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SOLUȚII PERIODICE ALE SISTEMELOR HIPERBOLICE DE ORDINUL I CONȚINÎND UN PARAMETRU MIC

Rezumat

Autorul analizează problema existenței soluțiilor periodice pentru problema slab liniară (2.1). (2.2). Teorema 1 este o teoremă de existență pentru problema modificată. Teorema 2 introduce ecuațiile de bifurcație. Teorema 4 este o teoremă de existență pentru soluțiile problemei (2.1). (2.2). Rezultatele sînt aplicate la ecuațiile din exemplul final.

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PARTIAL DIFFERENTIAL EQUATIONS OF FUCHS TYPE

BY

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§1. **Formulation of the problem.** As it is known [5], an equation of the form.

$$y'' + p_1(x)\, y' + p_2(x)\, y = 0$$

has an analytic solution in the neighbourhood of a singular point x_0 of the functions p_1 and p_2, if and only if in x_0, p_1 has a pole of first degree at most and p_2 has a pole of second degree at most.

In this paper we shall give a result of that sort, for linear partial differential systems, which generalizes the one of [2].

Let $I, J \subset N$, be two finite sets of indices, and $\mathbf{I} = N^I$. Let K be a nondiscrete complete, valuation field.

We shall consider linear systems of the form.

$$x^{q(j)} D^{r(j)} u_j(x) = g_j(x) + \tag{1}$$
$$+ \sum_{k \in J} \sum_{\substack{s(j,k) \in \\ \in S(j,k)}} f_{s(j,k)}(x)\, D^{s(j,k)} u_k(x) \quad (j \in J).$$

where $q(j), r(j) \in \mathbf{I}$, $S(j, k) \subset \mathbf{I}$, $D^r = \partial^{|r|} / \prod_{i \in I} \partial x_i^{r_i}$ is the differentiation in K^I ($|r| = \sum_{i \in I} r_i$, $r \in \mathbf{I}$).

In what follows we shall suppose.

(A) *the functions* $g_j, f_{s(j,k)} : K^I \to K$, *are analytic in a certain neighbourhood of the origin of* K^I.

We wish to get conditions for the existence of a family of functions $u_j : K^I \to K$, analytic in a certain neighbourhood of the origin of K^I, satisfying (1) and the „initial conditions".

$$D_i^t u_j(x)|_{x_i = 0} = 0, \; 0 \leqslant t < r_i(j), \; j \in J, \; i \in I, \tag{2}$$

where $D_i^t = \partial^t / \partial x_i^t$.

We shall deal with this problem in the ring of formal series, with weak topology, and then in an adequate ring of convergent series.

§2. Treatment of the problem in the ring of formal series. Le $K[[X_i; i \in I]] = F$, be the ring of the formal series in indeterminates $(X_i : i \in I)$, over K, with weak topology, i.e. product topology in $\mathbf{K}^{\mathbf{I}}$.

We recall that the order of the formal series $u = (u_m : m \in \mathbf{I}) \in F$, is defined by $\omega(u) = \inf\{|m| : u_m \neq 0\}$, if $u \neq 0$, and $\omega(u) = +\infty$ if $u = 0$, [1, 6]. We remark that one can define similarly the partial order of the formal series u with regard to any subset of indeterminates. The strict order of a formal series u, is defined by $\Omega(u) = r$, $r = (r_i : i \in I) \in \mathbf{I}$ if r_i represents for every $i \in I$, the order of the $\mathbf{I}$ formal series u with regard to the indeterminate X_i.

In F, the problem (1), (2) becomes,

$$X^{q(j)} D^{r(j)} u_j(X) = g_j(X) +$$

(3)
$$+ \sum_{k \in J} \sum_{\substack{s(j,k) \in \\ \in S(j,k)}} f_{s(j,k)}(X) D^{s(j,k)} u_k(X) \quad (j \in J).$$

(4)
$$\Omega(u_j) = r(j) \quad (j \in J).$$

where $g_j, f_{s(j,k)} \in F$, represent now the formal series corresponding to analytic functions $g_j, f_{s(j,k)}$, and D^r is algebraic derivative in F [1, 6].

We shall show that if certain conditions are fulfilled, then this problem has an unique solution,

$$u_j = (u_{j,m} : m \in \mathbf{I}) \in F, \; u_{j,m} = \frac{D^m u_j(0)}{m!} (j \in J).$$

the projections $u_{j,m}$ being determined by the method of successive approximations in K, which means that the series u_j, are limits of certain iterative sequence in the weak topology of F.

First, we remark that from (4) it follows,

(5)
$$D^m u_j(0) = 0, \; j \in J, \; m \in \mathbf{I}, \; r(j) - m \neq 0$$

(we set $m - m' = 0$ if $m_i < m'_i$, for every $i \in I$). It means that every projection $u_{j,m} = 0$, $j \in J$, $m \in \mathbf{I}$ for which $r(j) - m \neq 0$.

Let us follow the way of determining the projections of indice $m = r(j)$, $j \in J$. For that we shall apply to the relations (3) the operators of partial derivative $D^{q(j)}$, $j \in J$, and we shall substitute $X_i \to 0$, $i \in I$, in the formal series which are obtained. Thus we get,

$$q(j)!\, D^{r(j)} u_j(0) = D^{q(j)} g_j(0) +$$

(6)
$$+ \sum_{k \in J} \sum_{\substack{s(j,k) \in \\ \in S(j,k)}} \sum_{\substack{m' + m'' = \\ = q(j)}} C^{m'}_{q(j)} D^{m'} f_{s(j,k)}(0) D^{s(j,k)+m''} u_k(0) \, (j \in J).$$

where we have denoted $C^{m'}_m = \prod_{i \in I} C^{m'_i}_{m_i}$, $m, m' \in \mathbf{I}$, and $C^q_p = p!/q!(p-q)!$, $p, q \in N$, $p \leqslant q$.

To solve these relations with regard to the unknowns $v_j = D^{r(j)} u_j(0)$, $j \in J$, we must suppose $\omega(f_{s(j,k)}) \neq 0$ for every indice $s(j,k) \in S(j,k)$, which satisfies the relations $|r(k)| < |s(j,k) + q(j)|$, and $\omega(f_{s(j,k)}) = 0$ possibly only for those indices $s(j,k)$ so that $|r(k)| \geqslant |s(j,k) + q(j)|$.

In this case, (6), is a linear nonhomogenous system in $D^{r(j)} u_j(0)$, $j \in J$, if it is supposed that $D^{q(j)} g_j(0) \neq 0$, at least for an indice $j \in J$.

We remark that if in the vectorial space K^E ($E \subset N$, is a finite set of indices), we consider the vectorial norm $\|v\| = \sup\{|v_e| ; e \in E\}$, then the matricial norm subordinated to this, is [3].

$$\|F\| = \sup \sum_{e_2 \in E} \{|F_{e_1, e_2}| ; e_1 \in E\}.$$

The system (6) has a unique solution, which can be obtained by the method of successive approximations if we suppose

(7)
$$\sup_{j \in J} \sum_{k \in J} \sum_{s(j,k)} \frac{1}{q(j)!} |f_{s(j,k)}(0)| \leqslant \lambda < 1,$$

where $\sum_{s(j,k)}$, is extended over those indices $s(j,k)$ $|s(j,k) + q(j)| = |r(k)|$, $j, k \in J$.

Generally, if we assume that for every formal series u_j, $j \in J$, we have got all projections $u_{j,m'}$, $m' \in \mathbf{I}$, $|m'| < |r(j) + m|$, $|m| =$ constant, let us find now the conditions of determining the projections $u_{j,m'}$, for $m' \in \mathbf{I}$, $|m'| = |r(j) + m|$. To this aim we apply the operator $D^{q(j)+m}$ in both of members of the relations (3). We get as above,

$$C^{q(j)}_{m+q(j)}\, q(j)!\, D^{r(j)+m} u_j(0) = D^{q(j)+m} g_j(0) +$$

(8)
$$+ \sum_{k \in J} \sum_{\substack{s(j,k) \in \\ \in S(j,k)}} \sum_{\substack{m' + m'' = \\ = m + q(j)}} C^{m'}_{m+q(j)} D^{m'} f_{s(j,k)}(0) D^{s(j,k)+m''} u_k(0).$$

These relations represent a linear system in the unknowns $D^{r(j)+m} u_j(0)$, $j \in J$, $m \in \mathbf{I}$, with $|m| =$ constant, if we assume that the orders of formal series $f_{s(j,k)}$ satisfy the condition, $|s(j,k) + q(j)| \leqslant |r(k)| + \omega(f_{s(j,k)})$, $j, k \in J$. We notice that the system (6) contains as many unknowns as solutions equation $|m| =$ constant, has in $\mathbf{I}$, and the same number of equations. Let $E \subset N$, be a set the cardinal of which is the number mentioned above.

In order that the system (6) has a unique solution in K^E, we must suppose,

(9)
$$\sup_{\substack{m \in \mathbf{I} \\ |m| = n \\ j \in J}} \sum_{k \in K} \sum_{s(j,k)} \sum_{p} \frac{C^p_{m+q(j)}}{C^{q(j)}_{m+q(j)}\, q(j)!} |D^p f_{s(j,k)}(0)| \leqslant \lambda < 1.$$

where the sum $\sum_{s(j,k)}$ is extended over those indices $s(j,k)$ for which $|s(j,k)+q(j)| = |r(k)| + \omega(f_{s(j,k)})$ and the sum $\sum_p$ is extended over every $p \in \mathbf{I}$, for which $p = \Omega(f_{s(j,k)})$. Finally this conditions has to be fulfilled for every $n \in N$. But as we can see immediately the sequence $\{a_m : m \in \mathbf{I}\}$, where

$$a_m = \frac{C^p_{m+q(j)}}{q(j)!\,C^{q(j)}_{m+q(j)}} = \frac{m!}{p!\,(m+q(j)-p)!},\; m \in \mathbf{I}$$

considered as a simple sequence (owing to the relation of canonical order in $\mathbf{I}$), is unbounded except the case $p \leqslant q(j)$ when we have $a_m \leqslant 1$. We admit the last hypothesis. It means that we must use the strict order of the formal series $f_{s(j,k)}$, and assume that $\Omega(f_{s(j,k)}) \leqslant q(j)$. In this case the condition (9) is fulfilled obviously if

$$\sup_{j\in J} \sum_{k\in J} \sum_{s(j,k)} |D^p f_{s(j,k)}(0)| \leqslant \lambda < 1. \tag{10}$$

where $p = \Omega(f_{s(j,k)})$, the other symbols having the same significance as above.

If this condition is fulfilled, the system (8) can be solved by the method of successive approximations, and for every iterative sequence we have,

$$\lim_{t\to\infty} D^{r(j)+m} u_j^{(t)}(0) = D^{r(j)+m} u_j(0).$$

More exactly, from (10) we have,

$$\sup_{\substack{j\in J\\ |m|=n}} |D^{r(j)+m} u_j^{(t)}(0) - D^{r(j)+m} u_j(0)| \leqslant$$
$$\leqslant \frac{\lambda^t}{1-\lambda} \sup_{\substack{j\in J\\ |m|=n}} |D^{r(j)+m} u_j^{(1)}(0) - D^{r(j)+m} u_j^{(0)}(0)|.$$

If $u_j^{(0)} = 0$, then,

$$\begin{aligned} &\sup_{\substack{j\in J\\ |m|=n}} |D^{r(j)+m} u_j^{(t)}(0) - D^{r(j)+m} u_j(0)| \leqslant \\ &\leqslant \frac{\lambda^t}{1-\lambda} \sup_{\substack{j\in J\\ |m|=n}} \frac{m!}{(m+q(j))!} |D^{q(j)+m} g_j(0)|. \end{aligned} \tag{11}$$

We notice that if (10) is fulfilled (7) also is.

It follows that the solution of the problem (3), (4) is, in the weak topology of F, the limit of any iterative sequence $\{u_j^{(t)} ; t \in N\}$, $j \in J$, defined by

$$u_j^{(t)} \in F,\ \Omega(u_j^{(t)}) = r(j),\ j \in J,\ t \in N \tag{12}$$

$$\begin{aligned} &X^{q(j)} D^{r(j)} u_j^{(t+1)} = g_j(X) + \\ &+ \sum_{k\in J} \sum_{\substack{s(j,k)\in\\ \in S(j,k)}} f_{s(j,k)}(X)\, D^{s(j,k)} u_k^{(t)}(X),\ j \in J,\ t \in N, \end{aligned} \tag{13}$$

if we assume:

(B) *for every* $s(j,k) \in S(j,k)$, $j, k \in J$, *we have*, $|s(j,k)+q(j)| \leqslant |r(k)| + \omega(f_{s(j,k)})$, *and for every* $s(j,k) \in R(j)$ *we have* $\omega(f_{s(j,k)}) = |\Omega(f_{s(j,k)})|$, $\Omega(f_{s(j,k)}) = q(j)$, *where*,

$$R(j) = \{s(j,k) : s(j,k) \in S(j,k),\ |s(j,k)+q(j)| = |r(k)| + \omega(f_{s(j,k)})\}$$

(C) *there exists at least* $j \in J$ *such that*, $D^{q(j)} g_j(0) \neq 0$

(D) *we have*
$$\sup_{j\in J} \sum_{\substack{s(j,k)\in\\ \in R(j)}} |D^p f_{s(j,k)}(0)| \leqslant \lambda < 1,$$

where $p = \Omega(f_{s(j,k)})$.

From that it follows.

Theorem 1. *If the conditions* (B), (C), *and* (D) *are satisfied, then the problem* (3), (4) *has an unique solution* $(u_j ; j \in J)$, $u_j \in F$, $j \in J$. *This solution is in the weak topology of* F, *the limit of any sequences* $\{u_j^{(t)} ; t \in N\}$, $j \in J$, *defined by* (12), (13).

§3. Treatment of the problem in the ring of convergent series. If $u = (u_m / m! : m \in \mathbf{I}) \in F$, we say a series u is convergent, if it exists $\alpha \in \Gamma = (R^*_+)^J$, such that,

$$\|u\|_{|\alpha|} = \sum_{k\in N} \frac{M_k}{k!} |\alpha|^k < +\infty$$

(Garding's majorant [4]) where $M_k = \sup\{|u_m| : |m| = k\}$, $k \in N$, and $|\alpha| = \sum_{j\in J} \alpha_j$.

We denote,

$$C_{|\alpha|} = \{u : u \in F,\ \|u\|_{|\alpha|} < +\infty\}.$$

It can be seen easily that $(C_{|\alpha|}, \|\cdot\|_{|\alpha|})$ is a complete normed ring.

For every convergent series u, we call convergence bound, the real numbers $\rho(u)$, which is the convergence radius of the numerical power series $\|u\|_{|\alpha|}$, in $\tau = |\alpha|$. Obviously, if $u \in C_{|\alpha|}$, then $\rho(u) \geqslant |\alpha|$. Likewise,

$$\rho(uv) = \min\{\rho(u), \rho(v)\},\quad \rho(D^r u) = \rho(u),\ r \in \mathbf{I} \tag{14}$$

for every convergent series u, v.

We consider now the problem (3), (4) in a ring of convergent series. Let us assume $\rho(g_j) \neq 0$ and $\rho(f_{s(j,k)}) \neq 0$, $j, k \in J$. Let $\alpha \in \Gamma$,

$$|\alpha| < \min\{\rho(g_j),\ \rho(f_{s(j,k)})\} ; j, k \in J$$

From (14) it follows that for every iterative sequence, defined by (12) (13), we have $u_j^{(t)} \in C_{|\alpha|}$, $j \in J, t \in N$, if $u_j^{(0)} \in C_{|\alpha|}, j \in J$.

By virtue of the theorem 1, if the conditions (B), (C), (D) are fulfilled, then every iterative sequence $\{u_j^{(t)}, t \in N\}$, $j \in J$, defined by (12), (13) is convergent in the weak topology of F, for any initial approximation. In what follows we consider the iterative sequence with the initial approximation $u_j^{(0)} = 0$, $j \in J$.

On the other hand, from (11), by using another $\alpha \in \Gamma$, with $|\alpha|$ less than the one considered above, it follows,

$$\| u_j^{(t)} - u_j \|_{|\alpha|} \leqslant \frac{\lambda^t}{1-\lambda} \cdot \mu, \tag{15}$$

where μ is a constant depending only on α and g_j;

From that it follows $u_j^{(t)} - u_j \in C_{|\alpha|}$ and since $\| u_j \|_{|\alpha|} \leqslant \| u_j - u_j^{(t)} \|_{|\alpha|} + \| u_j^{(t)} \|_{|\alpha|}$, one gets $u \in C_{|\alpha|}$. In addition, obviously $\lim u_j^{(t)} = u_j$. Then,

Theorem 2. *Let be the system* (1), (2). *If the conditions* (A), (B), (C), (D) *are fulfilled, then there exists an unique family of functions* $u_j : K^I \to K$, *analytic in a certain neighbourhood of the origin of* K^I, *which satisfy the relations* (1), *and the initial conditions* (2). *Power series solutions in the neighbourhood of the origin of* K^I, *are limits, in the weak topology of* F, *of any sequences of power series defined by* (12), (13), *where* $u_j^{(0)}, j \in J$, *are convergent series.*

Remarks. 1) When Card (I) = Card $(J) = 1$, one gets again the well-known result for ordinary equations of order r, of Fuchs type, if one supposes that all indices s in the right-hand members are less than r (the condition (D) is then fulfilled obviously). If Card $(J) = n$ and Card $(I) = 1$ then one obtains a result of the same sort for systems of ordinary differential equations. Hence, our theorem generalized these results for partial differential equations.

2) If (B) is replaced by the stronger condition

(B′) *for every* $j, k \in J$, *we have,*

$$|s(j,k) + q(j)| < |r(k)| + \omega(f_{s(j,k)}),$$

then $R(j) = \emptyset$, and the condition (D) is superfluous. One obtains in this way the result from [2].

3) The theorem 2 can be adapted for system with an infinity of unknown functions.

4) If $g_j = 0$ (the case of homogenous systems) or even if the condition (C) is not fulfilled, then we have to admit that the system (6) has non vanishing solution. Obviously this takes place when.

(E) *the rank of the matrix,*

$$F = \left(\frac{1}{q(j)!} f_{s(j,k)}(0) : j, k \in J \,|\, s(j,k) + q(j)| = |r(k)| \right)$$

is less than Card (J).

In this way one obtains a result concerning the existence of an analytic nonzero solution for a homogenous linear system, as follows.

Theorem 3. *Let be the problem* (1), (2), *where* $g_j = 0$. *If the conditions* (A), (B), (D), (E) *are fulfilled, then there exists at least a family of nonzero functions* $u_j : K^I \to K$, *analytic in a neighbourhood of the origin of* K^I, *which satisfy* (1), (2).

5) The conditions $\omega(f_{s(j,k)}) = |\Omega(f_{s(j,k)})|$, $\Omega(f_{s(j,k)}) = q(j)$, $s(j,k) \in R(j)$, can not be weakend. When for example $\omega(f_{s(j,k)}) = |q(j)|$, one can get that the sequence $\{a_m : m \in \Gamma\}$, is unbounded, such that the condition (9) can not be satisfied for any system of functions.

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ECUAŢII CU DERIVATE PARŢIALE DE TIP FUCHS

Rezumat

În această lucrare se obţine o generalizare a rezultatului din [2]. Sînt considerate sisteme de ecuaţii cu derivate parţiale de forma (1) cu condiţii iniţiale (2). Problema este abordată în inelul seriilor formale, înzestrat cu topologia slabă (ceea ce face ca rezultatele să fie mai generale decît în [2] unde se utiliza topologia tare) şi într-un inel convenabil de serii convergente. Sînt prinse în acest mod sisteme de ecuaţii cu derivate parţiale de tip Fuchs (teorema 2), precum şi anumite clase de sisteme (condiţia (B)) în care în membrul drept intervin derivate de ordin superior celor din membrul sting, soluţia rămînînd analitică. Teorema 3, se referă la sisteme omogene care au soluţie nenulă (condiţia (E)).