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ON FINITE-TIME STABILITY OF DIFFERENCE SYSTEMS

BY

PAVEL TALPALARU

1. Introduction. In certain practical situation, stability in the Liapunov sense may be irrelevant. It may be more pertinent to require some function of state to remain bounded in a particular way over a fixed finite interval rather than to consider asymptotic properties. The theory of finite-time stability has been created in reponse to such types of problems within a continuous time context. A qualitative theory for finite time stability parallel to the classical Liapunov stability theory was developed by L. Weiss and E. F. Infante [5] starting from the discussion of „practical stability" in the monograph by J. P. La Salle and S. Lefschetz [1].

We now develop some results in the theory of finite time stability fo difference control systems. Our results will be related to the works of L. Weiss and E. F. Infante [4], A. N. Michel [2] and C. P. Tsokos and S. Leela [7]. Sufficiency conditions for finite-time stability of difference systems were first obtained by A. N. Michel and S. H. Wu [3].

2. Notations and definitions. Consider the difference system

$$x(n+1) = f(n, x(n), u), \tag{1}$$

where $n \in Z^+$, the set of all positive integers, $x \in R^p$ and $u \in R^q$ is a vector representing a forcing function and in general, $u = u(n, x(n))$.

Then as long as f is a well-defined function on $Z^+ \times R^p \times R^q$, there is no problem regarding existence and uniqueness of solutions to (1) starting from any given initial condition. Let be $I_N = \{n_0, n_0 + 1, \ldots, n_0 + N\}$, where $n_0, N \in Z^+$ and $\|.\|$ the euclidean norm of a vector from R^p or R^q.

We use the following notations:

$B(a) = \{x \in R^p : \|x\| < a\}$, $\overline{B}(a) = \{x \in R^p : \|x\| \leqslant a\}$. The investigation will be carried out by using a certain class of real-valued scalar functions $V(n, x)$ defined on $I_N \times R$, continuous with respect to x. Furthermore, corresponding to such a function, we define its total difference by

$$\Delta V(n, x) = V(n+1, f(n, x(n), 0)) - V(n, x(n)),$$

where $f(n, x, u)$ is the function in the right hand side of (1). Let

$$V_M^a(n) = \max_{\|x\| \le a} V(n, x), \quad V_m^b = \min_{\|x\| \le b} V(n, x).$$

We define a discrete interval $[n_1, n_2]$ where $n_1, n_2 \in Z^+$, $n_1 < n_2$, as the set of positive integers $\{n_1, n_1 + 1, \dots, n_2\}$.
We now give the following definitions of stability on finite time discrete interval $I_N = [n_0, n_0 + N]$.

Definition 1. *The system (1) is said to be stable with respect to* $(\alpha, \gamma, \varepsilon, I_N)$, $\alpha \leq \gamma$, *if, for any trajectory* $x(n; n_0, x_0)$ *of (1), the conditions* $\|x_0\| < \alpha$ *and* $\|u(n, x)\| \leq \varepsilon$ *for all* $n \in I_N$, *and* $x \in (\bar{B}(\gamma) - B(\alpha))$, *imply that* $\|x(n; n_0, x_0)\| < \gamma$ *for all* $n \in I_N$;

Definition 2. *The system (1) is said to be quasi-contractively stable with respect to* $(\alpha, \beta, \gamma, \varepsilon, I_N)$, $\alpha \leq \beta < \gamma$, *if, for any trajectory* $x(n; n_0, x_0)$ *of* (1), *the conditions* $\|x_0\| < \alpha$ *and* $\|u(n, x)\| \leq \varepsilon$ *for all* $n \in I_N$, *and* $x \in (\bar{B}(\gamma) - B(\alpha))$, *imply*

i) *stability with respect to* $(\alpha, \gamma, \varepsilon, I_N)$, *and*

ii) *there exists* $n_1 \in I_N$ *such that* $\|x(n; n_0, x_0)\| < \beta$ *for all* $n \in (n_1, n_0 + N]$.

Definition 3. *The system (1) is said to be contractively stable with respect to* $(\alpha, \beta, \gamma, \varepsilon, I_N)$, $\beta < \alpha \leq \gamma$, *if, for any trajectory* $x(n; n_0, x_0)$ *of (1), the conditions* $\|x_0\| < \alpha$ *and* $\|u(n, x)\| \leq \varepsilon$ *for all* $n \in I_N$, *and* $x \in (\bar{B}(\gamma) - B(\beta))$, *imply*

i) *stability with respect to* $(\alpha, \gamma, \varepsilon, I_N)$, *and*

ii) *there exists* $n_1 \in I_N$ *such that* $\|x(n; n_0, x_0)\| < \beta$ *for all* $n \in (n_1, n_0 + N]$.

3. Main results. In the following we shall use an extension to finite-difference equations of a result concerning differential inequalities, given by B. G. Pachpatte [6].

Lemma. *Suppose that the scalar functions* $\omega(n, v)$ *is defined and nonnegative on* $L_N \times R$, *and monotone increasing in* v, *such that*

$$\Delta r(n) \leq \omega(n, r(n)),$$

where $\Delta r(n) = r(n + 1) - r(n)$.

Then, if $r(n_0) \leq v_0$, *we have*

$$r(n) \leq v(n; n_0, v_0),$$

for all $n \in I_N$, *where* $v(n; n_0, v_0)$ *is the solution of*

$$\Delta v(n) = \omega(n, v(n)), \quad v(n_0) = v_0 .$$

We give now a result concerning the stability in the sense of Definition 1.

Theorem 1. *Assume that the following conditions hold:*

1) *there exists a function* $V(n, x)$ *defined on* $I_N \times R^p$, *continuous in* x *satisfying*

$$|V(n, x) - V(n, y)| \leq K(n) \|x - y\|, \tag{2}$$

for $n \in I_N$, $x, y \in (\bar{B}(\gamma) - B(\alpha))$, *where* $K(n) \geq 0$ *for* $n \in I_N$;

2) *the function* $\omega(n, v)$ *is defined on* $I_N \times R$ *and the difference inequality*

$$\Delta V(n, x)) \leq \omega(n, V(n, x)) \tag{3}$$

is verified for $n \in I_N$, $x \in (\bar{B}(\gamma) - B(\alpha))$;

3) *for* $n \in I_N$, $x \in (\bar{B}(\gamma) - B(\alpha))$.

$$\|f(n, x, u) - f(n, x, 0)\| \leq \varphi(\|u\|), \tag{4}$$

where $\varphi(t)$ *is nondecreasing in* t *and* $\varphi(0) = 0$;

4) *every solution* $v(n; n_1, v_0)$ *of*

$$\Delta v(n) = \omega(n, v(n)) + \varphi(\varepsilon) K(n), \quad v(n_1) = v_0, \quad n_1 \in I_N, \text{ satisfies} \tag{5}$$

$$v(n; n_1, v_0) < V_m^\gamma(n), \quad n > n_1 . \tag{6}$$

Then, if $v_0 = V_M^\alpha(n_1)$, *the difference control system* (1) *is stable with respect to* $(\alpha, \gamma, \varepsilon, I_N)$.

Proof. Let $0 < \alpha < \gamma$, $\varepsilon > 0$, $n_0 \in I_N$ be given and suppose that

$$\|x_0\| < \alpha \text{ and } \|u(n, x)\| \leq \varepsilon \text{ for } n \in I_N, \ x \in (\bar{B}(\gamma) - B(\alpha)).$$

The proof is by contradiction. Suppose $\|x_0\| < \alpha$ and that there exist n_1, $n_2 \in I_N$ such that for the solution $x(n) = x(n; n_0, x_0)$ we have

$$\|x(n_1; n_0, x_0)\| \leq \alpha, \qquad \|x(n_2; n_0, x_0)\| \geq \gamma$$

and

$$\alpha < \|x(n; n_0, x_0)\| < \gamma \text{ for } n \in (n_1, n_2).$$

For $n \in (n_1, n_2)$, $x(n; n_0, x_0) \in (\bar{B}(\gamma) - B(\alpha))$ and hence $\|u(n, x(n; n_0, x_0))\| \leq \varepsilon$, $n \in (n_1, n_2]$. Define, for $n \in (n_1, n_2]$, $r(n) = V(n, x(n))$. Then, for $n \in (n_1, n_2]$ we have:

$$r(n + 1) = V(n + 1, x(n + 1)) = V(n + 1, x(n + 1)) -$$
$$- V(n + 1, f(n, x(n), 0)) + V(n + 1, f(n, x(n), 0)) \leq$$
$$\leq K(n) \|x(n + 1) - f(n, x(n), 0)\| + V(n + 1, f(n, x(n), 0)).$$

and by (3) and (4) we obtain

$$\Delta r(n) \leq K(n) \varphi(\varepsilon) + \omega(n, r(n)). \tag{7}$$

Choose $v_0 = V_M^\alpha(n_1)$. Then, since $V(n_1, x(n_1)) \leq V_M^\alpha(n_1) = v_0$, by application of Lemma, it follows that

$$r(n) = V(n, x(n)) \leq v(n; n_1, v_0), \text{ for } n \in (n_1, n_2].$$

By the definition of $V_m^{\gamma}(n)$ and the relation (6), we have at $n = n_2$.

$$V_m^{\gamma}(n_2) \leqslant V(n_2, x(n_2)) \leqslant v(n_2; n_1, x_0) < V_m^{\gamma}(n_2),$$

but this last inequality is a contradiction, which proves the theorem.

Corollaries. 1. If $\omega(n, v) \equiv \psi(n)$, and if we suppose that the hypotheses 1) and 3) hold and in addition

2′) $\qquad \Delta V(n, x) \leqslant \psi(n)$, for $n \in I_N$ and $x \in (\bar{B}(\gamma) - B(\alpha))$;

3′) $\qquad \sum\limits_{j=n_1}^{n-1} [\varphi(\varepsilon) K(j) + \psi(j)] < V_m^{\gamma}(n) - V_M^{\alpha}(n_1)$, for all $n \in I_N$;

the theorem is also true.

Indeed, in this case we have:

$$\Delta v(n) = \varphi(\varepsilon) K(n) + \psi(n),$$

with the solution $v(n; n_1, v_0)$ given by

$$v(n; n_1, v_0) = \sum_{j=n_1}^{n-1} [\varphi(\varepsilon) K(j) + \psi(j)] + V_M^{\alpha}(n_1),$$

if we choose $v_0 = V_M^{\alpha}(n_1)$.

2. In the particular case $K(n) \equiv K = \text{const.}$, the hypothesis 3′) becomes

3″) $\qquad \sum\limits_{j=n_1}^{n-1} \psi(j) \leqslant V_m^{\gamma}(n) - V_M^{\alpha}(n_1) - (n - n_1 - 1) K\varphi(\varepsilon)$, for all $n \in I_N$.

Remark 1. When $\alpha = \gamma$, the Theorem can be formulated as follows: a) *for every* δ *such that* $0 < \delta < \alpha$, *the hypotheses* 1), 2) *and* 3) *of Theorem* 1 *hold except that the inequality* (2) *is valid for* $x, y \in (\bar{B}(\alpha) - B(\delta))$ *and the relations* (3), (4) *are true for* $x \in (\bar{B}(\alpha) - B(\delta))$;
b) *every solution* $v(n; n_1, v_0)$ of (5) *satisfies*

$$v(n; n_1, v_0) < V_m^{\alpha}(n), \quad n > n_1,$$

if $v_0 = V_M^{\delta}(n_1)$.

We present now a theorem on quasi-contractive stability

Theorem 2. *Assume that the following conditions hold:*
1) *the hypotheses* 1), 2) *and* 3) *of Theorem 1*;
2) *every solution* $v(n; n_1, v_0)$ *of* (5) *satisfies.*

(8) $\quad v(n; n_1, v_0) < V_m^{\gamma}(n), \; n > n_1, \; n_1 \in I_N$, *provided* $v_0 = V_M^{\alpha}(n_1)$;

(9) $\quad v(n_0 + N; n_1, v_0) < V_m^{\beta}(n_0 + N), \; n_1 \in I_N$, *provided* $v_0 = V_M^{\beta}(n_1)$;

(3) $\qquad V(n_0 + N, x) \geqslant V_m^{\beta}(n_0 + N), \; x \in (B(\gamma) - B(\beta))$.

Then, the difference control system (1) *is quasi-contractively stable with respect to* $(\alpha, \beta, \gamma, \varepsilon, I_N)$, $\alpha \leqslant \beta < \gamma$.

Proof. The system is stable with respect to $(\alpha, \gamma, \varepsilon, I_N)$ by hypotheses 1) and 2) — relation (8). Now, if there is no trajectory $x(n) = x(n; n_0, x_0)$, where $\| x_0 \| < \alpha$, which passes the boundary of $B(\beta)$, there is nothing to prove. Hence, consider an arbitrary trajectory $x(n)$, where $\| x(n_0) \| = \| x_0 \| < \alpha$ and suppose there exists $n_1 \in I_N$ such that $\| x(n_1; n, x_0) \| \leqslant \beta$ and $\| x(n; n_0, x_0) \| > \beta$ for all $n \in (n_1, n_0 + N]$. For $n \in (n_1, n_0 + N]$, $x(n; n_0, x_0) \in (B(\gamma) - B(\alpha))$ and therefore $\| u(n, x(n; x_0, x_0)) \| \leqslant \varepsilon$. If we consider the difference inequality (7), where $r(n) = V(n, x(n))$ for $n \in (n_1, n_0 + N]$, and if we choose $v_0 = V_M^{\beta}(n_1)$, by the application of Lemma it follows that

(10) $\qquad r(n) = V(n, x(n)) \leqslant v(n; n_1, x_0)$, for $n \in (n_1, n_0 + N]$.

Taking $n = n_0 + N$, and using the hypotheses 2) — relation (9), and (3), since $x(n_0 + N) \leqslant (B(\gamma) - B(\beta))$, one obtains $V_m^{\beta}(n_0 + N) \leqslant V(n_0 + N, x(n_0 + N)) \leqslant v(n_0 + N; n_1, v_0) < V_m^{\beta}(n_0 + N)$, which is a contradiction.

Corollary 3. When $\omega(n, v) \equiv \psi(n)$, if we suppose that the hypotheses 1) and 3) of the Theorem 1 hold and in addition,

$\bar{2}$) $\Delta V(n, x) \leqslant \psi(n)$, for $n \in I_N$ and $x \in (\bar{B}(\gamma) - B(\alpha))$;

$\bar{3}$) $\sum\limits_{j=n_1}^{n-1} [\varphi(\varepsilon) K(j) + \psi(j)] < V_m^{\gamma}(n) - V_M^{\alpha}(n_1)$, for all $n \in I_N$;

$\bar{4}$) $\sum\limits_{j=n_1}^{n_1+N} [\varphi(\varepsilon) K(j) + \psi(j)] < V_m^{\beta}(n_0 + N) - V_M^{\beta}(n_1)$, for $n_1 \in I_N$;

$\bar{5}$) $V(n_0 + n, x) \geqslant V_m^{\beta}(n_0 + N)$, for $x \in (B(\gamma) - B(\beta))$:

then the theorem is also valid.

Finally, we give a theorem on contractive stability for systems of the form (1).

Theorem 3. *Assume that the following conditions hold:*
1) *the hypotheses* 1), 2) *and* 3) *of Theorem 1 except that the inequality* (2) *is valid for* $x, y \in (\bar{B}(\gamma) - B(\beta))$ *and* (3), (4) *are true for* $x \in (\bar{B}(\gamma) - B(\beta))$;
2) *the relations* (8) *and* (9) *and the hypothesis* 3) *of Theorem* 2 *together with*

(11) $\qquad v(n_0 + N; n_0, V_{\beta}^{\alpha}(n_0)) < V_m^{\beta}(n_0 + N),$

where $V_{\beta}^{\alpha}(n_0) = \max\limits_{x \in (\bar{B}(\alpha) - B(\beta))} V(n_0, x)$;

Then, the difference control system (1) *is contractively stable with respect to* $(\alpha, \beta, \gamma, \varepsilon, I_N)$, $\beta < \alpha < \gamma$.

Proof. By hypotheses 1), 2) and 3) of Theorem 1 and the relation (8), the sistem is stable with respect to $(\alpha, \gamma, \varepsilon, I_N)$. Consider an arbitrary trajectory

$x(n) = x(n; n_0, x_0)$, where $\|x_0\| < \alpha$ and suppose $\|x(n)\| \geqslant \beta$ for all $n \in I_N$. Then, for $n \in I_N$, we have $x(n; n_0, x_0) \in (\bar{B}(\gamma) - B(\beta))$ and hence $\|u(n, x(n; n_0, x_0))\| \leqslant \varepsilon$. If we consider the difference inequality (7), where $v(n) = V(n, x(n))$, for $n \in I_N$, and if we choose $v_0 = V_\beta^\alpha(n_0)$, by the application of Lemma it follows

$$r(n) = V(n, x(n)) \leqslant v(n; n_0, V_\beta^\alpha(n_0)), \text{ for } n \in I_N .$$

Taking $n = n_0 + N$, and using the relation (11), since $x(n_0 + N) \in$ $\in (\bar{B}(\gamma) - B(\beta))$, we have

$$V(n_0 + N, x(n_0 + N)) \leqslant v(n_0 + N; n_0, V_\beta^\alpha(n_0)) < V_m^\beta(n_0 + N).$$

But, by hypothesis 3) of Theorem 2, this is a contradiction: hence, there exists $n_1 \in I_N$ for which $\|x(n_1)\| < \beta$. The remaider of the proof follows that of Theorem 2.

Remark 2. In the case $\alpha = \gamma$ and $\beta < \alpha$, a result analogue to that given in Remark 1 can be formulated.

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ASUPRA STABILITĂŢII SISTEMELOR CU DIFERENŢE PE INTERVAL FINIT

Rezumat

În această lucrare se consideră sistemul de control de forma (1) în diferenţe finite pentru care se definesc noţiunile de stabilitate în raport cu $(\alpha, \gamma, \varepsilon, I)$, stabilitate cuasicontractivă şi stabilitate contractivă în raport cu $(\alpha, \beta, \gamma, \varepsilon, I)$.

Utilizînd o funcţie $V(n, x)$ cu anumite proprietăţi, mai puţin acelea de a fi pozitiv definită sau descrescentă, se dau criterii de stabilitate în sensul introdus.

Rezultatele obţinute extind, la sistemele în diferenţe finite, unele rezultate asupra stabilităţii sistemelor diferenţiale ordinare, pe interval finit din [4] şi [7].

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ON A GENERALIZATION OF MEIJER TRANSFORM

BY

GOPAL JI SAHAI

1. Introduction. The generalized Meijer transform is defined by the integral relation*)

$$\Phi(p) = p\int_0^\infty e^{-hpx}(px)^l W_{k+\frac{1}{2}, m}(px)\, G_{\gamma,\delta}^{\alpha,\beta}\left[cpx \,\middle|\, \begin{matrix} a_\gamma \\ b_\delta \end{matrix}\right] f(x)\, dx = \tag{1.1}$$

$$= p\int_0^\infty Z(px) f(x)\, dx,$$

where $W_{k+\frac{1}{2},m}(z)$ is Whittaker function and $G_{\gamma,\delta}^{\alpha,\beta}\left[x \,\middle|\, \begin{matrix} a_\gamma \\ b_\delta \end{matrix}\right] = \partial$ is the well known Meijer's G-function. The relation (1.1) exists provided

(i) $\qquad f(x) = \mathbf{0}(x^{\lambda'})$ for small x,

$\mathrm{Re}(l + \lambda' \pm m + b_j + 3/2) > 0$, $j = 1, 2, \ldots, \alpha$.

(ii) $\qquad \delta > \gamma$; $2(\alpha + \beta) \geqslant \gamma + \delta$, $\mathrm{Re}\left(h + \dfrac{1}{2}\right) > 0$, $|\arg c| < \left[\alpha + \beta - \dfrac{1}{2}(\gamma + \delta)\right]\pi$,

(iii) $\qquad Z(p_0 x)$ is bounded for $x \geqslant 0$; $\mathrm{Re}(p) > R(P_0) > 0$.

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*) $G_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix}\right]$ denote $G_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix}\right]$

$\mathrm{Re}(\alpha \pm \beta) > 0$ denote $\mathrm{Re}(\alpha + \beta) > 0$, $\mathrm{Re}(\alpha - \beta) > 0$