This equality can be easily verified using (3.8).

Uniqueness. Let $\Gamma'_{\varphi(p)}$ be another local connector which verifies i) and ii). It follows that $\Gamma'_{\varphi(p)}$ must satisfy equation (3.8) i.e. $\Gamma'_{\varphi(p)} = \Gamma_{\varphi(p)}$.

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STRUCTURI CONFORME PE FIBRATE VECTORIALE BANACH

Rezumat

În această Notă se definește structura conformă pe fibrate vectoriale Banach și în particular, structura Weyl.

Se arată că există o corespondență bijectivă între mulțimea structurilor conforme pe un fibrat vectorial Banach și mulțimea reducerilor grupului structural al acestui fibrat vectorial Banach la grupul conform. Se demonstrează că există și este unică, o conexiune liniară fără torsiune, compatibilă cu o structură Weyl pe o varietate diferențiabilă Banach.

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PERIODIC SOLUTIONS OF FIRST ORDER HYPERBOLIC SYSTEMS CONTAINING A SMALL PARAMETER

BY

DAN PETROVANU

§0. This paper handles the problem of periodic solutions of the hyperbolic systems

$$u_x = \varepsilon F(x, y, u, v, \varepsilon), \quad v_y = \varepsilon G(x, y, u, v, \varepsilon), \tag{0.1}$$

where ε is a (small) parameter, u and v are the unknown vector functions (in general of different dimensions). By this same it is a continuation to author's paper [15]. It was already clear in [15] that there is a close analogy between the investigation of periodic solutions of Tricomi's problem (characteristic problem) for the system $u_x = F(x, y, u, v)$, $v_y = G(x, y, u, v)$ and the investigation of periodic solutions of the characteristic problem for the equation $u_{xy} = f(x, y, u, u_x, u_y)$. (See Cesari's papers [2], [3], [4], [5], see also Hale's paper [7]).

As in [15], we use the method of Cesari and Hale. The theoretic ground of this method should be found in [1] and also in [8], [9]. For further details and for different references, we send to [6], [9], [10]. The papers [6], [10] give an excellent survey on the use of this method in all the fields.

The reader should compare the results of this paper with the ones of [2], [3], [4], [5], [6], [7], [10], [11], [12], [13], [19], [22], [23], [26]. The papers [11], [12], [13], [14], [16], [17], [18], [19], [20], [21], [22], [23], [24], [25], [26] give a partial, but good indication of the results obtained in the field of periodic solutions for hyperbolic systems.

In handling the problem of periodic solutions for system (0.1) we could follow closer the method of Cesari-Hale as described in [2], [3], [4], that is, in our case to proceed as in [4] (as we did in [15], where we tried to keep close to the general lines of [2], [3]). For the sake of variety we choosed to keep closer to the general line of [7]. Furthermore, it was in the intention of the author to follow this line as close as possible, in order to make easy the comparison between the results of the present paper and the ones of [7] (and by this same the comparison with the results of [4] or those of [23] also).

§1. The linear problem. We consider in the following column-vector functions of the type col $[\varphi(x, y), \psi(x, y)]$, where $\varphi = (\varphi_1, \ldots, \varphi_n)$ is an n-dimensional and $\psi = (\psi_1, \ldots, \psi_m)$ an m-dimensional vector function. Denote, as usual by C_k the Banach space of the vector functions $\operatorname{col}(\varphi, \psi)$ having continuous derivatives of order k and for which the norm

$$\|\operatorname{col}(\varphi, \psi)\|_k = \sum_{j=0}^{k} \sum_{\mu+\nu=j} \left\{ \sup_{R^2} \left| \frac{\partial^{\mu+\nu} \varphi(x, y)}{\partial x^\mu \partial y^\nu} \right| + \sup_{R^2} \left| \frac{\partial^{\mu+\nu} \psi(x, y)}{\partial x^\mu \partial y^\nu} \right| \right\} < \infty .$$

Denote by C^*_{k+1} the Banach space

$$C^*_{k+1} = \{\operatorname{col}(\varphi, \psi) : \varphi_x(x, y) \in C_k,\ \psi_y(x, y) \in C_k,\ -\infty < x, y < \infty\}$$

of the vector functions $\operatorname{col}(\varphi, \psi)$ for which the norm

$$\|\operatorname{col}(\varphi, \psi)\|^*_{k+1} = \|\varphi\|_k + \|\psi\|_k + \sum_{\mu+\nu=k} \sup_{R^2} \left| \frac{\partial^{\mu+\nu+1} \varphi(x, y)}{\partial x^{\mu+1} \partial y^\nu} \right| +$$

$$+ \sum_{\mu+\nu=k} \sup_{R^2} \left| \frac{\partial^{\mu+\nu+1} \psi(x, y)}{\partial x^{\mu} \partial y^{\nu+1}} \right| < \infty .$$

(Obviously, for $k = 0$, $\operatorname{col}(\varphi, \psi) \in C^*_1$, we have

$$\|\operatorname{col}(\varphi, \psi)\|^*_1 = \sup|\varphi| + \sup|\varphi_x| + \sup|\psi| + \sup|\psi_y|).$$

Consider now the classes of functions

$$S = \{\operatorname{col}(\varphi, \psi) : \varphi(x + 2\pi, y) = \varphi(x, y) = \varphi(x, y + 2\pi),$$
$$\psi(x + 2\pi, y) = \psi(x, y) = \psi(x, y + 2\pi)\}.$$

$$N = \{\operatorname{col}(\varphi, \psi) : \varphi = \beta(y),\ \psi = \alpha(x),\ \beta(y + 2\pi) = \beta(y),\ \alpha(x + 2\pi) = \alpha(x)\},$$

Remark 1. Obviously, $N \subset S$. Moreover, $\operatorname{col}(\varphi, \psi) \in N \cap C_0$ implies $\operatorname{col}(\varphi, \psi) \in N \cap C^*_1$ and

$$\|\operatorname{col}(\varphi, \psi)\|^*_1 = \|\operatorname{col}(\varphi, \psi)\|_0$$

for any $\operatorname{col}(\varphi, \psi) \in N \cap C_0$.

Definition: *For any* $\operatorname{col}(\varphi, \psi) \in S \cap C_k$ *define the element* $P(\operatorname{col}(\varphi, \psi)) \in N \cap C^*_{k+1}$ *by*

$$[P \operatorname{col}(\varphi, \psi)](x, y) \overset{\text{def}}{=} \operatorname{col}\left[\frac{1}{2\pi} \int_0^{2\pi} \varphi(\xi, y)\, d\xi,\ \frac{1}{2\pi} \int_0^{2\pi} \psi(x, \eta)\, d\eta \right] = \tag{1.1}$$
$$= \operatorname{col}[P_1 \varphi, P_2 \psi].$$

Remark 2. Obviously $P^2 = P$ and P is a projection operator mapping $S \cap C_k$ onto $N \cap C_k$.

Denote now

$$N^\perp = \{\operatorname{col}(\varphi, \psi) : \operatorname{col}(\varphi, \psi) \in S \cap C_0,\ P(\operatorname{col}(\varphi, \psi)) = 0\}.$$

Remark 3. We have obviously $N^\perp \cap (N \cap C_k) = 0$ for any $k = 0, 1, \ldots$

Lemma 1. *The following statements are equivalent*

i) $\operatorname{col}(\varphi, \psi) \in N^\perp$,

ii) $$\operatorname{col}\left[\int_0^{2\pi} \varphi(\xi, y)\, d\xi,\ \int_0^{2\pi} \psi(x, \eta)\, d\eta \right] = \operatorname{col}(0, 0) = 0,$$

iii) $$\int_0^{2\pi}\int_0^{2\pi} \varphi(\xi, \eta)\, \delta(\eta)\, d\eta\, d\xi = 0, \qquad \int_0^{2\pi}\int_0^{2\pi} \psi(\xi, \eta)\, \gamma(\xi)\, d\xi\, d\eta = 0,$$

for any $\operatorname{col}(\delta, \gamma) \in N \cap C_0$.

The proof is almost trivial.

Lemma 2. *The set* $N \cap C_0$ *coincides with the set of solutions in* C^*_1 *of the homogeneous boundary-value problem*

$$u_x = 0, \quad v_y = 0, \tag{1.2}$$

$$u(x + 2\pi, y) = u(x, y) = u(x, y + 2\pi),\ v(x + 2\pi, y) = v(x, y) = \tag{1.3}$$
$$= v(x, y + 2\pi),$$

$(-\infty < x < \infty)$. The proof is obvious.

Lemma 3. *For every given* $k = 0, 1, \ldots$ *and a given* $\operatorname{col}(\varphi, \psi) \in S \cap C_k$, *the boundary value problem*

$$u_x = \varphi(x, y), \quad v_y = \psi(x, y), \tag{1.4}$$

$$u(x + 2\pi, y) = u(x, y) = u(x, y + 2\pi),\ v(x + 2\pi, y) = v(x, y) = \tag{1.5}$$
$$= v(x, y + 2\pi)$$

has a solution if and only if $\operatorname{col}(\varphi, \psi) \in N^\perp \cap C_k$. *Moreover, if* $\operatorname{col}(\varphi, \psi) \in N^\perp \cap C_k$, *then there exists a unique solution of* (1.4), (1.5), *which belongs to* $N^\perp \cap C^*_{k+1}$. *If this unique solution is denoted*

$$\operatorname{col}(u, v) = \mathcal{L}(x, y)(\operatorname{col}(\varphi, \psi)), \qquad 0 \leqslant x, y \leqslant 2\pi, \tag{1.6}$$

then $\mathcal{L}(\ldots)$ *is a linear operator from* $N^\perp \cap C_k$ *to* $N^\perp \cap C^*_{k+1}$ *and there exists a constant* K *such that*

$$\|\mathcal{L}(\ldots)(\operatorname{col}(\varphi, \psi))\|^*_{k+1} \leqslant K \|\operatorname{col}(\varphi, \psi)\|_k. \tag{1.7}$$

Proof. We obtain from (1.4)

$$u(x, y) = u(0, y) + \int_0^x \varphi(\xi, y)\, d\xi, \quad v(x, y) = v(0, y) + \int_0^y \psi(x, \eta)\, d\eta$$

and imposing conditions (1.5) we get

$$\int_0^{2\pi} \varphi(\xi, y)\, d\xi = 0, \qquad \int_0^{2\pi} \psi(x, \eta)\, d\eta = 0,$$

thus $\mathrm{col}(\varphi, \psi) \in N^{\perp} \cap C_k$. For the sufficiency consider $\mathrm{col}(\varphi, \psi) \in N^{\perp} \cap C_k$ and take

$$u(x, y) = \int_0^x \varphi(\xi, y)\, d\xi, \quad v(x, y) = \int_0^y \psi(x, \eta)\, d\eta.$$

Then $u(x, y), v(x, y)$ defined this way satisfy clearly (1.4), (1.5). Now, the uniqueness follows because the difference of any two solutions of (1.4), (1.5) would be both in $N \cap C_0$ and $N^{\perp} \cap C_0$, thus it is zero. Define

$$\mathrm{col}(u, v) \overset{\mathrm{def}}{=} \mathcal{L}(x, y)(\mathrm{col}(\varphi, \psi)) =$$

$$(1.8) \qquad = \mathrm{col}\left\{\int_0^x \varphi(\xi, y)\, d\xi - \frac{1}{2\pi}\int_0^{2\pi}\left[\int_0^s \varphi(\xi, y)\, d\xi\right] ds,\right.$$

$$\left.\int_0^y \psi(x, \eta)\, d\eta - \frac{1}{2\pi}\int_0^{2\pi}\left[\int_0^t \psi(x, \eta)\, d\eta\right] dt\right\}.$$

Obviously $\mathcal{L}(\ldots)(\mathrm{col}(\varphi, \psi)) \in N^{\perp} \cap C^*_{k+1}$, it is the solution of (1.4)–(1.5) and there exists a constant K such as to have (1.7).

§ 2. **The nonlinear system.** Consider now the problem

$$(2.1) \qquad u_x = \varepsilon F(x, y, u, v), \qquad v_y = \varepsilon G(x, y, u, v),$$

$$(2.2) \qquad u(x + 2\pi, y) = u(x, y) = u(x, y + 2\pi), \qquad v(x + 2\pi, y) = = v(x, y) = v(x, y + 2\pi).$$

We suppose throughout this section that $F(x, y, u, v)$ *and* $G(x, y, u, v)$ *are continuous in*

$$\Omega(R) = \{(x, y, u, v) : 0 \leqslant x, y \leqslant 2\pi, \ |u| + |v| \leqslant R\}$$

and satisfy in $\Omega(R)$ *the Lipschitz conditions and periodicity conditions in* x, y:

$$(2.3) \qquad \begin{aligned} |F(x, y, \bar{u}, \bar{v}) - F(x, y, u, v)| &\leqslant L(|\bar{u} - u| + |\bar{v} - v|), \\ |G(x, y, \bar{u}, \bar{v}) - G(x, y, u, v)| &\leqslant M(|\bar{u} - u| + |\bar{v} - v|), \end{aligned}$$

$$(2.4) \qquad \begin{aligned} F(x + 2\pi, y, u, v) &= F(x, y, u, v) = F(x, y + 2\pi, u, v), \\ G(x + 2\pi, y, u, v) &= G(x, y, u, v) = G(x, y + 2\pi, u, v). \end{aligned}$$

Consider for $\mathrm{col}(\beta, \alpha) \in N \cap C_0$ and for a, b such that $0 < a < b < R$,

$$V_1(\beta, \alpha : a, b) = \{\mathrm{col}(\varphi, \psi) : \mathrm{col}(\varphi, \psi) \in S \cap C_0,$$

$$P\,\mathrm{col}(\varphi, \psi) = \mathrm{col}(\beta, \alpha), \ \|\mathrm{col}(\beta, \alpha)\|_0 \leqslant a, \ \|\mathrm{col}(\varphi, \psi)\|_0 \leqslant b\}.$$

For $\mathrm{col}(\varphi, \psi) \in V_1(\beta, \alpha : a, b)$ denote

$$(2.5) \qquad \begin{aligned} f(x, y, \varphi, \psi) &\overset{\mathrm{def}}{=} F(x, y, \varphi(x, y), \psi(x, y)), \\ g(x, y, \varphi, \psi) &\overset{\mathrm{def}}{=} G(x, y, \varphi(x, y), \psi(x, y)). \end{aligned} \qquad (0 \leqslant x, y \leqslant 2\pi).$$

Theorem 1. *Assume* F, G *are continuous in* $\Omega(R)$ *and satisfy* (2.3). *Let* $0 < a < b < R$ *be given constants. Then there exists an* $\varepsilon_1 > 0$ *with the following property: For any* $\mathrm{col}(\beta, \alpha) \in N \cap C_0$ *with* $\|\mathrm{col}(\beta, \alpha)\|_0 \leqslant a$ *and for any* ε *with* $|\varepsilon| \leqslant \varepsilon_1$ *there is a unique vector function* $\mathrm{col}(A, B) \in V_1(\beta, \alpha : a, b)$, *(where* $A = A(\beta, \alpha : \varepsilon)$, $B = B(\beta, \alpha : \varepsilon)$*) such that* A_x, B_y *are continuous in* (x, y) *(that is* $\mathrm{col}(A, B) \in C^*_1$*) and*

$$(2.6) \qquad \begin{aligned} A_x &= \varepsilon f(x, y, A, B) - \varepsilon P_1 f(\ldots, A, B)(x, y), \\ B_y &= \varepsilon g(x, y, A, B) - \varepsilon P_2\, g(\ldots, A, B)(x, y), \end{aligned}$$

where $f(x, y, A, B), g(x, y, A, B)$ *are defined in* (2.5).

Furthermore, $\mathrm{col}(A(\beta, \alpha : \varepsilon), B(\beta, \alpha ; \varepsilon))$ *can be obtained by the method of successive approximations*

$$\mathrm{col}(u^0, v^0) = \mathrm{col}(\beta, \alpha),$$

$$(2.7) \qquad \mathrm{col}(u^{n+1}(x, y), v^{n+1}(x, y)) = \mathrm{col}(\beta(y), \alpha(x)) + \\ + \varepsilon \mathcal{L}(x, y)(I - P)(\mathrm{col}[f(\ldots, u^n, v^n), g(\ldots, u^n, v^n)]),$$

$n = 0, 1, 2, \ldots$

Moreover, $\mathrm{col}[A(\beta, \alpha : \varepsilon), B(\beta, \alpha : \varepsilon)]$ *is continuous in* $\beta, \alpha, \varepsilon$ *and Lipschitzian in* $\mathrm{col}(\beta, \alpha)$, *uniformly with respect to* ε *for* $\|\mathrm{col}(\beta, \alpha)\|_0 \leqslant a$, $|\varepsilon| \leqslant \varepsilon_1$. *Finally,* $A(\beta, \alpha : 0) = \beta(y)$, $B(\beta, \alpha : 0) = \alpha(x)$.

Proof. From our assumption it is obvious that there are two constants K_1, K_2 such that for any $\mathrm{col}(\varphi, \psi) \in V_1$ we have

$$(2.8) \qquad \|f(x, y, \varphi, \psi)\|_0 \leqslant K_1, \quad \|g(x, y, \varphi, \psi)\|_0 \leqslant K_2,$$

and, for any $\mathrm{col}(\varphi, \psi)$, $\mathrm{col}(\tilde{\varphi}, \tilde{\psi}) \in V_1$,

$$(2.9)\qquad \begin{aligned} &\| f(x, y, \tilde{\varphi}, \tilde{\psi}) - f(x, y, \varphi, \psi)\|_0 \leqslant L(\|\tilde{\varphi} - \varphi\|_0 + \|\tilde{\psi} - \psi\|_0),\\ &\| g(x, y, \tilde{\varphi}, \tilde{\psi}) - g(x, y, \varphi, \psi)\|_0 \leqslant M(\|\tilde{\varphi} - \varphi\|_0 + \|\tilde{\psi} - \psi\|_0).\end{aligned}$$

Denote

$$\mathcal{M}_1 [\operatorname{col}(u, v)](x, y) \overset{\text{def}}{=} \beta(y) + \int_0^x \left\{ f(\xi, y, u, v) - \frac{1}{2\pi}\int_0^{2\pi} f(t, y, u, v)\, dt \right\} d\xi -$$

$$- \frac{\varepsilon}{2\pi}\int_0^{2\pi}\left\{\int_0^s \left[f(\xi, y, u, v) - \frac{1}{2\pi}\int_0^{2\pi} f(t, y, u, v)\, dt \right] d\xi\right\} ds,$$

$$\mathcal{M}_2 [\operatorname{col}(u, v)](x, y) \overset{\text{def}}{=} \alpha(x) + \int_0^y \left\{ g(x, \eta, u, v) - \frac{1}{2\pi}\int_0^{2\pi} g(x, t, u, v)\, dt \right\} d\eta -$$

$$- \frac{\varepsilon}{2\pi}\int_0^{2\pi}\left\{\int_0^s \left[g(x, \eta, u, v) - \frac{1}{2\pi}\int_0^{2\pi} g(x, t, u, v)\, dt \right] d\eta\right\} ds,$$

thus

$$\mathcal{M}(\operatorname{col}(u, v)) \overset{\text{def}}{=} \operatorname{col}[\mathcal{M}_1(\operatorname{col}(u, v)), \mathcal{M}_2(\operatorname{col}(u, v))] =$$

$$= \operatorname{col}(\beta, \alpha) + \varepsilon \mathcal{L}(x, y)(I - P)(\operatorname{col}(f(\ldots, u, v), g(\ldots, u, v))).$$

Consider the equation

$$(2.10)\qquad \operatorname{col}(u, v) = \mathcal{M}(\operatorname{col}(u, v)).$$

We obtain immediately from (2.8), (2.9)

$$(2.11)\qquad \|\mathcal{M}(\operatorname{col}(u, v))\|_1^* \leqslant \|\alpha\|_0 + \|\beta\|_0 + 2|\varepsilon| K(K_1 + K_2) \leqslant a + 2\varepsilon_1 K(K_1 + K_2) < b$$

for every $\operatorname{col}(u, v) \in V_1$, provided

$$(2.12)\qquad \varepsilon_1 < (b - a)[2K(K_1 + K_2)]^{-1}.$$

Then, for any $\operatorname{col}(u, v)$, $\operatorname{col}(\tilde{u}, \tilde{v}) \in V_1$, we have

$$(2.13)\qquad \|\mathcal{M}(\operatorname{col}(\tilde{u}, \tilde{v})) - \mathcal{M}(\operatorname{col}(u, v))\|_1^* \leqslant 2\varepsilon_1 K(L + M)\|\operatorname{col}(\tilde{u}, \tilde{v}) - \operatorname{col}(u, v)\|_0 = 0.$$

Thus $\mathcal{M}$ is a contraction mapping $V_1 \cap C_0$ into $V_1 \cap C_1$, provided

$$(2.14)\qquad \varepsilon_1 < [2K(L + M)]^{-1}.$$

Denote by $\operatorname{col}(A, B)$ the fixed point of $\mathcal{M}$, that is the solution of (2.10). Obviously, $A = A(\beta, \alpha; \varepsilon)$, $B = B(\beta, \alpha; \varepsilon)$ satisfy the first part of the statement. It is also clear that $A(\beta, \alpha; 0) = \beta(y)$, $B(\beta, \alpha; 0) = \alpha(x)$. Furthermore, A and B are obviously continuous in ε for $|\varepsilon| \leqslant \varepsilon_1$. Now, for any $\operatorname{col}(\beta^1, \alpha^1)$, $\operatorname{col}(\beta^2, \alpha^2) \in N \cap C_0$ with $\|\beta^2\|_0 + \|\alpha^2\|_0 \leqslant a$, $\|\beta^1\|_0 + \|\alpha^1\|_0 \leqslant a$, we have

$$\|\operatorname{col}[A(\beta^1, \alpha^1; \varepsilon), B(\beta^1, \alpha^1; \varepsilon)] - \operatorname{col}[A(\beta^2, \alpha^2; \varepsilon), B(\beta^2, \alpha^2; \varepsilon)]\|_1^* \leqslant$$

$$\leqslant \|\operatorname{col}(\beta^1 - \beta^2, \alpha^1 - \alpha^2)\|_0 +$$

$$+ 2\varepsilon_1 K(L + M)\|\operatorname{col}[A(\beta^1, \alpha^1; \varepsilon), B(\beta^1, \alpha^1; \varepsilon)] - \operatorname{col}[A(\beta^2, \alpha^2; \varepsilon), B(\beta^2, \alpha^2; \varepsilon)]\|_1^*.$$

Since we have (2.14), it is obvious that $\operatorname{col}[A(\beta, \alpha; \varepsilon), B(\beta, \alpha; \varepsilon)]$ is Lipschitzian with respect to (β, α), uniformly with respect to $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$. The theorem is proved.

Theorem 2. *Suppose F, G are continuous in $\Omega(R)$ and satisfy (2.3) in $\Omega(R)$. Let $a < b < R$, ε_1 and $A(\beta, \alpha; \varepsilon)$, $B(\beta, \alpha; \varepsilon)$ be the numbers and functions of Theorem 1. If there exist an $\varepsilon_2 \leqslant \varepsilon_1$, a vector function $\operatorname{col}(\beta(\varepsilon), \alpha(\varepsilon)) \in N \cap C_0$, $\|\beta\|_0 + \|\alpha\|_0 \leqslant a$, $|\varepsilon| \leqslant \varepsilon_2$, such that*

$$(2.16)\qquad \begin{aligned} &P_1 f(\ldots, A(\beta(\varepsilon), \alpha(\varepsilon); \varepsilon), B(\beta(\varepsilon), \alpha(\varepsilon); \varepsilon)) = 0,\\ &P_2 g(\ldots, A(\beta(\varepsilon), \alpha(\varepsilon); \varepsilon), B(\beta(\varepsilon), \alpha(\varepsilon); \varepsilon)) = 0,\end{aligned}$$

then $\operatorname{col}[A(\beta(\varepsilon), \alpha(\varepsilon); \varepsilon), B(\beta(\varepsilon), \alpha(\varepsilon); \varepsilon)]$ is a solution of (2.1), (2.2) for $|\varepsilon| \leqslant \varepsilon_2$. Conversely, if (2.1), (2.2) has a solution $\operatorname{col}[u(x, y; \varepsilon), v(x, y; \varepsilon)]$ which is continuous together with u_x, v_y in x, y, ε, for $0 \leqslant x, y \leqslant 2\pi$, $|\varepsilon| \leqslant \varepsilon_2$, and

$$\|\operatorname{col}(u(\ldots, \varepsilon), v(\ldots; \varepsilon))\|_1^* < b, \quad \|P \operatorname{col}[u(\ldots; \varepsilon), v(\ldots; \varepsilon)]\|_0 \leqslant a,$$

then $u(x, y; \varepsilon) = A(x, y, \beta(\varepsilon), \alpha(\varepsilon); \varepsilon)$, $v(x, y, \varepsilon) = B(x, y, \beta(\varepsilon), \alpha(\varepsilon); \varepsilon)$, where A, B are the functions given by Theorem 1,

$$P_1 u(\ldots; \varepsilon) = \beta(\varepsilon), \quad P_2 v(\ldots; \varepsilon) = \alpha(\varepsilon),$$

and $\beta(\varepsilon)$, $\alpha(\varepsilon)$ satisfy the „bifurcation" (determining) equations (2.16).

Proof. The first part of the theorem is trivial. Now, if $\operatorname{col}(u(x, y; \varepsilon), v(x, y; \varepsilon))$ is a solution of (2.1), (2.2), denote $\beta(\varepsilon) = P_1 u(\ldots; \varepsilon)$, $\alpha(\varepsilon) = P_2 v(\ldots; \varepsilon)$. Since $\operatorname{col}(u, v)$ is a solution of (2.1), (2.2) we have obviously

$$(I - P_1)(u_x - \varepsilon f(\ldots, u, v)) = 0, \quad (I - P_2)(v_y - \varepsilon g(\ldots, u, v)) = 0,$$

$$P_1(u_x - \varepsilon f(\ldots, u, v)) = 0, \quad P_2(v_y - \varepsilon g(\ldots, u, v)) = 0,$$

and also $P_1 u_x = 0$, $P_2 v_y = 0$. Thus $P_1 f(\ldots, u, v) = 0$, $P_2 g(\ldots, u, v) = 0$, and thus (2.1) are equivalent to

$$
\begin{aligned}
u_x &= \varepsilon f(\ldots, u, v) - \varepsilon P_1 f(\ldots, u, v), \\
v_y &= \varepsilon g(\ldots, u, v) - \varepsilon P_2 g(\ldots, u, v), \\
P_1 f(\ldots, u, v) &= 0, \quad P_2 g(\ldots, u, v) = 0,
\end{aligned} \tag{2.17}
$$

for $|\varepsilon| \leq \varepsilon_2$. Then Theorem 1 implies $u = A(\beta(\varepsilon), \alpha(\varepsilon); \varepsilon)$, $v = B(\beta(\varepsilon), \alpha(\varepsilon); \varepsilon)$ and the last two equations in (2.17) imply $\beta(\varepsilon)$, $\alpha(\varepsilon)$ satisfy (2.16).

Theorem 3. *Suppose $F(x, y, u, v)$, $G(x, y, u, v)$ have continuous derivatives F_u, F_v, G_u, G_v, which are also Lipschitzian with respect to u, v in $\Omega(R)$. Suppose $a < b < R$, ε_1, $A(\beta, \alpha; \varepsilon)$, $B(\beta, \alpha; \varepsilon)$ are the numbers and functions of Theorem 1. Then there exists ε_3, $0 < \varepsilon_3 \leq \varepsilon_1$ such that* col $[A(\beta, \alpha; \varepsilon), B(\beta, \alpha; \varepsilon)]$ *is continuously differentiable with respect to* col (β, α), *for $\|\beta\|_0 + \|\alpha\|_0 < a$, $|\varepsilon| \leq \varepsilon_3$. Moreover, the derivative $D(A, B)/D(\beta, \alpha)$ is at $\varepsilon = 0$ the identity operator.*

Proof. All we have to show is, in fact, that for any fixed col $(\beta, \alpha) \in X \cap C_0$, with $\|\beta\|_0 + \|\alpha\|_0 < a$, there exists a linear continuous operator $W_{\beta\alpha}: X \cap C_0 \to S \cap C_1^*$, such that for every $r > 0$, there is a $s > 0$ with the property that

$$
\begin{aligned}
&\|\operatorname{col}[A(\beta + k, \alpha + h), B(\beta + k, \alpha + h)] - \\
&- \operatorname{col}[A(\beta, \alpha), B(\beta, \alpha)] - W_{\beta\alpha}(\operatorname{col}(h, k))\|_1^* \leq \\
&\leq r(\|k\|_0 + \|h\|_0) = r\|\operatorname{col}(k, h)\|_1^*.
\end{aligned} \tag{2.18}
$$

for col $(k, h) \in X \cap C_0$ and $\|\operatorname{col}(k, h)\|_0 \leq s$.

If we choose $\|h\|_0 + \|k\|_0$ sufficiently small, then $A(\beta + k, \alpha + h)$ and $B(\beta + k, \alpha + h)$ are defined by Theorem 1. We have now, using (2.5),

$$
\begin{aligned}
&f[x, y, A(\beta + k, \alpha + h), B(\beta + k, \alpha + h)] - f[x, y, A(\beta, \alpha), B(\beta, \alpha)] = \\
&\qquad = F_u^{kh}(x, y) E^{kh} + F_v^{kh}(x, y) H^{kh}, \\
&g[x, y, A(\beta + k, \alpha + h), B(\beta + k, \alpha + h)] - g[x, y, A(\beta, \alpha), B(\beta, \alpha)] = \\
&\qquad G_u^{kh}(x, y) E^{kh} + G_v^{kh}(x, y) H^{kh},
\end{aligned} \tag{2.19}
$$

where

$$
\begin{aligned}
F_u^{kh}(x, y) &= \frac{\partial F(x, y, A(\beta + k_1, \alpha + h_1), B(\beta + k, \alpha + h))}{\partial u}, \\
F_v^{kh}(x, y) &= \frac{\partial F(x, y, A(\beta, \alpha), B(\beta + k_2, \alpha + h_2))}{\partial v}, \\
G_u^{kh}(x, y) &= \frac{\partial G(x, y, A(\beta + k_3, \alpha + h_3), B(\beta + k, \alpha + h))}{\partial u}, \\
G_v^{kh}(x, y) &= \frac{\partial G(x, y, A(\beta, \alpha), B(\beta + k_4, \alpha + h_4))}{\partial v},
\end{aligned} \tag{2.20}
$$

$$
A(\beta + k, \alpha + h) - A(\beta, \alpha) = E^{kh}, \quad B(\beta + k, \alpha + h) - B(\beta, \alpha) = H^{kh},
$$

where h_i, k_i ($i = 1, 2, 3, 4$) tend to zero as h, k approach zero. On the other hand F_u^{00}, F_v^{00}, G_u^{00}, G_v^{00}, depend only on β and α and, under our assumptions, there are two constants K_3, K_4 such that

$$
\|F_u^{kh}\|_0, \|F_v^{kh}\|_0 \leq K_3; \qquad \|G_u^{kh}\|_0, \|G_v^{kh}\|_0 \leq K_4; \tag{2.21}
$$

for $\|k\|_0 + \|h\|_0 \leq s_1$. From the continuity of A, B in (β, α) we also have

$$
\begin{aligned}
&\|F_u^{kh} - F_u^{00}\|_0, \|F_v^{kh} - F_v^{00}\|_0, \|G_u^{kh} - G_u^{00}\|_0, \|G_v^{kh} - G_v^{00}\| \leq \\
&\qquad \leq \nu(\|k\|_0 + \|h\|_0),
\end{aligned} \tag{2.22}
$$

with $\lim \nu(s) = 0$ as $s \to 0$.

Consider now the equation

$$
\begin{aligned}
&\operatorname{col}[U(x, y), V(x, y)] = \operatorname{col}(\tilde{\beta}(y), \tilde{\alpha}(x)) + \\
&+ \varepsilon \mathcal{L}(x, y)(I - P)(\operatorname{col}[F_u^{00} U + F_v^{00} V, G_u^{00} U + G_v^{00} V]),
\end{aligned} \tag{2.23}
$$

with $\mathcal{L}$ defined in Lemma 3, and col $(\tilde{\beta}, \tilde{\alpha})$ arbitrary in $X \cap C_0$.

One shows, exactly as in Theorem 1, that for $|\varepsilon| \leq \varepsilon_3 < \varepsilon_1$, $\varepsilon_3 < [2K(K_3 + K_4)]^{-1}$ and $\varepsilon_3 < (b - a)[2K(K_3 + K_4)]^{-1}$ the equation (2.23) has a unique solution col $[U(\tilde{\beta}, \tilde{\alpha}), V(\tilde{\beta}, \tilde{\alpha})]$ in $S \cap C_1$ which satisfies

$$
\|\operatorname{col}(U(\tilde{\beta}, \tilde{\alpha}), V(\tilde{\beta}, \tilde{\alpha}))\|_1^* \leq (1 - 2K(K_3 + K_4)\varepsilon_3)^{-1}\|\operatorname{col}(\tilde{\beta}, \tilde{\alpha})\|_1^*. \tag{2.24}
$$

Now, due to the uniqueness of the solution of (2.23), col $(U(\beta, \alpha), V(\beta, \alpha))$ is an operator from $X \cap C_0$ (or, according to Remark 1, from $X \cap C_1^*$) into $S \cap C_1^*$. One sees also, because of the uniqueness of the solution of (2.23) for $\tilde{\beta}$, $\tilde{\alpha}$ fixed, that this operator is linear. Thus, it is a linear and, according to (2.24), a continuous operator. Denote this operator by

$$
W_{\beta\alpha} \overset{\mathrm{def}}{=} \operatorname{col}(U(\beta, \alpha), V(\beta, \alpha)) = \operatorname{col}(U_{\beta\alpha}, V_{\beta\alpha}).
$$

We obviously have at $\varepsilon = 0$ $W_{\beta\alpha} = \operatorname{col}(\beta, \alpha)$, thus $W_{\beta\alpha}$ is for $\varepsilon = 0$ the identity operator.

We obtain now from (2.19), (2.20), taking into account that $A(\beta, \alpha)$, $B(\beta, \alpha)$ satisfy (2.10):

$$
\begin{aligned}
&\operatorname{col}(E^{kh}, H^{kh}) - W_{\beta\alpha}(k, h) = \operatorname{col}(E^{kh} - U_{\beta\alpha}(k, h), H^{kh} - V_{\beta\alpha}(k, h)) = \\
&= \varepsilon \mathcal{L}(\ldots)(I - P)\{\operatorname{col}[(F_u^{kh} - F_v^{00})E^{kh} + (F_v^{kh} - F_v^{00})H^{kh}, \\
&\qquad (G_u^{kh} - G_u^{00})E^{kh} + (G_v^{kh} - G_v^{00})H^{kh}] + \\
&+ \operatorname{col}[F_u^{00}(E^{kh} - U_{\beta\alpha}(k, h)) + F_v^{00}(H^{kh} - V_{\beta\alpha}(k, h)), \\
&\qquad G_u^{00}(E^{kh} - U_{\beta\alpha}(k, h)) + G_v^{00}(H^{kh} - V_{\beta\alpha}(k, h))]\}.
\end{aligned} \tag{2.25}
$$

Now, by Theorem 1 we have also

$$\| \operatorname{col} (E^{kh}, H^{kh}) \|_1^* \leqslant K_5 (\| k \|_0 + \| h \|_0). \tag{2.26}$$

Thus, by (2.25), (2.26), (2.21), (2.22) we obtain after some computations

$$\| \operatorname{col} (E^{kh} - U_{\beta\alpha} (k, h), H^{kh} - V_{\beta\alpha} (k, h)) \|_1^* \leqslant$$
$$\leqslant [1 - 2 \varepsilon_3 K (K_3 + K_1)]^{-1} 2K_5 K \varepsilon_3 (\| k \|_0 + \| h \|_0) \cdot \nu [\| k \|_0 + \| h \|_0],$$

which proves (2.18), that is Theorem 3.

Remark 1. Nothing is changed in Theorems 1, 2, 3 if F and G depend continuously on ε.

Consider now the bifurcation equations of problem (2.1), (2.2), that is equations (2.16). They can be written in the form

$$2\pi H (\beta, \alpha : \varepsilon) \overset{\text{def}}{=} 2\pi \operatorname{col} [R (\beta, \alpha : \varepsilon), T (\beta, \alpha : \varepsilon)] = 0. \tag{2.27}$$

where

$$\begin{aligned} 2\pi R (\beta, \alpha : \varepsilon) (y) &\overset{\text{def}}{=} \int_0^{2\pi} F (\xi, y, A (\beta, \alpha : \varepsilon), B (\beta, \alpha : \varepsilon)) \, d\xi, \\ 2\pi T (\beta, \alpha ; \varepsilon) (x) &\overset{\text{def}}{=} \int_0^{2\pi} G (x, \eta, A (\beta, \alpha : \varepsilon) \, B (\beta, \alpha : \varepsilon)) \, d\eta. \end{aligned} \tag{2.28}$$

We have now obviously

$$\begin{aligned} 2\pi R (\beta, \alpha : 0) (y) &= \int_0^{2\pi} F (\xi, y, \beta (y), \alpha (\xi)) \, d\xi, \\ 2\pi T (\beta, \alpha ; 0) \; (x) &= \int_0^{2\pi} G (x, \eta, \beta (\eta), \; \alpha (x)) \, d\eta. \end{aligned} \tag{2.29}$$

Also, if the conditions of Theorem 3 are fulfilled we immediately find, denoting $p = \operatorname{col} (\beta, \alpha)$,

$$2\pi H'_p (p, 0) (k, h) = 2\pi \frac{D (\operatorname{col} [R (\beta, \alpha : 0), T (\beta, \alpha : 0)])}{D (\beta, \alpha)} (k, h) =$$

$$\text{(2.30)} = \operatorname{col} \Big\{ \int_0^{2\pi} [F_u (\xi, y, \beta (y), \alpha (\xi)) \, k (y) + F_v (\xi, y, \beta (y), \alpha (\xi)) \, h (\xi)] \, d\xi,$$
$$\int_0^{2\pi} [G_u (x, \eta, \beta (\eta), \alpha (x)) \, k (\eta) + G_v (x, \eta, \beta (\eta), \alpha (x)) \, h (x)] \, d\eta \Big\}.$$

Now it is known (see O. Vejvoda [23], Th. 2.2 of that paper) that one has the following implicit functions theorem in Banach spaces

Theorem. (Implicit functions): *Consider an equation (I)* $H(p, \varepsilon) = 0$, *where* $H (p, \varepsilon): \mathcal{P} \times [-\varepsilon_1, \varepsilon_1] \to Q$, $\mathcal{P}$ *and* Q *being Banach spaces. Suppose i)* $p_0^* \in \mathcal{P}$ *is a solution of* $H (p_0, 0) = 0$, *ii)* $H (p, \varepsilon)$ *is continuous in* p *and* ε *and its derivative* $H'_p (p, \varepsilon)$ *exists and is continuous in* p *and* ε *for* $p \in B (p_0^*, \delta)$ *and* $\varepsilon \in [-\varepsilon_1, \varepsilon_1]$ *(*δ *being conveniently choosen, such as to have* $B (p_0^*, \delta) \subset \mathcal{P}$*). iii) The operator* $M = [H'_p (p_0^*, 0)]^{-1}$ *exists and is continuous from* Q *to* $\mathcal{P}$ *Then there exists an* $\varepsilon^* > 0$ *such that equation (I) has for* $-\varepsilon^* \leqslant \varepsilon \leqslant \varepsilon^*$ *a unique solution* $p = p^* (\varepsilon) \in \mathcal{P}$, *continuous in* ε *and such that* $p^* (0) = p_0^*$.

From this implicit functions theorem and from the preceeding results we can immediately prove

Theorem 4. *Suppose the conditions of Theorem 3 are satisfied and* $H (p : \varepsilon) = H (\beta, \alpha : \varepsilon)$ *is defined by (2.27), (2.28). If there is a vector function* $\operatorname{col} (\beta^0, \alpha^0) \in N \cap C_0$ *with*

$$\| \operatorname{col} (\beta^0, \alpha^0) \|_1^* = \| \operatorname{col} (\beta^0, \alpha^0) \|_0 = \| \beta^0 \|_0 + \| \alpha^0 \|_0 < a,$$

such that

$$H (\beta^0, \alpha^0 : 0) = \operatorname{col} \Big[\frac{1}{2\pi} \int_0^{2\pi} F (\xi, y, \beta^0 (y), \alpha^0 (\xi)) \, d\xi, \frac{1}{2\pi} \int_0^{2\pi} G (x, \eta, \beta^0 (\eta),$$
$$(\alpha^0 (x)) \, d\eta \Big] = 0,$$

and the linear operator $H'_p (p_0, 0)$, (where $p_0 = (\beta^0, \alpha^0)$) *has a continuous inverse taking* $N \cap C_0$ *into* $N \cap C_0$, *then there exist an* $\varepsilon_1 > 0$ *and a vector function* $\operatorname{col} [u (x, y, \beta^0, \alpha^0 : \varepsilon), v (x, y, \beta^0, \alpha^0, \varepsilon)]$ *continuous in* x, y, ε *for* $| \varepsilon | \leqslant \varepsilon_1$, $0 \leqslant x, y \leqslant 2\pi$, *such that* $\operatorname{col} [u (x, y, \beta^0, \alpha^0, \varepsilon), v (x, y, \beta^0, \alpha^0, \varepsilon)]$ *satisfies (2.1), (2.2) and* $\operatorname{col} [u (x, y, \beta^0, \alpha^0 : 0), v (x, y, \beta^0, \alpha^0 : 0)] = \operatorname{col} [\beta^0, \alpha^0]$.

Nothing is changed if F and G in (2.1), (2.2) depend continuously on a parameter ε.

Example. Consider the system

$$\begin{aligned} u_x &= \varepsilon F (x, y, u, v) = \varepsilon [D_1 (x, y) + C_1 (x) \, u + L (y) \, v + \varepsilon F_1 (x, y, u, v)], \\ v_y &= \varepsilon G (x, y, u, v) = \varepsilon [D_2 (x, y) + M (x) \, u + C_2 (y) \, v + \varepsilon G_2 (x, y, u, v)], \end{aligned}$$

with $D_1, C_1, L, F_1, D_2, M, C_2, G_2$ ω-periodic in x and y. Furthermore, C_1 is an $n \times n$ matrix, L is a $n \times m$ one; M an $m \times n$ matrix, C_2 an $m \times m$ one; u, D_1 and F_1 are $n \times 1$ vectors, v, D_2, G_2 are $m \times 1$ vectors. Suppose D_1, D_2, C_1, M, L, C_2 are continuous and F_1 and G_2 have continuous and Lipschitzian with respect to u, v derivatives $\partial F_1/\partial u$, $\partial F_1/\partial v$, $\partial G_2/\partial u$, $\partial G_2/\partial v$. Denote

$$d_1 = \int_0^{2\pi}\int_0^{2\pi} D_1(\xi, \eta)\, d\xi d\eta, \quad d_2 = \int_0^{2\pi}\int_0^{2\pi} D_2(\xi, \eta)\, d\xi d\eta,$$

$$\overline{C}_1 = \int_0^{2\pi} C_1(\xi)\, d\xi,\ \overline{C}_2 = \int_0^{2\pi} C_2(\eta)\, d\eta,\ \overline{L} = \int_0^{2\pi} L(\eta)\, d\eta,$$

$$\overline{M} = \int_0^{2\pi} M(\xi)\, d\xi,\ \overline{\beta} = \int_0^{2\pi} \beta(\eta)\, d\eta,\ \overline{\alpha} = \int_0^{2\pi} \alpha(\xi)\, d\xi.$$

Suppose we have hypotheses

(H): *$\overline{C}_1$ and $\overline{C}_2$ are nonsingular matrices and $[\overline{C}_2 - \overline{M}(\overline{C}_1)^{-1}\overline{L}]$ is a nonsingular matrix also.*

We then obtain, according to (2.29) and Remark 4

$$\text{(I)}\quad \begin{aligned} &2\pi R(\beta, \alpha;\ 0)(y) = \overline{C}_1\beta(y) + L(Y)\,\overline{\alpha} + \int_0^{2\pi} D_1(\xi, y)\, d\xi = 0, \\ &2\pi T(\beta, \alpha; 0)(x) = M(x)\,\overline{\beta} + \overline{C}_2\,\alpha(x) + \int_0^{2\pi} D_2(x, \eta)\, d\eta = 0. \end{aligned}$$

Under hypotheses (H) above, we can immediately find $\overline{\alpha}$ and $\overline{\beta}$. We integrate the first of (I) with respect to y between 0 and 2π, and the second one with respect to x between 0 and 2π. We obtain this way

$$\overline{C}_1\,\overline{\beta} + \overline{L}\,\overline{\alpha} + d_1 = 0, \quad \overline{M}\,\overline{\beta} + \overline{C}_2\,\overline{\alpha} + d_2 = 0,$$

thus, after some computations

$$\overline{\alpha} = (\overline{C}_2 - \overline{M}(\overline{C}_1)^{-1}\overline{L})^{-1}[\overline{M}(\overline{C}_1)^{-1} d_1 - d_2],$$

$$\overline{\beta} = -(\overline{C}_1)^{-1}\ d_1 - (\overline{C}_1)^{-1}\overline{L}\,[\overline{C}_2 - \overline{M}(\overline{C}_1)^{-1}\overline{L}\,]^{-1}[\overline{M}(\overline{C}_1)^{-1} d_1 - d_2].$$

We introduce $\overline{\alpha}, \overline{\beta}$ determined this way in (I) and find obviously

$$\beta(y) = -(\overline{C}_1)^{-1} L(y)\,\overline{\alpha} - (\overline{C}_1)^{-1}\int_0^{2\pi} D_1(\xi, y)\, d\xi,$$

$$\alpha(x) = -(\overline{C}_2)^{-1} M(x)\,\overline{\beta} - (\overline{C}_2)^{-1}\int_0^{2\pi} D_2(x, \eta)\, d\eta.$$

On the other hand, according to (2.30) we find

$$2\pi H'_p(p, 0)(k, h) = \begin{bmatrix} \overline{C}_1 k(y) + L(y)\,\overline{h} \\ M(x)\,\overline{k} + \overline{C}_2 h(x) \end{bmatrix}.$$

To show that $H'_p(p, 0)\,.\,(k, h)$ has a continuous inverse taking $N \cap C_0$ into $N \cap C_0$ is now a very simple problem. We have only to show that system

$$\overline{C}_1 k(y) + L(y)\,\overline{h} = k_1(y),\ M(x)\,\overline{k} + \overline{C}_2 h(x) = h_1(x)$$

has a unique solution for any col $(k_1, h_1) \in N \cap C_0$, and that this solution depends continuously on col (k_1, h_1). Under assumptions (H) this is almost obvious. We have only to apply the same procedure to determine first $\overline{k}, \overline{h}$ in terms of $\overline{C}_1, \overline{L}, \overline{k}_1, \overline{M}, \overline{C}_2, \overline{h}_1$ and then $k(y), h(x)$ in terms of $\overline{k}, \overline{h}$ etc... So the main result of Theorem 4 applies to the example under consideration.

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SOLUȚII PERIODICE ALE SISTEMELOR HIPERBOLICE DE ORDINUL I CONȚINÎND UN PARAMETRU MIC

Rezumat

Autorul analizează problema existenței soluțiilor periodice pentru problema slab liniară (2.1), (2.2). Teorema 1 este o teoremă de existență pentru problema modificată. Teorema 2 introduce ecuațiile de bifurcație. Teorema 4 este o teoremă de existență pentru soluțiile problemei (2.1), (2.2). Rezultatele sînt aplicate la ecuațiile din exemplul final.

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PARTIAL DIFFERENTIAL EQUATIONS OF FUCHS TYPE

BY

DORIN CIUMAȘU

§1. Formulation of the problem. As it is known [5], an equation of the form.

$$y'' + p_1(x)\,y' + p_2(x)\,y = 0$$

has an analytic solution in the neighbourhood of a singular point x_0 of the functions p_1 and p_2, if and only if in x_0, p_1 has a pole of first degree at most and p_2 has a pole of second degree at most.

In this paper we shall give a result of that sort, for linear partial differential systems, which generalizes the one of [2].

Let $I, J \subset N$, be two finite sets of indices, and $\mathbf{I} = N^I$. Let K be a nondiscrete complete, valuation field.

We shall consider linear systems of the form.

$$x^{q(j)} D^{r(j)} u_j(x) = g_j(x) +$$

$$+ \sum_{k\in J} \sum_{\substack{s(j,k)\in \\ \in S(j,k)}} f_{s(j,k)}(x)\, D^{s(j,k)} u_k(x) \quad (j \in J). \tag{1}$$

where $q(j),\ r(j) \in \mathbf{I}$, $S(j,k) \subset \mathbf{I}$, $D^r = \partial^{|r|} / \prod_{i\in I} \partial x_i^{r_i}$ is the differentiation in K^I ($|r| = \sum_{i\in I} r_i$, $r \in \mathbf{I}$).

In what follows we shall suppose.

(A) *the functions* $g_j, f_{s(j,k)} : K^I \to K$, *are analytic in a certain neighbourhood of the origin of* K^I.

We wish to get conditions for the existence of a family of functions $u_j : K^I \to K$, analytic in a certain neighbourhood of the origin of K^I, satisfying (1) and the „initial conditions".

$$D_i^t u_j(x)|_{x_i=0} = 0,\ 0 \leqslant t < r_i(j),\ j \in J,\ i \in I. \tag{2}$$

where $D_i^t = \partial^t / \partial x_i^t$.