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SOLUTION OF AN INTEGRAL EQUATION BY L AND L^{-1} OPERATORS

BY

R. U. VERMA

1. Introduction. In a recent paper Fox [2] has obtained solutions of certain integral equations by using the technique of L and L^{-1} operators. These solutions need a comprehensive table of Laplace transforms and its inverse, which exist. In this work it has been shown by the application of L and L^{-1} operators that the given integral equations can be transformed into another integral equation involving the H-function, as a symmetrical Fourier kernel given in an earlier paper by the author [8] and the solution is then immediate. We can express the H-function by means of the Mellin-Barnes integral [1, p. 49]:

$$H_{p,q}^{m,n}\left(x\left|\begin{matrix}(a_i, \alpha_i)\\(b_i, \beta_i)\end{matrix}\right.\right) = H_{p,q}^{m,n}\left(x\left|\begin{matrix}(a_i, \alpha_i)_{1,p}\\(b_i, \beta_i)_{1,q}\end{matrix}\right.\right) \tag{1}$$

$$= (2\pi i)^{-1} \int_C \frac{\prod_1^m \Gamma(b_i + \beta_i s) \prod_1^n \Gamma(1 - a_i - \alpha_i s)}{\prod_{m+1}^q \Gamma(1 - b_i - \beta_i s) \prod_{n+1}^p \Gamma(a_i + \alpha_i s)} x^{-s}\, ds, \tag{2}$$

where an empty product is to be interpreted as unity and the following simplifying assumptions are made:

(i) $0 \leqslant m \leqslant q$, $0 \leqslant n \leqslant p$;

(ii) α_i and β_j are positive for $i = 1, \ldots, p$ and $j = 1, \ldots, q$;

(iii) All the poles of the integrand in (2) are simple;

(iv) The contour C is a straight line parallel to the imaginary axis in the s ($= \sigma + it$, σ and t being real) plane such that all the poles of $\Gamma(b_i + \beta_i s)$, for $i = 1, \ldots, m$ lie to the left and those of $\Gamma(1 - a_i - \alpha_i s)$ for $i = 1, \ldots, n$ lie to the right of it.

A detailed account of the convergence conditions of the integral (2) can be found in [1, p. 49]. $H_{p,q}^{m,n}(x)$ can be expressed as a sum of power series by computing the residues at the poles. These are usually an entire function. For the sake of brevity, we shall write

$H^{m,n}_{p,q}(x)$ or $H(x)$.

Verma [8] has shown that the function

(3) $$H^{m+n,\,p+q}_{2p+2q,\,2m+2n}(x) = (2\pi i)^{-1}\int_C M_{m,n,p,q}(s)\, x^{-s}\, ds,$$

where

(4) $$\begin{aligned} M_{m,n,p,q}(s) = {} & \prod_1^m \Gamma(c_j+\gamma_j(s-1/2)) \prod_1^p \Gamma(a_j-\alpha_j(s-1/2)) \\ & \cdot \prod_1^n \Gamma(d_j+\delta_j(s-1/2)) \prod_1^q \Gamma(b_j-\beta_j(s-1/2)) \\ & \cdot \left\{\prod_1^n \Gamma(d_j-\delta_j(s-1/2)) \prod_1^q \Gamma(b_j+\beta_j(s-1/2))\right\}^{-1} \\ & \left\{\prod_1^m \Gamma(c_j-\gamma_j(s-1/2)) \prod_1^p \Gamma(a_j+\alpha_j(s-1/2))\right\}^{-1}, \end{aligned}$$

behaves as a symmetrical Fourier kernel.

The following theorem of Fox [2, p. 300] will be required in finding the solution:

Theorem. *If*

(i) $\alpha > 0$, $1/2\,\alpha + \beta > 0$, $t > 0$, (ii) $s = \sigma + i\mu$, σ *and* μ *both real;*

then

(5) $$L^{-1}\left[(2\pi i)^{-1}\int_C \Gamma(\alpha s+\beta)\, F(s)\, t^{-\alpha s-\beta}\, ds\right] = (2\pi i)^{-1}\int_C F(s)\, x^{\alpha s+\beta-1}\, ds,$$

where, for both integrals, the contour C may be the line $\sigma = 1/2$, a line parallel to the imaginary axis in the complex s-plane.

The solution of the following integral equation will be developed here:

(6) $$\int_0^\infty H^{m+n+1,\,p+q}_{2p+2q+1,\,2m+2n+1}(xu)\, f(u)\, du = \Phi(x), \qquad (x > 0),$$

where $\varphi(x)$ is given and $f(x)$ is to be found. The H-function will be expressed by the following definition:

(7) $$H^{m+n+1,\,p+q}_{2p+2q+1,\,2m+2n+1}(x) = (2\pi i)^{-1}\int_C M_{m,n,p,q,1}(s)\, x^{-s}\, ds,$$

where

(8) $$\begin{aligned} M_{m,n,p,q,1}(s) = {} & \prod_1^m \Gamma(c_j+\gamma_j(s-1/2)) \prod_1^p \Gamma(a_j-\alpha_j(s-1/2)) \\ & \cdot \prod_1^n \Gamma(d_j+\delta_j(s-1/2)) \prod_1^q \Gamma(b_j-\beta_j s(-1/2)) \\ & \cdot \left\{\prod_1^n \Gamma(d_j-\delta_j(s-1/2)) \prod_1^q \Gamma(b_j+\beta_j(s-1/2))\right\}^{-1} \\ & \cdot \left\{\prod_1^m \Gamma(c_j-\gamma_j(s-1/2)) \prod_1^p \Gamma(a_j+\alpha_j(s-1/2))\right\}^{-1} \\ & \cdot \Gamma(\lambda+s)\{\Gamma(\mu+s)\}^{-1}. \end{aligned}$$

From (7) it follows that

(9) $$\mathcal{M}\,[H^{m+n+1,\,p+q}_{2p+2q+1,\,2m+2n+1}(x)] = M_{m,n,p,q,1}(s),$$

where $\mathcal{M}$ denotes the Mellin transform.

We assume the following conditions of validity are satisfied.

(i) Let $s = \sigma + it$, where σ and t are real; then the contour C along which the integral is taken is a straight line parallel to the imaginary axis in the complex s-plane whose equation is $\sigma = 1/2$.

(ii) All the poles of the integrand in (7) are simple. The contour C is such that all the poles of $\Gamma(c_j \in \gamma_j(s-1/2))$, $\Gamma(d_j+\delta_j(s-1/2))$, $\Gamma(\lambda+s)$ lie to the left and those of $\Gamma(a_j-\alpha_j(s-1/2))$, $\Gamma(b_j-\beta_j(s-1/2))$ to the right of it.

(iii) $$D = 2\left(\sum_1^m \gamma_j - \sum_1^p \alpha_j + \sum_1^n \delta_j - \sum_1^q \beta_j\right) > 0.$$

(iv) $(1/D)(\lambda-\mu) < 0$, for (7).

It may be seen by considering the asymptotic behaviour of the gamma function that the condition (iv) ensures the convergence of integral (7).

2. The Laplace and Mellin transforms. The Laplace transform of $\varphi(x)$ denoted by $L\{\varphi(x)\}$ is defined by

(10) $$L\{\varphi(x)\} = \int_0^\infty e^{-xt}\varphi(x)\, dx = \psi(t).$$

With $\varphi(x)$ and $\psi(t)$ related as in (10) the inverse Laplace transform of $\psi(t)$ is written as

(11) $$L^{-1}[\psi(t)] = \varphi(x).$$

Symbolically, we can write

$$L^{-1} L = L L^{-1} = 1. \tag{12}$$

Usually L will change the variable t to x and L^{-1} will change x to t.

If $\mathscr{M}[h(u)] = H(s)$ and $\mathscr{M}[f(u)] = F(s)$,

then the Mellin-Parseval Theorem states that

$$\int_0^\infty h(u) f(u)\, du = (2\pi i)^{-1} \int_C H(s) F(1-s)\, ds, \tag{13}$$

where C is a suitable contour in the s-plane.

3. The solution of (6) as an integral equation. In order to solve (6), we first apply (13) to the left hand side of (6). We assume $f(u) \in L_2(0, \infty)$. Then by considering the asymptotic expansion of $H^{m+n+1,\, p+q}_{2p+2q+1,\, 2m+2n+1}(u)$ for large positive u and $x > 0$ in the first case, and for small positive u and $x > 0$ in the second case, it can be easily shown that the use of Theorem 72 [6, p. 95], is justified with $k = 1/2$. Thus using (13) to the left side of (6), we have

$$\varphi(x) = (2\pi i)^{-1} \int_C M_{m,n,p,q,1}(s)\, x^{-s} F(1-s)\, ds, \quad (x > 0), \tag{14}$$

where $M_{m,n,p,q,1}(s)\, x^{-s}$ and $F(s)$ are Mellin transforms of $H^{m+n+1,\, p+q}_{2p+2q+1,\, 2m+2n+1}(xu)$ and $f(u)$ respectively.

We can write (14) in the form:

$$\varphi(x) = (2\pi i)^{-1} \int_C M_{m,n,p,q}(s) F(1-s) \frac{\Gamma(\lambda+s)}{\Gamma(\mu+s)} x^{-s}\, ds. \tag{15}$$

In this part of the problem, we shall eliminate the two gamma functions from the integrand of (15) by the methods of L and L^{-1} operators. We shall make further the simplifying asumption that : $F(1-s) \in L_2(1/2 - i\infty, 1/2 + i\infty)$. In an earlier paper of the author [8], it has been shown that, $M_{m,n,p,q}(s)$ is bounded on the line $\sigma = 1/2$. Thus, by using asymptotic expansion of the gamma function [7] along the line $s = 1/2 + it$, for large $|t|$, we can infer

$$M_{m,n,p,q}(s) F(1-s) \{\Gamma(\mu+s)\}^{-1} \in L_2(1/2 - i\infty, 1/2 + i\infty). \tag{16}$$

Hence we can use equation (5), with $\alpha = 1$, and $\beta = \lambda$, to eliminate $\Gamma(\lambda+s)$ from the integrand of (15) by means of the operator L^{-1}. Thus, we can write (15) in the form:

$$x^{-\lambda} \varphi(x) = (2\pi i)^{-1} \int_C M_{m,n,p,q}(s)\, F(1-s) \frac{\Gamma(\lambda+s)}{\Gamma(\mu+s)} \cdot x^{-s-\lambda}\, ds. \tag{17}$$

Now by applying L^{-1} operator, we have

$$L^{-1}[x^{-\lambda} \varphi(x)] = (2\pi i)^{-1} \int_C M_{m,n,p,q}(s) \frac{F(1-s)}{\Gamma(\mu+s)} t^{s+\lambda-1}\, ds. \tag{18}$$

Again, (18) can be written in the form

$$t^{\mu-\lambda} L^{-1}[x^{-\lambda} \varphi(x)] = (2\pi i)^{-1} t^{\mu} \int_C M_{m,n,p,q}(s) F(1-s) \cdot \{\Gamma(\mu+s)\}^{-1} t^{s-1}\, ds \tag{19}$$

Now we use, L operator to (19), and we have

$$L\{t^{\mu-\lambda} L^{-1}[x^{-\lambda} \varphi(x)]\}$$

$$= \int_0^\infty e^{-xt} t^{\mu} \left\{ \{(2\pi i)^{-1} \int_C M_{m,n,p,q}(s) \frac{F(1-s)}{\Gamma(\mu+s)} t^{s-1} ds \right\} dt \tag{20}$$

$$= (2\pi i)^{-1} \int_C M_{m,n,p,q}(s) F(1-s)\, x^{-s-\mu}\, ds.$$

We can write (20) in the following form:

$$x^{\mu} L\{t^{\mu-\lambda} L^{-1}[x^{-\lambda} \varphi(x)]\} = (2\pi i)^{-1} \int_C M_{m,n,p,q}(s) F(1-s)\, x^{-s}\, ds. \tag{21}$$

If we write

$$\psi(x) = \{x^{\mu} L\{t^{\mu-\lambda} L^{-1}[x^{-\lambda} \varphi(x)]\}\}, \tag{22}$$

(21) can be put into a concise form:

$$(2\pi i)^{-1} \int_C M_{m,n,p,q}(s)\, x^{-s} F(1-s)\, ds = \psi(x), \tag{23}$$

or

$$(2\pi i)^{-1} \int_C \mathscr{M}[H^{m+n,\, p+q}_{2p+2q,\, 2m+2n}(u)]\, x^{-s} F(1-s)\, ds = \psi(x), \tag{24}$$

in the light of (9).

On using [5, p. 391, eqn. (12)] to the left side of (24), it can be expressed by an integral involving the product of $H^{m+n,\, p+q}_{2p+2q,\, 2m+2n}(x)$ and $f(u)$. The result, thus found is:

$$\int_0^\infty H^{m+n,\,p+q}_{2p+2q,\,2m+2n}(xu)\,f(u)\,du = \psi(x), \tag{25}$$

where $H(xu)$ is defined in (3).

Since $H(xu)$ is a symmetrical Fourier kernel, shown in an earlier work of the author [8], we are thus led, formally to the solution as:

$$f(x) = \int_0^\infty \psi(u)\,H^{m+n,\,p+q}_{2p+2q,\,2m+2n}(xu)\,du \tag{26}$$

$$= \int_0^\infty \{u^\mu L\,\{t^{\mu-\lambda}\,L^{-1}\,[u_{-\lambda}\,\varphi(u)]\}\}\,H^{m+n,\,p+q}_{2p+2q,\,2m+2n}(xu)\,du\,. \tag{27}$$

4. **Particular cases.** (i) By taking $p = 0$, $q = 0$, $n = 0$, $m = 1$, $\lambda = 0$, $\mu = 1/2$, $\gamma_j = 1$, $j = 1,\dots, m$; $c_1 - 1/2 = a$, in (6), and using [1, p. 217, eqn. (25)], we get

$$\pi^{1/2}\,(2\sin a\pi)^{-1}\int_0^\infty [J^2_{-a}\,\{(xu)^{1/2}\} - J^2_a\,\{(xu)^{1/2}\}]\,f(u)\,du = \varphi(x),\quad (x > 0), \tag{28}$$

whose solution is

$$f(x) = \pi^{1/2}\,(2\sin a\pi)^{-1}\int_0^\infty \{u^\mu L\,\{t^{\mu-\lambda}\,L^{-1}\,[u^{-\lambda}\,\varphi(u)]\}\}\,.\;[J^2_a\,\{(xu)^{1/2}\} - J^2_a\,\{(xu)^{1/2}\}]\,du, \tag{29}$$

where $J_\nu(x)$ is the ordinary Bessel function.

(ii) Taking $n = 0$, $q = 0$ in (6), we obtain

$$\int_0^\infty H^{m+1,\,p}_{2p+1,\,2m+1}(xu)\,f(x)\,dx = \varphi,\,(x)\quad (x > 0), \tag{30}$$

whose solution is

$$f(x) = \int_0^\infty \{x^\mu L\,\{t^{\mu-\lambda}\,L^{-1}\,[u^{-\lambda}\,\varphi(u)]\}\}\,H^{m,\,p}_{2p,\,2m}(xu)\,du, \tag{31}$$

where $H^{m,\,p}_{2p,\,2m}(x)$ is the symmetrical Fourier kernel introduced by F o x [4]

Remark. Such integral equations can also be solved by following the method of usual fractional integration operators. The given integral equation can be transformed, by the application of fractional integration operators, to another integral equation with symmetrical Fourier kernel and the formal solution is then immediate. The method of fractional integration operators has been frequently employed in finding the formal solution of certain dual integral equations.

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University of Cape Coast,
Cape Coast, Ghana

SOLUŢII ALE UNOR ECUAŢII INTEGRALE CU AJUTORUL OPERATORILOR L şi L^{-1}

Rezumat

În această lucrare, folosim puterile anulatoare ale operatorilor L şi L^{-1}, pentru a găsi soluţia unor ecuaţii integrale. Ecuaţia integrală dată a fost transformată, prin aplicarea operatorilor L şi L^{-1}, într-o altă ecuaţie integrală cu nucleu Fourier simetric şi soluţia e imediată. Apoi sînt discutate unele cazuri particulare.