th} order connection as above. Then, the connection is called *adapted* if its lift sends foliate $(k-1)$-jets to foliate k-jets. This is equivalent to the existence of a differentiable homomorphism $\lambda : J^{k-1}_f E \to J^k_f E$ such that

$$(1.20) \qquad \omega i^{k-1} = i^k \lambda.$$

Since i^k is a monomorphism, λ is uniquely defined by (1.20). Since ω is a right inverse of ρ^{k-1} we get from (1.20) and using the diagram $(\Delta^k E)$

$$i^{k-1} = \rho^{k-1} i^k \lambda = i^{k-1} \rho^{k-1}_f \lambda,$$

whence, because i^{k-1} is a monomorphism, we see that λ is a right inverse of $\rho_f{}^{k-1}$ and it defines a differentiable splitting of the first row of $(\Delta^k E)$. This will be called the *induced splitting* of the connection.

If μ is the corresponding retraction, i.e. we have the relations

$$(1.21) \qquad \mu\varepsilon_f = \mathrm{id}, \; \rho^{k-1}_f \lambda = \mathrm{id}, \; \varepsilon_f \mu + \lambda\rho^{k-1}_f = \mathrm{id}, \; \rho^{k-1}_f \varepsilon_f = 0, \; \mu\lambda = 0,$$

it is easy to establish that the following diagram has exact rows and commutative squares

$$(C^kE) \qquad \begin{array}{ccccccccc} 0 & \to & J^{k-1}_f E & \xrightarrow{\lambda} & J^k_f E & \xrightarrow{\mu} & L^k_{\mathrm{sym}}(Tr, E) & \to & 0 \\ & & \downarrow{\scriptstyle i^{k-1}} & & \downarrow{\scriptstyle i^k} & & \downarrow{\scriptstyle \cong} & & \\ 0 & \to & J^{k-1} E & \xrightarrow{\omega} & J^k E & \xrightarrow{\tilde{\omega}} & L^k_{\mathrm{sym}}(T, E) & \to & 0. \end{array}$$

Conversely, suppose that the first row of $(C^k E)$ is given by a differentiable splitting of the first row of $(\Delta^k E)$. Consider a splitting (a^{k-1}, q^{k-1})

of the sequence $(A^{k-1} E)$, an arbitrary k^{th} order connection on E of lift ζ and put

$$(1.22) \qquad \omega = i^k \lambda a^{k-1} + \zeta q^{k-1} p^{k-1}.$$

Then, using (1.15) and (1.21) one gets easily $\rho^{k-1}\omega = \mathrm{id}$. and $\omega i^{k-1} = i^k\lambda$, hence ω defines an adapted connection on E whose restriction to the foliate jets (the induced splitting) is just λ.

The formula (1.22) defines a set of solutions of the equations (1.20) and $\rho^{k-1}\omega = \mathrm{id}.$, for a given λ. Let ω' be an arbitrary solution of the same equations. Then we have

$$(\omega' - \omega)\, i^{k-1} = 0,$$

whence

$$\omega' - \omega = \nu p^{k-1}$$

and by right multiplication with q^{k-1}

$$\nu = (\omega' - \omega)\, q^{k-1}.$$

It follows that ω' can be put under the form

$$(1.23) \qquad \omega' = i^k \lambda a^{k-1} + \tilde{\zeta} q^{k-1} p^{k-1}$$

and it is easy to see that this is a solution of the considered equations if and only if

$$(1.24) \qquad \rho^{k-1}\tilde{\zeta} = \mathrm{id}. - \theta a^{k-1}$$

for some homomorphism θ. The case of a connection $\tilde{\zeta}$ is contained here by taking $\theta = 0$.

Hence (1.23) and (1.24) give all the adapted connections with the induced splitting λ.

We summarize the previous results in

Proposition 1.4. *Every adapted connection on the f.v.b. E defines a unique induced differentiable splitting of the first row of $(\Delta^k E)$ in such a manner that the diagram $(C^k E)$ has exact rows and commutative squares. Every differentiable splitting of the first row of $(\Delta^k E)$ is the induced splitting of the set of adapted k^{th} order connections on E, given by (1.23) and (1.24).*

Corollary. *For every f.v.b. E and every $k = 1, 2, \ldots$ there are adapted k^{th} order connections.*

This follows from the second part of proposition 1.4 since, in view of Whitney's theorem, there are differentiable splittings of the first row of $(\Delta^k E)$.

Finally, we shall give one more definition: *a k^{th} order adapted connection on the f.v.b. E is called a foliate (projectable [9]) connection if the lift λ of the induced splitting of this connectioni is foliate. The corresponding retraction μ is then foliate too.*

From the previous considerations it follows

Proposition 1.5. *The f.v.b. E admits a foliate k^{th} order connection if and only if the first row of the diagram $(\Delta^k E)$ admits a foliate splitting. All the connections which correspond to such a splitting are defined by (1.23) and (1.24).*

As M o l i n o [9] and K a m b e r - T o n d e u r [6] pointed out and as we shall see later (section 3) it is not necessarily true that every f.v.b. admits foliate connections.

2. Differential operators on foliate vector bundles.

2.1. In this section, we want to consider differential operators on foliate vector bundles and we begin by recalling the following definitions [11, 6].

If E and F are two differentiable vector bundles over the same basis V, a k^{th} order differential operator D is an additive operator

$$D : \underline{E} \to \underline{F} \tag{2.1}$$

such that

$$D = \Phi(D) j^k \tag{2.2}$$

for some unique $F(V)$-module homomorphism

$$\Phi(D) : \underline{J^k E} \to \underline{F} \tag{2.3}$$

which, of course, can be identified with a vector bundle homomorphism $\Phi(D) : J^k E \to F$.

The k-symbol of the operator D is defined by [6]

$$\sigma(D) = \Phi(D)\,\varepsilon : L^k_{\mathrm{sym}}(T, E) \to F, \tag{2.4}$$

where ε is defined by the diagram $(\Delta^k E)$. $\sigma(D)$ can also be considered as an $F(V)$-module homomorphism of the corresponding sheaves of germs.

Since $L^k_{\mathrm{sym}}(T, E) \approx L^k_{\mathrm{sym}}(T, R) \otimes E$, where R is the trivial line bundle, it follows that for every $x \in V$ and every $\alpha_x \in L^k_{\mathrm{sym}}(T, R)_x$, $\sigma(D)$ induces a homomorphism $\sigma(D)(\alpha_x) : E_x \to F_x$. Particularly, every tangent covector τ at x defines, by the help of the symmetric product, an α_x and we get the homomorphism $\sigma(D)(\tau_x) : E_x \to F_x$. If this is an isomorphism for every $\tau \neq 0$ and every $x \in V$, D is called a *k-elliptic operator.*

Suppose that we have a k^{th} order connection on E, which is represented by the second line of the diagram $(C^k E)$. Then

$$\nabla = \tilde{\omega} j^k : \underline{E} \to \underline{L^k_{\mathrm{sym}}(T, E)} \tag{2.5}$$

is a k^{th} order differential operator called the *covariant differential* with respect to the given connection. For its symbol we have $\sigma(\nabla) = \mathrm{id}$, and this characterizes the covariant differentials. Hence the definition of a connection is equivalent to the definition of some differential operator of identical symbol: the corresponding covariant differential.

This is a very important example of a differential operator, since any other k^{th} order operator can be expressed by the help of ∇. In fact, using (1.18) and considering

$$\gamma(D) = \Phi(D)\,\omega : J^{k-1} E \to F, \tag{2.6}$$

we immediately obtain

$$D = \sigma(D)\nabla + \gamma(D)\,\rho^{k-1} j^k. \tag{2.7}$$

This also implies that a differential operator D is „divisible" by a covariant differential if and only if

$$\gamma(D) = 0. \tag{2.8}$$

2.2. Now, let (2.1) be a differential operator between the f.v.b. E and F. Then D is called a *foliate operator* if it sends foliate sections to foliate sections or, equivalently, if there is the following commutative diagram of sheaves:

$$\begin{tikzcd} \underline{\underline{E}} \arrow[r, "D'"] \arrow[d, "\iota"] & \underline{\underline{F}} \arrow[d, "\iota"] \\ \underline{E} \arrow[r, "D"'] & \underline{F} \end{tikzcd} \tag{2.9}$$

where ι is the inclusion map. D' which is an additive operator is called the *foliate part* of D. We can also say that D is a foliate operator if it has a foliate part.

Proposition 2.1. *In the above notation, D is a foliate differential operator if and only if one of the following equivalent properties is satisfied: a) for all the structural vector fields X on V one has $(DX - XD)|_{\underline{\underline{E}}} = 0$; b) the f.v.b. homomorphism $\Phi(D) i^k : J^k_f E \to F$ is foliate.*

In fact, the equivalence of the condition a) and the definition of foliate operators follows immediately from $\underline{\underline{E}} = \cap \ker X$ (section 1.3) and the equivalence of b) to the same definition follows by expressing the homomorphism $\Phi(D) i^k$ in local coordinates, using the atlas $\mathcal{A}$ on V.

We shall denote

$$\Phi'(D) = \Phi(D) i^k \tag{2.10}$$

and call it the *foliate part* of $\Phi(D)$.

Now, if s is a section of $\underline{\underline{E}}$, we have in view of (2.9) and (2.2)

$$D's = Ds = \Phi(D) i^k j^k_f s = \Phi'(D) j^k_f(s).$$

which shows that one has the factorization

$$D' = \Phi'(D)\, j^k_f \tag{2.11}$$

and it is clear that $\Phi'(D)$ is uniquely defined by this relation. This enables us to prove

Proposition 2.2. *An additive operator $D' : \underline{\underline{E}} \to \underline{\underline{F}}$ is the foliate part of a k^{th} order foliate operator if and only if there is a unique factorization $D' = \alpha j^k_f$ where α is a foliate homomorphism $\alpha : J^k_f E \to F$.*

The necessity has already been proved. As for the sufficiency let a^k be a left inverse of i^k (section 1.4) and take

$$D = (\alpha a^k + \beta p^k)\, j^k, \tag{2.12}$$

where β is arbitrary and p^k is the canonical projection of the sequence $(A^k E)$ of 1.4. It is easy to see that D is the desired operator and, moreover, the formula (2.12) defines all the foliate operators D for which D' is the foliate part.

Let us also remark from (2.11) that $\Phi'(D)$ depends only on the foliate part D' of D. This suggests the introduction of a *foliate symbol* for the foliate operator D, defined by

$$\sigma'(D) = \Phi'(D)\varepsilon_f : L^k_{\text{sym}}(Tr, E) \to F. \tag{2.13}$$

where ε_f is defined in the diagram $(\Delta^k E)$. From that diagram and from (2.10) and (2.4), we get

$$\sigma'(D) = \sigma(D)\,\pi^*. \tag{2.14}$$

$\sigma'(D)$ depends only on the foliate part D' of D. It is to be remarked that, just like for $\sigma(D)$, we get homomorphisms $\sigma'(D)(v_x) : E_x \to F_x$ for every $v_x \in Tr^*(\mathfrak{F})_x$ and if they are isomorphisms for all $x \in V$ and all $v_x \neq 0$, D is said to be *foliately k-elliptic.*

For $k = 1$, we have another characterization of the foliate parts of operators too.

Proposition 2.3. *An additive operator $D' : \underline{\underline{E}} \to \underline{\underline{F}}$ is the foliate part of a first order foliate operator of foliate symbol σ' if and only if for any $f \in \Phi(V)$, and $s \in \underline{\underline{E_x}}$ one has*

$$D'(fs) = fD's + \sigma'(df \otimes s). \tag{2.15}$$

$\sigma' : L(Tr, E) \to F$ being a foliate homomorphism.

Here $df \otimes s$, regarded as an element of $L(Tr, E)$, is given by

$$df \otimes s\,(v) = df(\pi^{*-1} v)\, s \quad (v \in Tr), \tag{2.16}$$

which is obviously meaningful.

The necessity follows by a straightforward computation using (2.9), (2.2) etc. and the obvious general formula [6].

$$j^1(fs) = fj^1 s + \varepsilon(df \otimes s). \tag{2.17}$$

For the sufficiency, let $l : L(T, E) \to L(Tr, E)$ be a differentiable left inverse of π^* and define $D : \underline{E} \to \underline{F}$ by the formula

$$Ds = D(s^\alpha \sigma_\alpha) = s^\alpha D'\sigma_\alpha + \sigma' l\,(ds^\alpha \otimes \sigma_\alpha) \quad (s \in \underline{E}). \tag{2.18}$$

where σ^α is a local basis of $\underline{E}$.

Straightforward computations show that D is independent of the chosen basis and that one has

$$D(fs) = fDs + \sigma' l\,(df \otimes s) \qquad (f \in F(V)). \tag{2.19}$$

Then, by the similar proposition for differential operators (lemma 2.12 of [11]), D is a first order operator of symbol $\sigma' l$ and we see from (2.18) and (2.15) that D has a foliate part which is just D'. Moreover, from (2.14) we see that the foliate symbol of D is just σ'. Hence, proposition 2.3 is proved.

2.3. In this section we want to consider some examples of foliate differential operators.

The first which we can take is

$$\pi^{*-1} d\pi^* = \underline{L^k_{\text{alt}}(Tr, R)} \to \underline{L^{k+1}_{\text{alt}}(Tr, R)}, \tag{2.20}$$

where d is the exterior differential and π^{*-1} is the inverse of the isomorphism defined by π^*. Using local coordinates, it is easy to see that it sends foliate forms to foliate forms. The foliate part d' of (2.20) is just the exterior differential of foliate forms and R e i n h a r t [12] used it to define the cohomology spaces of this kind of differential forms.

If the foliation $\mathfrak{F}$ has a Riemann bundle-like metric, one can construct an adjoint operator δ' and a laplacian $\Delta' = d'\delta' + \delta' d'$ defined for foliate forms. According to proposition 2.2, δ' and Δ' are the foliate parts of some foliate operators and they play an important role in the study of the previously mentioned cohomology [12].

Another example is offered by Spencer's operator

$$\tilde{D} : \underline{J^{k+1} E} \to \underline{L(T, J^k E)}.$$

In fact, from the definition of $\tilde{D}$ as reproduced, for instance, in [6] it follows immediately that $\tilde{D}' = \pi^{*-1} D_f^{k+1}$ is a well defined first order differential operator

$$\tilde{D}' : \underline{J^{k+1}_f E} \to \underline{L(Tr, J^k E)}$$

and this operator is foliate.

Now, suppose that the f.v.b. E is endowed with a k^{th} order foliate connection which is represented by the diagram $(C^k E)$ of section 1.5. Then, we can define

$$\nabla' : \underline{E} \to \underline{L^k_{\mathrm{sym}}(Tr, E)} \tag{2.21}$$

by $\nabla' = \mu j^k_f$ and it follows from proposition 2.2 that ∇' is the foliate part of a set of k^{th} order foliate operators.

The operator ∇' plays in some sense the role of the covariant differential ∇ for the foliate parts of operators. In fact, suppose that $D' : \underline{\underline{E}} \to \underline{\underline{F}}$ is the foliate part of some operator. Then, $D' = \Phi' j^k_f$ and using the third relation (1.21) we get $\Phi' = \sigma'\mu + \gamma' \rho^{k-1}_f$, where σ' is the corresponding foliate symbol and $\gamma' = \Phi' \lambda$. Hence

$$D' = \sigma' \nabla' + \gamma' \rho^{k-1}_f j^k_f, \tag{2.22}$$

which is an analogon of the formula (2.6). D' is „divisible" by ∇' if and only if $\gamma' = 0$.

It is to be remarked that one can characterize the adapted and foliate connections by the help of the covariant differential ∇ too. In fact, we have

Proposition 2.4. *A k^{th} order connection on the f.v.b. E is adapted if and only if $\nabla/_{\underline{E}}$ takes values in $\underline{L^k_{\mathrm{sym}}(Tr, E)}$ and it is foliate if and only if these values are in $\underline{\underline{L^k_{\mathrm{sym}}(Tr, E)}}$.*

In fact, if we have the adapted connection of the diagram $(C^k E)$, we get

$$\nabla s = \pi^* \mu j^k_f(s) \qquad (s \in \underline{E}).$$

Conversely, if for a connection we have the indicated condition, this implies that $\tilde{\omega} i^k$ is divisible by π^*, whence (using $(\Delta^k E)$) we get the induced splitting μ and the connection is adapted.

The second part of the proposition is obvious.

2.4. Let us insist a little on some questions relative to the first order connections on an f.v.b. E. As we already know, they can be defined by the corresponding covariant differential, which is a first order differential operator $\nabla : E \to L(T, E)$ of identical symbol.

For the questions which we have in mind it is worth to consider a slightly generalized kind of operators, introduced by the following definition (see also [6], [16] and [15]). If E and F are two vector bundles on V and if we have two homomorphisms $\varkappa : E \to F$, $\varphi : T(V) \to T(V)$, a *$(\varphi, \varkappa)$-connection* of (E, F) is a first order differential operator

$$D : \underline{E} \to \underline{L(T, F)} \tag{2.23}$$

whose symbol is $\sigma(D) : L(T, E) \to L(T, F)$ given by

$$\sigma(D)(a)(t) = \varkappa(a(\varphi(t))) \qquad (a \in L(T, E),\ t \in T). \tag{2.24}$$

If $\varphi = \mathrm{id.}$ and $\varkappa = \mathrm{id.}$, we have just the usual connections on E.

Recalling the natural isomorphisms $L(T, M) \approx T^* \otimes M$, we could write $\sigma(D) = \varphi^* \otimes \varkappa$, hence a $(\varphi, \varkappa)$-connection is a differential operator of „decomposable" symbol. Then, according to lemma 2.12 of [11], D is characterized as an additive operator with the following property

$$D(fs) = fDs + \varphi^*(df) \otimes \varkappa(s) \qquad (f \in F(V), \quad s \in \underline{E}). \tag{2.25}$$

Suppose that E and F are foliate vector bundles. Then, according to proposition 2.4, we shall say that D is an *adapted* $(\varphi, \varkappa)$-connection if D sends $\underline{E}$ to $\underline{L(Tr, F)}$ and that D is a *foliate* $(\varphi, \varkappa)$-connection if D sends $\underline{\underline{E}}$ to $\underline{\underline{L(Tr, F)}}$. Taking in (2.25) a foliate section s and a foliate function f we see that in the case of an adapted connection φ sends necessarily $T(\mathfrak{F})$ to $T(\mathfrak{F})$, hence it induces a homomorphism $\varphi' : Tr(V) \to Tr(V)$. From the same formula (2.25), it follows that in the case of a foliate $(\varphi, \varkappa)$-connection $\varkappa$ and φ' are necessarily foliate homomorphisms. In this last case, the induced operator

$$D' : \underline{\underline{E}} \to \underline{\underline{L(Tr, F)}} \tag{2.26}$$

satisfies the property

$$D'(fs) = f D's + \varphi'^*(df) \otimes \varkappa(s) \qquad (s \in \underline{\underline{E}},\ f \in \Phi(V)_c), \tag{2.27}$$

whence, according to proposition (2.3), it is the foliate part of some first order foliate operators of foliate symbol $\sigma' = \varphi'^* \otimes \varkappa$, i.e.

$$\sigma'(a)(t) = \varkappa(a(\varphi'(t))) \qquad (a \in L(Tr, E),\ t \in Tr). \tag{2.28}$$

If φ induces φ' as above, we can always obtain an adapted $(\varphi, \varkappa)$-connection for (E, F). In fact, we can construct it locally and then use, in the usual way, a partition of unity on V.

Proposition 2.5. *A $(\varphi, \varkappa)$-connection D is adapted if and only if*

$$Ds(X) = 0 \tag{2.29}$$

for all $s \in \underline{\underline{E}}$ and all the structural vector fields X. D is foliate if and only if besides (2.29) we have

$$XDs(Y) = 0 \tag{2.30}$$

for all $s \in \underline{\underline{E}}$, all the structural vector fields X and all the foliate vector fields Y on V.

The proof follows immediately from the definitions.

If (2.23) is a $(\varphi, \varkappa)$-connection and φ, $\varkappa$ are isomorphisms, the *curvature* of D is the $L(E, F)$-valued two-form Ω given by [15, 16]

$$\begin{aligned} \Omega(X, Y)(s) = {} & D(\varkappa^{-1} Ds(Y))(X) - D(\varkappa^{-1} Ds(X))(Y) - \\ & - Ds(\varphi^{-1}[\varphi X, \varphi Y]), \end{aligned} \tag{2.31}$$

where X, Y are vector fields on V and $s \in \underline{E}$.

In the case of a foliate (φ, α)-connection and for foliate s, X, Y we see by proposition 2.5 that the first two terms of $\Omega(X, Y)(s)$ depend only on $\pi X, \pi Y$. If, moreover, φ' is a foliate isomorphism, which happens if and only if φ sends the vectors on $\mathfrak{F}$ and only them to vectors on $\mathfrak{F}$ and φ' is foliate, the last term of $\Omega(X, Y)(s)$ also depends only on $\pi X, \pi Y$, in view of proposition 1.1. Then it follows that we can write

$$\begin{aligned}(2.32)\qquad \Omega(X, Y)(s) &= D'(\alpha^{-1} D' s(\pi Y))(\pi X) - D'(\alpha^{-1} D'(\pi X))(\pi Y) - \\ &- D's(\pi\varphi^{-1}[\varphi X, \varphi Y]),\end{aligned}$$

for foliate s, X, Y. This means that we have

$$(2.33)\qquad \Omega = \pi^* \bar{\Omega}$$

for some section $\bar{\Omega}$ of $L^2_{\mathrm{alt}}(Tr, L(E, F))$.

Proposition 2.6. *For an adapted (φ, α)-connection, $\Omega(X, Y) = 0$ if X and Y are structural fields. For a foliate (φ, α)-connection this is true even if only one argument is structural.*

Of course, we suppose that φ and α satisfy all the mentioned necessary conditions. The first assertion is an immediate consequence of proposition 2.5 and formula (2.31). For the second assertion, we remark that Ω being a section of $L^2_{\mathrm{alt}}(T, L(E, F))$, $\Omega(X, Y)(s)$ depends only on the punctual values of X, Y, s. This means that, if we want to calculate $\Omega(X, Y)(s)$ at $x \in V$, we have to consider $X(x)$, $Y(x)$, $s(x)$, next to take local fields X, Y, s around x which have the given values at x and finally to apply (2.31). Now, if X, for instance, is structural we can obviously choose, locally, X-structural and Y, s-foliate. Then, using propositions 2.5 and 1.1 and the formula (2.31) we get the desired result.

We recall that proposition 2.6 is known for the usual connections and that it implies the important consequences on the characteristic classes of an f.v.b. given by B o t t [2], M o l i n o [9, 10] and K a m b e r - T o n d e u r [6 a].

3. Existence problems

3.1. An important problem of the theory of differential operators is the existence of operators of a given symbol and satisfying some other supplementary condition. For instance, in [1] this condition is analyticity, in [6] it is invariance and here this condition is that the operator be foliate. The method to answer such problems [1,6] consists in writing down some exact cohomology sequence, which allows to define a cohomological obstruction to the existence of the operators looked for and this method will be adapted here to the case which we are interested in.

If no supplementary condition, besides differentiability, is asked, the mentioned obstruction vanishes and the operators looked for always exist [6]. Moreover, we can see that this is also a consequence of the existence of k^{th}

order connections on any differentiable vector bundle E. In fact, if $\sigma : L^k_{\mathrm{sym}}(T, E) \to F$ is a given „abstract" symbol, the formula (2.7), offers, as one easily gets, the set of all the operators D of symbol σ.

Now, the problem which we want to discuss in this section is : let be E, F two f.v.b. and

$$(3.1)\qquad \sigma : L^k_{\mathrm{sym}}(T, E) \to F$$

a k^{th} order abstract symbol, under what conditions do there exist foliate differential operators $D : \underline{E} \to \underline{F}$ such that $\sigma(D) = \sigma$ and which are these operators ?

We begin by establishing the following preliminary result :

Lemma 3.1. *The notation being as above, there exist foliate operators D of symbol σ if and only if there is a foliate homomorphism $\Phi' : J^k_f E \to F$ such that $\Phi' \varepsilon_f = \sigma\pi^*$.*

Suppose that D exists. This means that there exists $\Phi(D)$ of (2.3) such that (2.4) is satisfied. Let us consider the following diagram, which is partly contained in $(\Delta^k E)$

$$(3.2)\qquad \begin{tikzcd}
0 \arrow[r] & L^k_{sym}(Tr,E) \arrow[rr, shift left, "\pi^*"] \arrow[dd, "\varepsilon_f"'] \arrow[dr, "\sigma'"] & & L^k_{sym}(T,E) \arrow[ll, dashed, shift left, "\zeta"] \arrow[r, shift left, "\chi"] \arrow[dl, "\sigma"'] \arrow[dd, "\varepsilon"] & L^k_{sym}(T,E)/L^k_{sym}(Tr,E) \arrow[l, dashed, shift left, "\psi"] \arrow[r] \arrow[dll, "\sigma''"'] \arrow[dd, "\bar{\varepsilon}"] & 0 \\
& & F & & & \\
0 \arrow[r] & J^k_f E \arrow[rr, shift left, "\iota^k"] \arrow[ur, "\Phi'"] & & J^k E \arrow[ll, dashed, shift left, "a^k"] \arrow[r, shift left, "p^k"] \arrow[ul, "\Phi(D)"'] & Q^k E \arrow[l, dashed, shift left, "q^k"] \arrow[r] \arrow[ull, "\Phi''"] & 0
\end{tikzcd}$$

Here, the squares of full arrows are commutative, the rows are exact and split, which means that we have (1.15) and

$$(3.3)\qquad \zeta\pi^* = \mathrm{id.},\ \chi\psi = \mathrm{id.},\ \pi^*\zeta + \psi\chi = \mathrm{id.},\ \chi\pi^* = 0,\ \zeta\psi = 0.$$

Also, we suppose that the following commutativity relations hold

$$(3.4)\qquad \Phi(D)\varepsilon = \sigma,\ \sigma\pi^* = \sigma',\ \sigma\psi = \sigma'',\ \Phi(D)\, i^k = \Phi',\qquad \Phi(D)\, q^k = \Phi''.$$

Now, from (1.15), (3.3) and (3.4) we get

$$(3.5)\qquad \sigma = \sigma'\zeta + \sigma''\chi,\ \Phi(D) = \Phi' a^k + \Phi'' p^k$$

and the first relation (3.4) becomes

(3.6) $$\sigma'\zeta + \sigma''\chi = \Phi' a^k \varepsilon + \Phi'' p^k \varepsilon.$$

Whence, by right multiplication with π^* and ψ

(3.7) $$\sigma' = \Phi' \varepsilon_f, \quad \sigma'' = \Phi' a^k \varepsilon\psi + \Phi'' \bar{\varepsilon}.$$

Since D is foliate, Φ' is foliate by proposition 2.1 and the first relation (3.7) shows that the announced condition is necessary.

Now, suppose the condition is satisfied, hence we can find Φ' satisfying the first relation (3.7). Then, we can find a differentiable homomorphism Φ'' such that the second relation (3.7) holds too. In fact, we know from the diagram $(\Delta^k E)$ that $\bar{\varepsilon}$ is a monomorphism, hence it has a differentiable left inverse ε' and if we take

(3.8) $$\Phi'' = (\sigma'' - \Phi' a^k \varepsilon\psi)\, \varepsilon',$$

the second relation (3.7) will be verified.

But in this case

(3.9) $$D = (\Phi' a^k + \Phi'' p^k)\, j^k$$

is a k^{th} order foliate differential operator of symbol σ and the lemma is proved.

Let us remark from this lemma that the problem of finding a foliate D of symbol σ is meaningful only if the homomorphism $\sigma\pi^*$ is foliate and in the sequel this condition is supposed to be always satisfied. Moreover, we always denote $\sigma' = \sigma\pi^*$ and call it a *foliate abstract symbol*.

It is simple to see that if we know all the Φ' satisfying the first relation (3.7), we get all the corresponding Φ'' by adding to (3.8) a term of the form $\lambda\bar{\rho}^{k-1}$, where $\bar{\rho}^{k-1}$ is defined by $(\Delta^k E)$. Then, by the second relation (3.5), we obtain all the foliate operators of symbol σ.

Hence, since Φ' of (3.2) is just the foliate part of $\Phi(D)$ of the foliate operators looked for our problem reduces to the following: given a foliate abstract symbol σ', does there exist a foliate part D' of a k^{th} order foliate operator of foliate symbol σ' and, in the affirmative case, which are all these operators D'?

3.2. Now, we shall study this *reduced existence problem*. As we already indicated, its solution depends on some cohomological obstruction which we shall define following the general schema of [1,6]. As a matter of fact, this problem is contained in the general theory of [6].

For the category of $\Phi(V)$-modules, $\mathrm{Hom}(\underline{\underline{A}},.)$ is an exact functor, with values in the same category, whenever $\underline{\underline{A}}$ is locally free and of finite rank. This implies that the sheaves $\mathrm{Ext}^p(\underline{\underline{A}},.)$ vanish for $p \geqslant 1$, whence, using a known exact sequence for Ext, we see that the exact sequence $(S^k E)$ of proposition 1.3 gives rise to the following exact sequence of $\Phi(V)$-modules

(3.10) $$0 \to \underline{L}(J_f^{k-1}E, F) \to \underline{L}(J_f^k E, F) \xrightarrow{\varepsilon'_f} \underline{L}(\underline{L}^k_{\mathrm{sym}}(Tr, E), F) \to 0$$

for every f.v.b. E and F.

Hence, we can write down the following exact cohomology sequence

(3.11) $$\ldots \to H^0(V, \underline{L}(J_f^k E, F)) \xrightarrow{\varepsilon^*_f} H^0(V, \underline{L}(\underline{L}^k_{\mathrm{sym}}(Tr, E), F))$$

$$\xrightarrow{\delta} H^1(V, \underline{L}(J_f^{k-1}E, F)) \to \ldots$$

Now, let $\sigma' = L^k_{\mathrm{sym}}(Tr, E) \to F$ be a foliate abstract symbol. Obviously, σ' can be considered as an element of $H^0(V, \underline{L}(\underline{L}^k_{\mathrm{sym}}(Tr, E), F))$ and it follows that we can associate to σ' an element

(3.12) $$\zeta(\sigma') = \delta(\sigma') \in H^1(V, \underline{L}(J_f^{k-1}E, F)).$$

By definition, $\zeta(\sigma')$ is the announced *obstruction* to the existence of operators D' as needed in the reduced existence problem.

Proposition 3.2. *The reduced existence problem has a solution for a given* σ' *if and only if* $\zeta(\sigma') = 0$. *In this case, if* $D'_0 : \underline{\underline{E}} \to \underline{\underline{F}}$ *is a solution, any other solution* D' *is of the form*

(3.13) $$D' = D'_0 + \alpha j_f^{k-1}$$

where $\alpha : J_f^{k-1}(E) \to F$.

In fact, from the exact sequence (3.11), we see that $\zeta(\sigma') = 0$ if and only if $\sigma' = \varepsilon_f^*(\Phi')$ for $\Phi' \in H^0(V, \underline{L}(J^k E, F))$ which means just $\sigma' = \Phi' \varepsilon_f$ for a foliate homomorphism $\Phi' : J_f^k E \to F$ and this is the desired existence result.

If one also has $\sigma' = \tilde{\Phi}' \varepsilon_f$ it follows $(\tilde{\Phi}' - \Phi')\,\varepsilon_f = 0$, whence $\tilde{\Phi}' = \Phi' + \alpha\rho_f^{k-1}$ which, in turn, implies the formula (3.13).

3.3. In this section we shall calculate the obstruction $\zeta(\sigma')$ for an arbitrary foliate abstract symbol $\sigma' : L^k_{\mathrm{sym}}(Tr, E) \to F$. A similar calculation for analytic connections is given in [1].

First, we give a straightforward representation of $\zeta(\sigma')$ using the classical definition of the homomorphism δ in the Čech cohomology of V with the corresponding sheaves of coefficients.

The given symbol σ' can be considered as represented by the 0-cocycle of an open covering $\mathfrak{U} = \{U_0\}$ of V with coefficients in $\underline{L}(\underline{L}^k_{\mathrm{sym}}(Tr, E), F)$, consisting of the system of sections $\{\sigma'/_{U_0}\}$. Moreover, we can obviously suppose that $\mathfrak{U}$ is such that over each U_0 one can construct a foliate k^{th}-order connection on $E/_{U_0}$ having the retraction $\bar{\omega}(U_0)$ and the induced splitting $\mu(U_0)$. (This follows from local coordinate expressions. For the connection, we use the notation of the diagram $(C^k E)$ together with the argument U_0 which indicates the corresponding neighbourhood).

Then, the system $\{\sigma' \mu(U_0)\}$ is a 0-cochain of $\mathfrak{U}$ with coefficients in $\underline{\underline{L(J_f^k E, F)}}$, which is send by the map ε_f' of (3.10) to the cocycle $\{\sigma'/_{U_0}\}$.

It follows that the obstruction $\zeta(\sigma')$ is represented by the 1-cocycle of $\mathfrak{U}$ with coefficients in $\underline{\underline{L(J_f^{k-1}E, F)}}$ defined by the system

$$\{[\sigma' \mu(U_{0_1}) - \sigma' \mu(U_{0_0})](\varepsilon_f^{k-1})^{-1}/_{U_{0_0} \cap U_{0_1}}\} \tag{3.14}$$

which is easily seen to be meaningful.

Now, we shall be able to represent $\zeta(\sigma')$ by a differential form on V, using our previous de Rham type theorem for the cohomology spaces $H^i(V, S)$, where S is an arbitrary f.v.b. on V [13, 14].

Let us begin by recalling the mentioned theorem. Consider an arbitrary Riemann metric on V. Then a decomposition $T(V) = T(\mathfrak{F}) \oplus T'(\mathfrak{F})$ is defined, where $T'(\mathfrak{F})$ is orthogonal to $T(\mathfrak{F})$ and, corresponding to this decomposition, there is a classical canonical decomposition of the S-valued differential forms on V as sums of forms of type (p, q) [13, 14]. Consequently, the (locally defined) exterior differential splits into a sum $d = d' + d'' + \partial$, where d' has the type $(1, 0)$, $d'' - (0, 1)$ and $\partial - (2, -1)$. One proves that d'' is globally defined for S-valued forms, $d''^2 = 0$ and d'' satisfies a Poincaré type lemma, whence, if we denote by $A^{pq}(S)$ the space of S-forms of type (p, q), by $Z^{pq}(S)$ the subspace of the d''-closed forms and by $\mathcal{A}^{pq}(S)$ the corresponding sheaf of germs,

$$0 \to S \to \mathcal{A}^{00}(S) \xrightarrow{d_0''} \mathcal{A}^{01}(S) \xrightarrow{d_1''} \ldots \tag{3.15}$$

is a fine resolution of S and

$$H^i(V, S) \approx Z^{0i}(S)/d''A^{0,\,i-1}(S). \tag{3.16}$$

It follows that the obstruction $\zeta(\sigma')$ can be represented by a d''-closed form of type $(0, 1)$ with values in $\underline{\underline{L}}(J^{k-1}E, F)$ and we want to calculate such a form.

To do this, we must put under an explicit form the isomorphism (3.16) for $i = 1$. From (3.15) we have the exact sequence of sheaves

$$0 \to S \to \mathcal{A}^{00}(S) \xrightarrow{d_0''} \ker d_1'' \to 0$$

(of course, d_i'' is d'' applied to the forms of the type $(0, i)$), which implies the exact cohomology sequence

$$\ldots \to H^0(V, \mathcal{A}^{00}(S)) \to H^0(V, \ker d_1'') \xrightarrow{\delta'} H^1(V, S) \to 0.$$

Now, let ω be a d''-closed S-valued form of type $(0, 1)$ on V. Using again a covering $\mathfrak{U}$, $\{\omega/_{U_0}\}$ is a 0-cocycle for $\ker d_1''$ which by the d''-Poincaré lemma can be put under the form $\{d''\varphi(U_0)\}$ for some 0-cochain of $\mathcal{A}^{00}(S)$ (hence $\varphi(U_0)$ is a cross section in $S/_{U_0}$ and $d''\varphi(U_0) = \omega/_{U_0}$). Then, by the definition of δ', we see that $\delta'\omega$ is represented by the 1-cocycle of $\mathcal{A}^{00}(S)$ given by $\{[\varphi(U_{0_1}) - \varphi(U_{0_0})]/_{U_{0_0} \cap U_{0_1}}\}$. The correspondence which sends $\delta'\omega$ to the d''-cohomology class of ω is just the isomorphism (3.16) for $i = 1$.

Now, we comme back to the obstruction $\zeta(\sigma')$, which is represented by the 1-cocycle (3.14). Let $\mu : J_f^k E \to L_{sym}^k(Tr, E)$ be the retraction of the induced splitting of an arbitrary adapted k^{th} order connection on E (which, as we already know, always exists). Then, over every U_0, we have

$$(\mu(U_0) - \mu/_{U_0})\, \varepsilon_f = 0,$$

where $\mu(U_0)$ are given by the local foliate connections used in (3.14). This relation assures that

$$\varphi(U_0) = \sigma'(\mu(U_0) - \mu/_{U_0})(\varepsilon_f^{k-1})^{-1} \tag{3.17}$$

is a well defined cross-section of $L(J_f^{k-1}E, F)/_{U_0}$ (which, generally, is not foliate).

Comparing with (3.14), we see that the 1-cocycle which defines $\zeta(\sigma')$ can be put under the form $[\varphi(U_{0_1}) - \varphi(U_{0_0})]/_{U_{0_0} \cap U_{0_1}}$, whence, in view of the previous explicitation of (3.16), we derive that the corresponding d''-closed form is

$$\omega = d''\varphi(U_0) \tag{3.18}$$

with $\varphi(U_0)$ given by (3.17). ω does not depend on the local foliate connections $\mu(U_0)$ since, if we replace them by $\mu'(U_0)$ and if we denote by $\varphi'(U_0)$ the corresponding cross-sections, $\varphi'(U_0) - \varphi(U_0)$ are foliate and $d''\varphi' = d''\varphi$. Hence, we have

Theorem 3.3. *If $\sigma' : L_{sym}^k(Tr, E) \to F$ is a foliate abstract symbol, the corresponding obstruction $\zeta(\sigma')$ is the d''-cohomology class of the form ω given by (3.18) and (3.17), which is a d''-closed $L(J^{k-1}E, F)$-valued form of type $(0, 1)$ on V.*

In the case $k = 1$, we can calculate ω more explicitely.

Remark that, generally, $\varphi(U_0)$ defines over U_0 the $(k-1)^{th}$ order operator $\bar{\varphi}(U_0) = \varphi(U_0)\, j^{k-1} : \underline{\underline{E}} \to \underline{\underline{F}}$, which is easily seen to be equal to

$$\bar{\varphi}(U_0) = \sigma'(\nabla'(U_0) - \nabla'/_{U_0}), \tag{3.19}$$

where ∇' are the operators (2.21) associated to the connections entering in the formula (3.17).

If $k = 1$, $\bar{\varphi}(U_0) = \varphi(U_0)$ and we can apply d'' to $\bar{\varphi}$, which gives the following expression of the form representing the obstruction

$$\omega = -\sigma' d'' \nabla'. \tag{3.20}$$

Here, we used the fact that $\nabla'(U_0)$ and σ' are foliate and $d'' \nabla'$ is regarded as an $L(E, L(Tr, E))$-valued 1-form in the following way

$$[d'' \nabla'(t)]_x (e)(v) = [d''(\nabla' \bar{e}(v))]_x (t), \tag{3.21}$$

where $t \in T(\mathcal{F})_x$, $v \in Tr(\mathcal{F})_x$, $e \in E_x$, $\bar{e} \in \underline{E}_x$ and $\bar{e}(x) = e$.

The form $d'' \nabla'$ has an interpretation related to the curvature of the used adapted connection. In fact, $Tr(\mathcal{F})$ can be identified with the bundle $T''(\mathcal{F})$ which is normal to the foliation, hence in (3.21) we can consider $t \in T(\mathcal{F})$ and $v \in T''(\mathcal{F})$ which shows that $d'' \nabla'$ can be considered as an $L(E, E)$-valued 2-form of the type (1,1). If Ω is the curvature form of the adapted connection which defines ∇', given by (2.31), then usual (but long enough) calculations in local coordinates show that

$$d'' \nabla' (X, Y) = \Omega(X, Y) \tag{3.22}$$

for any two vector fields X, Y which are respectively of the type (1,0) and (0,1). Formula (3.22) is just the announced interpretation.

3.4. The expression given for $\zeta(\sigma')$ in the previous section shows that the answer to the existence problem is related to the existence of a foliate connection on E. In fact, if the f.v.b. E has a k^{th} order foliate connection, we can take in (3.14) the retractions $\mu(U_0) = \mu/_{U_0}$ where μ is defined by that foliate connection, which gives $\zeta(\sigma') = 0$ for any σ'. Hence, *if E has a k^{th} order foliate connection there are k^{th} order foliate differential operators $\underline{E} \to \underline{F}$ of any given symbol σ such that $\sigma\pi^*$ is foliate.*

All these operators can be obtained, as it is indicated by proposition 3.2 and in section 3.1, if we obtain one foliate part D_0' of such an operator. As one easily sees, we can take for D_0' the operator given by the formula (2.22) with σ' replaced by $\sigma\pi^*$.

As for the problem of the existence of a k^{th} order foliate connection on an f.v.b. E, we can see that it enters into the sheme of the reduced existence problem of sections 3.2, 3.3. In fact, we know from proposition 1.5 that the existence of such a connection is equivalent to the existence of the induced foliate splitting, which in turn is equivalent to the existence of the corresponding operator ∇' of (2.21) which is an operator of foliate symbol $\sigma' = \text{id}$. *Hence, a foliate k^{th} order connection on E exists if and only if*

$$\zeta(\text{id.}) = 0. \tag{3.23}$$

$\zeta(\text{id.})$ is the obstruction to the existence of a k^{th} order foliate connection on E and, in the notation of the formula (3.17) it is represented by the differential form

$$\omega = d''[(\mu(U_0) - \mu/_{U_0})(\rho_f^{k-1})^{-1}]. \tag{3.24}$$

For $k = 1$, we get as at the end of section 3.3

$$\omega = -d'' \nabla' \tag{3.25}$$

and the formula (3.22) shows that the obstruction given here differs only in sign from the obstruction obtained in a straightforward manner by Molino [9].

Moreover, we remark that, when looking for the obstruction $\zeta(\varphi'^* \otimes \varkappa)$ to the existence of a $(\varphi, \varkappa)$-connection on E (see the notation of section 2.4), one gets a form ω which, just like $d'' \nabla'$ can also be interpreted as the (E, F)-valued 2-form of type (1,1) defined by $\omega(X, Y) = -\Omega(X, Y)$, where X has the type (1,0) and Y – (0,1), Ω being the curvature of an arbitrary adapted $(\varphi, \varkappa)$-connection on E.

Following Molino (up to a difference in sign), we call the d''-cohomology class of the form $-d'' \nabla'$ the Atiyah class of E and denote it by $a(E)$. Analogously, the class of the form (3.24) will be the *k^{th} Atiyah class $a^k(E)$*. Hence

Proposition 3.4. *The f.v.b. E admits a k^{th} order foliate connection if and only if its k^{th} Atiyah class vanishes.*

The foliate manifolds V such that $a(Tr(\mathcal{F})) = 0$ were studied by Molino [9, 10], who called them, *foliate manifolds with projectable transverse connection*. Such are, for instance, the foliate manifolds with bundle like metric (Reinhart's spaces [13]).

On the other hand, one can find examples of foliations which do not admit projectable transverse connections. In fact, let us consider Roussarie's foliation as described by Bott [2a]. In Bott's notation, one sees that the Atiyah class of the transverse bundle of this foliation is represented by $d\eta$ and this is a d''-exact form if and only if there is on the respective manifold $M = G \setminus SL(2, R)$ a globally defined function λ such that $d\eta = d(\lambda\theta)$. But this implies

$$\frac{1}{2} X^* \wedge Y^* \wedge Z^* = d(\lambda d\theta)$$

which is impossible in view of the compactness of M, as it is explained in [2a].

Using a known construction of a k^{th} order connection [11] we shall prove the following interesting property of the foliate manifolds with projectable transverse connection.

Theorem 3.5. *Let V be a foliate manifold with projectable transverse connection and E a foliate vector bundle on V with vanishing first Atiyah class. Then, for every f.v.b. F, every integer $k \geqslant 1$ and every abstract symbol $\sigma : L^k_{\text{sym}}(T, E) \to F$ such that $\sigma\pi^*$ is foliate, there are k^{th} order foliate differential operators $D : \underline{E} \to \underline{F}$ of symbol σ.*

The hypotheses imply that the f.v.b. $Tr(\mathcal{F})$ and E admit first order foliate connections. The theorem will be proved if we shall be able to construct from them, a foliate k^{th} order connection on E.

Let ∇_{Tr} be the covariant differential of a foliate connection on $Tr(\mathcal{F})$. Then, it is classical [7] that the formula

$$\nabla\alpha(X)(t) = X\alpha(t) - \alpha((\nabla_{Tr}\, t)(X)),$$

$\alpha \in \underline{Tr^*_x}$, $t \in \underline{Tr_x}$, $X \in T(V)_x$ defines a connection on Tr^*, which is easily seen to be foliate (as a consequence of proposition 2.5.

Now, chosing a decomposition $T^*(V) = Tr^*(\mathcal{F}) \oplus S$ and an arbitrary connection on S, we can extend the connection on $Tr^*(\mathcal{F})$ to a connection ∇_{T^*} on T^* and, using it, we obtain the connection $\nabla_E^{(k)}$ on $\otimes^k T^*(V) \otimes E$ given by the corollary of theorem 9.4 of [11]. From the expression of $\nabla_E^{(k)}$ given in [11], it is easy to see that $\nabla_E^{(k)}$ induces a foliate connection on the subbundle $\otimes^k Tr^*(\mathcal{F}) \otimes E$.

Finally, using the theorem 7 of [11], we get on E the k^{th} order connection with covariant differential

$$D_k = S^k \nabla_E^{(k-1)} \dots \nabla_E^{(1)} \nabla_E,$$

where S^k denotes symmetrization, and this connection is foliate in view of proposition 2.4.

This ends the proof of the announced theorem.

We also note that, if we start from adapted connections on Tr and E and use the same construction like in the previous proof, we get an adapted k^{th} order connection on E, which could be used in the expression of the obstruction of theorem 3.3. It would be interesting to look for the relations between the form ω of that obstruction and the curvatures of the used connections on Tr and E.

We end by the remark that the problems treated in this paper obviously admit a generalization in the following directions. Instead of the jet bundles of section 1.4. one can consider the bundles of nonholonomic and semiholonomic jets as defined in [7]. For the semiholonomic jet bundles, a diagram like $(\Delta^k E)$ can be constructed and the considerations of section 1.5 will apply to adapted and foliate semiholonomic connections. Then, instead to study differential operators one will consider the semiholonomic differential forms of [7] and one shall generalize for them the considerations of sections 2 and 3. The obstruction to the existence of a foliate semiholonomic differential form will be given by a formula like (3.17), where the γ will be semiholonomic connections. In this manner, one will get results regarding the existence of foliate semiholonomic differential forms of a given symbol, which are similar to those given previously for differential operators. Further generalizations to more general maps $E \to V$ then the vector bundles are also possible.

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OBSERVAŢII ASUPRA OPERATORILOR DIFERENŢIALI PE VARIETĂŢI FOLIATE

Rezumat

În lucrare, se fac o serie de consideraţii privind studiul operatorilor diferenţiali liniari între fibrări vectoriale foliate. În primul paragraf „Fibrări vectoriale foliate", se descrie categoria acestui tip de fibrări, se consi-

deră fibrările de jeturi asociate unui fibrat vectorial foliat, obținîndu-se diagramele importante ($\Delta^k E$) și se introduc conexiunile adaptate și foliate de un ordin oarecare pe un fibrat vectorial foliat. Cel de-al doilea paragraf „Operatori diferențiali pe fibrări vectoriale foliate" se ocupă de unele proprietăți generale ale operatorilor diferențiali liniari „foliați" între două fibrări vectoriale foliate cu aceeași bază, adică operatori a căror parte transversă" depinde numai de foi. Tot aici sînt date exemple de asemenea operatori și se cercetează legătura între operatori oarecare și conexiuni. În fine, în ultimul paragraf „Probleme de existență" se calculează obstrucția coomologică la existența unui operator diferențial foliat de simbol dat și se demonstrează existența unor asemenea operatori atunci cînd există conexiuni foliate corespunzătoare. Menționăm că tot aici se indică un exemplu de foliație (construită de R. Roussarie), care nu admite o conexiune transversală foliată. Lucrarea are la bază teoriile expuse în [1], [6], [9] și [13].

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CONFORMAL STRUCTURES ON BANACH VECTOR BUNDLES

BY

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In this paper, conformal structures, in particular Weyl structures on Banach vector bundles are defined. We prove that there is a one-to-one correspondence between the set of conformal structures on a Banach vector bundle and the set of reductions of its structural group to the conformal group.

The existence and the uniqueness of a connection without torsion, compatible with a Weyl structure on a Banach manifold are proved.

1. Linear conformal space. Conformal group. Let $\mathbf{E}$ be a real, infinit-dimensional linear space.

Definition 1.1. *A conformal structure on* $\mathbf{E}$ *is a set* $C(\mathbf{E})$ *of scalar products on* $\mathbf{E}$, *denoted by* $(,)_a$, $a \in \mathcal{J}$ *which satisfy*

$$(,)_a = \lambda_{ab} (,)_b \qquad a, b \in \mathcal{J}, \tag{1.1}$$

where λ_{ab} *is a positive real number, and* $\mathcal{J}$ *a set of indices.*

Definition 1.2. *The linear space* $\mathbf{E}$ *with conformal structure* $C(\mathbf{E})$ *is called a conformal space.*

Remark 1.1. In the conformal space $\mathbf{E}$ the angle between two vectors can be defined by

$$\cos (u, v) = \frac{(u, v)_a}{\| u \|_a \| v \|_a} \qquad \forall u, v \in \mathbf{E}, \ a \in \mathcal{J}, \tag{1.2}$$

where $\| u \|_a = \sqrt{(u, u)_a}$, the ratio of their lengths is well defined but their absolute lengths are not defined.

If $(,)$ is a fixed element of $C(\mathbf{E})$, (1.1) can be replaced by

$$(,)_a = \lambda_a (,)_a \qquad \forall a \in \mathcal{J}. \tag{1.3}$$

This implies that all norms $u \to \| u \|_a$ are equivalent to the fixed norm $u \to \| u \| = \sqrt{(u, u)}$. In the following, the space $\mathbf{E}$ with the norm $\| . \|$ will be supposed to be complete.