$x(n) = x(n; n_0, x_0)$, where $\| x_0 \| < \alpha$ and suppose $\| x(n) \| \geqslant \beta$ for all $n \in I_N$. Then, for $n \in I_N$, we have $x(n; n_0, x_0) \in (\bar{B}(\gamma) - B(\beta))$ and hence $\| u(n, x(n; n_0, x_0)) \| \leqslant \varepsilon$. If we consider the difference inequality (7), where $v(n) = V(n, x(n))$, for $n \in I_N$, and if we choose $v_0 = V_\beta^\alpha(n_0)$, by the application of Lemma it follows

$$r(n) = V(n, x(n)) \leqslant v(n; n_0, V_\beta^\alpha(n_0)), \text{ for } n \in I_N.$$

Taking $n = n_0 + N$, and using the relation (11), since $x(n_0 + N) \in (\bar{B}(\gamma) - B(\beta))$, we have

$$V(n_0 + N, x(n_0 + N)) \leqslant v(n_0 + N; n_0, V_\beta^\alpha(n_0)) < V_m^\beta(n_0 + N).$$

But, by hypothesis 3) of Theorem 2, this is a contradiction; hence, there exists $n_1 \in I_N$ for which $\| x(n_1) \| < \beta$. The remainder of the proof follows that of Theorem 2.

Remark 2. In the case $\alpha = \gamma$ and $\beta < \alpha$, a result analogue to that given in Remark 1 can be formulated.

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ASUPRA STABILITĂŢII SISTEMELOR CU DIFERENŢE PE INTERVAL FINIT

Rezumat

În această lucrare se consideră sistemul de control de forma (1) în diferențe finite pentru care se definesc noțiunile de stabilitate în raport cu (α, γ, ε, I), stabilitate cuasicontractivă și stabilitate contractivă în raport cu (α, β, γ, ε, I).

Utilizînd o funcție $V(n, x)$ cu anumite proprietăți, mai puțin acelea de a fi pozitiv definită sau descrescentă, se dau criterii de stabilitate în sensul introdus.

Rezultatele obținute extind, la sistemele în diferențe finite, unele rezultate asupra stabilității sistemelor diferențiale ordinare, pe interval finit din [4] și [7].

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ON A GENERALIZATION OF MEIJER TRANSFORM

BY

GOPAL JI SAHAI

1. Introduction. The generalized Meijer transform is defined by the integral relation*)

$$\Phi(p) = p \int_0^\infty e^{-hpx} (px)^l W_{k+\frac{1}{2}, m}(px)\, G_{\gamma,\delta}^{\alpha,\beta}\left[cpx \,\middle|\, \begin{matrix} a_\gamma \\ b_\delta \end{matrix} \right] f(x)\, dx = \tag{1.1}$$

$$= p \int_0^\infty Z(px) f(x)\, dx,$$

where $W_{k+\frac{1}{2},m}(z)$ is Whittaker function and $G_{\gamma,\delta}^{\alpha,\beta}\left[x \,\middle|\, \begin{matrix} a_\gamma \\ b_\delta \end{matrix} \right] = \partial$ is the well known Meijer's G-function. The relation (1.1) exists provided

(i) $\qquad f(x) = 0(x^{\lambda'})$ for small x,

$\mathrm{Re}(l + \lambda' \pm m + b_j + 3/2) > 0$, $j = 1, 2, \ldots, \alpha$.

(ii) $\qquad \delta > \gamma$; $2(\alpha + \beta) \geqslant \gamma + \delta$, $\mathrm{Re}\left(h + \dfrac{1}{2}\right) > 0$, $|\arg c| < \left[\alpha + \beta - \dfrac{1}{2}(\gamma + \delta)\right]\pi$,

(iii) $\qquad Z(p_0 x)$ is bounded for $x \geqslant 0$; $\mathrm{Re}(p) > R(P_0) > 0$.

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*) $G_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} a_p \\ b_q \end{matrix} \right]$ denote $G_{p,q}^{m,n}\left[x \,\middle|\, \begin{matrix} a_1, a_2, \ldots, a_p \\ b_1, b_2, \ldots, b_q \end{matrix} \right]$

$\mathrm{Re}(\alpha \pm \beta) > 0$ denote $\mathrm{Re}(\alpha + \beta) > 0$, $\mathrm{Re}(\alpha - \beta) > 0$

2. In this section we state the known results which will be used in this paper

$$x^l e^{-x/2} W_{k+\frac{1}{2},m}(px) = {}^*G^{2,0}_{1,2}\left[x\,\middle|\,\begin{matrix} l-k+\frac{1}{2} \\ l+m+\frac{1}{2}\end{matrix}\right], \tag{2.1}$$

$$e^{-x} = G^{1,0}_{0,1}\left[x\,\middle|\,\begin{matrix} - \\ 0\end{matrix}\right]. \tag{2.2}$$

The integral of [2, p. 198] may be written as :

$$\int_0^\infty G^{A,B}_{C,D}\left[ax\,\middle|\,\begin{matrix} e_C \\ f_D\end{matrix}\right] G^{h,o}_{q,r}\left[bx\,\middle|\,\begin{matrix} \alpha_q \\ \beta_r\end{matrix}\right] G^{\alpha,\beta}_{\gamma,\delta}\left[cx\,\middle|\,\begin{matrix} a_\gamma \\ b_\delta\end{matrix}\right] dx =$$

$$= \frac{1}{b} S\left[\begin{matrix} \frac{a}{b} \\ \frac{c}{b}\end{matrix}\,\middle|\,\left|\begin{matrix} h, & o \\ r-h, & q\end{matrix}\right]\begin{matrix}[\Delta(1, \beta_r+1)] \\ [\Delta(1, \alpha_q+1)]\end{matrix}\,\middle|\,\begin{pmatrix} B, & A \\ C-B, & D-A\end{pmatrix}\begin{matrix} e_C \\ f_D\end{matrix}\,\middle|\right. \tag{2.3}$$

$$\left.\left|\begin{pmatrix} \beta, & \alpha \\ \gamma-\beta, & \delta-\alpha\end{pmatrix}\begin{matrix} a_\gamma \\ b_\delta\end{matrix}\right|\right],$$

where $\Delta(s, a) = \frac{a}{s}, \frac{a+1}{s}, \ldots, \frac{a+s-1}{s}$;

$$[\Delta(s, a_r)] = \Delta(s, a_1), \Delta(s, a_2), \ldots \Delta(s, a_r)$$

and S-function is introduced by S h a r m a [4, p. 27] provided the conditions given in the result are satisfied.

The afore said transform (1.1) is of general nature in the sense that the other recent transforms may be obtained on specializing the parameters involved therein.

3. Particular cases. Taking $\alpha = \delta = 1$ and $\beta = \gamma = 0$ and using (2.2), (1.1) reduces to

$$\Phi(p) = p\int_0^\infty e^{-(h+c)px}(px)^l W_{k+\frac{1}{2},m}(px) f(x)\, dx. \tag{3.1}$$

Further if we take $h = a - C$ then (3.1) reduces to P a n d e y [3, p. 339]

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*) $G^{2,0}_{1,2}\left[x\,\middle|\,\begin{matrix} a_1 \\ b_1 \pm c_1\end{matrix}\right]$ denotes $G^{2,0}_{1,2}\left[x\,\middle|\,\begin{matrix} a_1 \\ b_1+c_1, \ b_1-c_1\end{matrix}\right]$.

Again when $h + c = \frac{1}{2}$, $l = -k - \frac{1}{2}$ and $k = \pm m$ in (3.1) the Laplace transform comes out as a particular case of (1.1).

4. In this section an inversion formula for the transform (1.1) is established.

Theorem. *If $\Phi(p)$ is the generalized Meijer transform of $f(x)$ defined as in (1.1), then*

$$\frac{1}{2}[f(x+) + f(x-)] = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{\sigma-iT}^{\sigma+iT} M(s)\, K(s)\, x^{-s}\, ds. \tag{4.1}$$

where

$$K(s) =$$

$$= \left\{S\left|\begin{matrix} h-1/2 \\ c\end{matrix}\,\middle|\,\begin{matrix} 2, 0 \\ 0, 1\end{matrix}\right|\begin{matrix} l-s+3/2 \pm m \\ l-s-k+3/2\end{matrix}\,\middle|\,\begin{pmatrix} 0, 1 \\ 0, 0\end{pmatrix}\begin{matrix} - \\ 0\end{matrix}\,\middle|\,\begin{pmatrix} \beta, & \gamma \\ \gamma-\beta, & \delta-\alpha\end{pmatrix}\begin{matrix} a_\gamma \\ b_\delta\end{matrix}\right|\right\}^{-1} \tag{4.A}$$

or

$$K(s) = \left\{\sum_{r=0}^{\infty}\frac{\left(\frac{1}{2}-h\right)^r}{r!} G^{\alpha,\beta+2}_{\gamma+2,\delta+1}\left[c\,\middle|\,\begin{matrix} -l-r\pm m-\frac{1}{2}-s, \ a_\gamma \\ b_\delta, -l-r+k-\frac{1}{2}-s,\end{matrix}\right]\right\}^{-1} \tag{4.B}$$

(The two values of $K(s)$ [(4.A) and (4.B)] have been evaluated by different methods which give an interesting relation between G-function and S-function of two variables) and

$$M(s) = \int_0^\infty \Phi(p)\, p^{-s-1}\, dp,$$

provided that the integral

(i) $\int_0^\infty t^{\sigma-1} f(t)\, dt$ *and* $\int_0^\infty t^{-\sigma-1}\, \Phi(t)\, dt$

are absolutely convergent [$s = \sigma + it$, $0 < T < \infty$]

(ii) *$f(t)$ is of bounded variation in the nhd of the point $t = x$.*

(iii) $f(t) = O(t)^{\lambda}$

$$\operatorname{Re}\left(l + \lambda' \pm m + b_j + \frac{3}{2}\right) > 0,\ j = 1, 2, \ldots, \alpha,$$

$$\delta > \gamma,\ 2(\alpha + \beta) \geqslant \gamma + \delta,\ \operatorname{Re}\left(h + \frac{1}{2}\right) > 0,$$

$$\left|\arg\left(h - \frac{1}{2}\right)\right| < \frac{1}{2}\pi,\ |\arg c| < \left[\alpha + \beta - \frac{1}{2}(\gamma + \delta)\right]\pi.$$

(iv) *$z(p_0 t)$ is bounded for $t \geqslant 0$.*

$$R(p) > R(p_0) > 0.$$

However if $f(t)$ is continuous at $t = x$ then

$$f(x) = \frac{1}{2\pi i} \lim_{T\to\infty} \int_{\sigma - iT}^{\sigma + iT} M(s)\, K(s)\, x^{-s}\, ds. \tag{4.2}$$

Proof. Multiplying both sides of (1.1) by p^{-s-1} and integrating with respect to p between the limits 0 and ∞, we get

$$M(s) = \int_0^\infty p^{-s} \left\{ \int_0^\infty e^{-hpx} (px)^l\, W_{k+\frac{1}{2}, m}(px)\, G^{\alpha,\beta}_{\gamma,\delta}\left[cpx \,\middle|\, \begin{matrix} a_\gamma \\ b_\delta \end{matrix}\right] f(x)\, dx \right\} dp =$$

$$= \int_0^\infty x^{s-1} f(x)\, dx \int_0^\infty e^{-hpx} (px)^{l-s}\, W_{k+\frac{1}{2}, m}(px)\, G^{\alpha,\beta}_{\gamma,\delta}\left[cpx \,\middle|\, \begin{matrix} a_\gamma \\ b_\delta \end{matrix}\right] d(px),$$

provided that the change in order of integration is justifiable.

Evaluating the later integral with the help of (2.1), (2.2) and (2.3) we get

$$M(s)\, K(s) = \int_0^\infty x^{s-1} f(x)\, dx, \tag{4.3}$$

provided that

$$\operatorname{Re}\left((l + m + b_j + \frac{3}{2} - s\right) > 0,\quad j = 1, 2, \ldots \alpha$$

$$\operatorname{Re}\left(h + \frac{1}{2}\right) > 0,\ |\arg c| < \left[\alpha + \beta - \frac{1}{2}(\gamma + \delta)\right]\pi,$$

$$\left|\arg\left(h - \frac{1}{2}\right)\right| < \pi/2,\ \delta > \gamma,\ 2(\alpha + \beta) \geqslant \gamma + \delta.$$

Now applying Mellin's inversion formula [5, p. 7] in (4.3) we get (4.1), provided the condition (i) and (ii) are satisfied, which completes the proof.

To justify the change of order of integration, we see that the x integral is absolutely convergent if the generalized Meijer Transform of $|f(x)|$ exist, the p integral converges absolutely if $\operatorname{Re}(l - s \pm m + b_j + 3/2) > 0$ $= 1, 2, \ldots, \alpha$.

$$\operatorname{Re}(h + 1/2) > 0,\ \delta > \gamma,\ 2(\alpha + \beta) \geqslant \gamma + \delta \text{ and } |\arg c| < \left[\alpha + \beta - \frac{1}{2}(\gamma + \delta)\right]\pi$$

and the resulting integral is absolutely convergent if $x^{-s} f(x)$ belongs to $L(0, \infty)$. Hence the change of order of integration is justifiable by de la Vallée Poussin Theorem [1, p. 204].

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ASUPRA UNEI GENERALIZĂRI A TRANSFORMATEI MEIJER

Rezumat

Obiectul acestui articol este de a defini prin (1.1) o nouă transformare Meijer. Relația (4.1) dă formula de inversiune.