Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XX, s. I a, fasc. 2, 1974

THE BENDING OF NATURALLY TWISTED AND ANISOTROPIC BEAMS BY A TRANSVERSE LOAD

BY

C. I. BORŞ

Let us consider a beam limited by two planes $x_3 = 0$, $x_3 = h$ $(h > 0)$ and a surface $\mathcal{F}$, the equation of which is given by

$$f(x_1 - kx_2 x_3,\; x_2 + kx_1 x_3) = 0, \tag{1}$$

where k is a small parameter such that the square and higher powers of k can be neglected.

We shall call such beams *slightly twisted beams*. Some problems concerning such beams were already studied in [1, 2], etc.

The problem of bending by a transverse load when the material of the beam is anisotropic with one plane of elastic symmetry is discussed in this paper.

As in the previous papers let us make up the transformation

$$\xi_1 = x_1 - kx_2 x_3, \qquad \xi_2 = x_2 + kx_1 x_3, \qquad \xi_3 = x_3. \tag{2}$$

After this transformation the surface (1) becomes

$$f(\xi_1, \xi_2) = 0, \tag{3}$$

which is a cylindrical surface $\mathcal{F}_1$ in the space ξ_1, ξ_2, ξ_3.

We shall denote by $\mathcal{V}$ the region occupied by the beam (1) and by $\mathcal{V}_1$ the region occupied by the beam (3). We shall denote by S the domain and also the area of the cross-section of the cylindrical beam (3) and by Γ the boundary of S.

It is easy to prove the following formulae:

$$\begin{cases} \dfrac{\partial}{\partial x_1} = \dfrac{\partial}{\partial \xi_1} + k\xi_3 \dfrac{\partial}{\partial \xi_2}, \qquad \dfrac{\partial}{\partial x_2} = \dfrac{\partial}{\partial \xi_2} - k\xi_3 \dfrac{\partial}{\partial \xi_1}, \\ \dfrac{\partial}{\partial x_3} = \dfrac{\partial}{\partial \xi_3} - k\left(\xi_2 \dfrac{\partial}{\partial \xi_1} - \xi_1 \dfrac{\partial}{\partial \xi_2}\right) \end{cases} \tag{4}$$

and

(5) $$n_1 = \cos\alpha + k\xi_3\cos\beta,\quad n_2 = \cos\beta - k\xi_3\cos\alpha,$$
$$n_3 = -k\,(\xi_2\cos\alpha - \xi_1\cos\beta),$$

where n_i are the direction cosines of the exterior normal to the surface (1) and $\cos\alpha$, $\cos\beta$ are the direction cosines of the exterior normal to the curve Γ.

We take the axis ξ_3 to be central-line of the beam and the axes of ξ_1 and ξ_2 to be the principal axes of inertia of the end $\xi_3 = 0$. In this case we have

(6) $$\iint_S \xi_1\,d\sigma = 0,\qquad \iint_S \xi_2\,d\sigma = 0,\qquad \iint_S \xi_1\,\xi_2\,d\sigma = 0.$$

Let the end $x_3 = 0$ be fixed and let the tractions applied on the end $x_3 = h$ be statically equivalent to a transverse force F_1 parallel to the ξ_1-axis and passing through the centroid of the end.

The lateral surface of the beam is supposed to be free from tractions so that we have no body forces in the domain $\mathcal{V}$.

Under these assumptions the stress components σ_{ij} must satisfy the equilibrium equations

(7) $$\sigma_{ij,j} = 0 \quad \text{in } \mathcal{V}^{*)}$$

and the boundary conditions

(8) $$\sigma_{ij}\,n_j = 0 \qquad \text{on } \mathcal{F}.$$

In our case the relations between the stress components σ_{ij} and the strain components γ_{ij} are given by

(9) $$\begin{cases} \sigma_{11} = A\,\gamma_{11} + H\,\gamma_{22} + G\,\gamma_{33} + Q\gamma_{12}, \\ \sigma_{22} = H\gamma_{11} + B\gamma_{22} + F\gamma_{33} + R\gamma_{12}, \\ \sigma_{33} = G\gamma_{11} + F\gamma_{22} + C\gamma_{33} + T\gamma_{12}, \\ \sigma_{12} = Q\gamma_{11} + R\gamma_{22} + T\gamma_{33} + D\gamma_{12}, \\ \sigma_{23} = L\gamma_{23} + N\gamma_{31}, \\ \sigma_{31} = N\gamma_{23} + M\gamma_{31}, \end{cases}$$

where $A, B, \ldots, L, M, N$ are moduli of elasticity. The inverse relations of (9) are taken in the form

*) We use the summation convention over the repeated indices. The index j, following comma, indicates partial differentiation with respect to x_j.

(10) $$\begin{cases} \gamma_{11} = \dfrac{1}{E}(\nu_{11}\,\sigma_{11} + \nu_{12}\,\sigma_{22} + \nu_{13}\,\sigma_{12} - \nu_1\,\sigma_{33}), \\ \gamma_{22} = \dfrac{1}{E}(\nu_{12}\,\sigma_{11} + \nu_{22}\,\sigma_{22} + \nu_{23}\,\sigma_{12} - \nu_2\,\sigma_{33}), \\ \gamma_{12} = \dfrac{1}{E}(\nu_{13}\,\sigma_{11} + \nu_{23}\,\sigma_{22} + \nu_{33}\,\sigma_{12} - \nu_3\,\sigma_{33}), \\ \gamma_{33} = \dfrac{1}{E}(-\nu_1\,\sigma_{11} - \nu_2\,\sigma_{22} - \nu_3\,\sigma_{12} + \sigma_{33}), \\ \gamma_{23} = \dfrac{1}{LM - N^2}(M\sigma_{23} - N\sigma_{31}), \\ \gamma_{31} = \dfrac{-1}{LM - N^2}(N\sigma_{23} - L\sigma_{31}), \end{cases}$$

where the strain coefficients E, ν_{ij}, ν_i can be expressed in terms of moduli of elasticity [2, p. 197].

The strains γ_{ij} are connected with the displacements u_i by

$$\gamma_{ii} = u_{i,i} \ \text{(not summed)}$$
$$\gamma_{ij} = u_{i,j} + u_{j,i}\,(i \neq j)$$

and they must satisfy the conditions of compatibility of Saint-Venant.

In order to solve our problem let us suppose that the displacements u_i are of the form

(11) $$\begin{cases} u_1 = -\tau\xi_2\,\xi_3 + a\left[\dfrac{1}{2}(\nu_1\,\xi_1^2 - \nu_2\,\xi_2^2)\,(h - \xi_3) + \dfrac{1}{2}\,h\xi_3^2 - \dfrac{1}{6}\,\xi_3^3\right] + ku_1^0, \\ u_3 = \tau\xi_1\,\xi_3 + a\,(\nu_2\,\xi_1\,\xi_2 + \dfrac{1}{2}\,\nu_3\,\xi_1^2)\,(h - \xi_3) + ku_3^0, \\ u_3 = \tau\varphi\,(\xi_1,\ \xi_2) - a\left[\chi\,(\xi_1, \xi_2) + \xi_1\,\xi_3\left(h - \dfrac{1}{2}\,\xi_3\right)\right] + ku_3^0, \end{cases}$$

where u_i^0 are unknown complementary displacements, τ and a are constants depending of the end conditions, φ and χ being the function of torsion and the function of flexure of the beam (3). They are defined by

(12) $$\begin{cases} \Delta\varphi = 0 \quad \text{in } S, \\ \mathcal{D}\,\varphi = (M\xi_2 - N\xi_1)\cos\alpha + (N\xi_2 - L\xi_1)\cos\beta \quad \text{on } \Gamma \end{cases}$$

and

$$(13)\qquad \begin{cases} \Delta\chi + (M\nu_1 + L\nu_2 + N\nu_3 - E)\,\xi_1 = 0 \quad \text{in } S. \\ \mathcal{D}\chi = -\Bigl[\frac{1}{2} M(\nu_1\xi_1^2 - \nu_2\xi_2^2) + N\Bigl((\nu_2\xi_1\xi_2 - \frac{1}{2}\nu_2\xi_1^2\Bigr)\Bigr]\cos\alpha - \\ -\Bigl[\frac{1}{2} N(\nu_1\xi_1^2 - \nu_2\xi_2^2) + L\Bigl(\nu_2\xi_1\xi_2 + \frac{1}{2}\nu_3\xi_1^2\Bigr)\Bigr]\cos\beta \quad \text{on } \Gamma, \end{cases}$$

where the operators Δ and $\mathcal{D}$ are given by

$$(14)\qquad \Delta = M\frac{\partial^2}{\partial\xi_1^2} + 2N\frac{\partial^2}{\partial\xi_1\,\partial\xi_2} + L\frac{\partial^2}{\partial\xi_2^2},$$

$$\mathcal{D} = \cos\alpha\Bigl(M\frac{\partial}{\partial\xi_1} + N\frac{\partial}{\partial\xi_2}\Bigr) + \cos\beta\Bigl(N\frac{\partial}{\partial\xi_1} + L\frac{\partial}{\partial\xi_2}\Bigr).$$

The stresses σ_{ij} corresponding to displacements (11) are

$$(15)\qquad \begin{cases} \sigma_{11} = -k\Bigl\{\tau(A+H)\xi_3^2 + GH_1 + a\xi_3(h-\xi_3)\bigl[(A+H)\nu_2\xi_2 + \\ + \{H\nu_3 + Q(\nu_1-\nu_2)\}\xi_1\bigr] - aG\xi_2\xi_3\Bigl(h - \frac{1}{2}\xi_3\Bigr)\Bigr\} + k\tau_{11}, \\ \sigma_{22} = -k\Bigl\{\tau(B+H)\xi_3^2 + FH_1 + a\xi_3(h-\xi_3)\bigl[(B+H)\nu_2\xi_2 + \\ + \{B\nu_3 + R(\nu_1-\nu_2)\}\xi_1\bigr] - aF\xi_2\xi_3\Bigl(h - \frac{1}{2}\xi_3\Bigr)\Bigr\} + k\tau_{22}, \\ \sigma_{12} = -k\Bigl\{\tau(Q+R)\xi_3^2 + TH_1 + a\xi_3(h-\xi_3)\bigl[(Q+R)\nu_2\xi_2 + \\ + \{R\nu_3 + D(\nu_1-\nu_2)\}\xi_1\bigr] - aT\xi_2\xi_3\Bigl(h - \frac{1}{2}\xi_3\Bigr)\Bigr\} + k\tau_{12}, \\ \sigma_{33} = -a\xi_1(h-\xi_3) - k\Bigl\{\tau(F+G)\xi_3^2 + CH_1 + a\xi_3(h - \\ -\xi_3)\bigl[(F+G)\nu_2\xi_2 + \{F\nu_3 + T(\nu_1-\nu_2)\}\xi_1\bigr] - aC\xi_2\xi_3\Bigl(h - \\ -\frac{1}{2}\xi_3\Bigr)\Bigr\} + k\tau_{33}, \end{cases}$$

$$\begin{cases} \sigma_{23} = L\Bigl[\frac{\partial h_1}{\partial\xi_2} + \tau\xi_1 - a\Bigl(\nu_2\xi_1\xi_2 + \frac{1}{2}\nu_3\xi_1^2\Bigr)\Bigr] + N\Bigl[\frac{\partial h_1}{\partial\xi_1} - \tau\xi_2 - \\ -\frac{1}{2}a(\nu_1\xi_1^2 - \nu_2\xi_2^2)\Bigr] - kL\Bigl[\frac{\partial h_1}{\partial\xi_1}\xi_3 + \tau\xi_2\xi_3 + a(h-\xi_3)\{\nu_2(\xi_2^2 - \\ -\xi_1^2) + \nu_3\xi_1\xi_2\} - a\xi_3^2\Bigl(h - \frac{1}{2}\xi_3\Bigr)\Bigr] + kN\Bigl[\frac{\partial h_1}{\partial\xi_2}\xi_3 - \tau\xi_1\xi_3 - \\ -a(h-\xi_3)(\nu_1+\nu_2)\xi_1\xi_2\Bigr] + k\tau_{23}. \\ \sigma_{31} = N\Bigl[\frac{\partial h_1}{\partial\xi_2} + \tau\xi_1 - a\Bigl(\nu_2\xi_1\xi_2 + \frac{1}{2}\nu_3\xi_1^2\Bigr)\Bigr] + M\Bigl[\frac{\partial h_1}{\partial\xi_1} - \\ -\tau\xi_2 - \frac{1}{2}a(\nu_1\xi_1^2 - \nu_2\xi_2^2)\Bigr] - kN\Bigl[\frac{\partial h_1}{\partial\xi_1}\xi_3 + \tau\xi_2\xi_3 + a(h-\xi_3)\{\nu_2(\xi_2^2 - \\ -\xi_1^2) + \nu_3\xi_1\xi_2\} - a\xi_3^2\Bigl(h - \frac{1}{2}\xi_3\Bigr)\Bigr] + kM\Bigl[\Bigl(\frac{\partial h_1}{\partial\xi_2} - \tau\xi_1\Bigr)\xi_3 - \\ -a(h-\xi_3)(\nu_1+\nu_2)\xi_1\xi_2\Bigr] + k\tau_{31}. \end{cases}$$

where τ_{ij} are the stresses corresponding to the displacements u_i^0 and

$$h_1 = \tau\varphi(\xi_1, \xi_2) - a\chi(\xi_1, \xi_2).$$

$$H_1 = \xi_2\frac{\partial h_1}{\partial\xi_1} - \xi_1\frac{\partial h_1}{\partial\xi_2}.$$

Taking into account the stresses (15) from the equations of equilibrium (7) we find that the stresses τ_{ij} satisfy the equations

$$\begin{aligned} &\frac{\partial\tau_{11}}{\partial\xi_1} + \frac{\partial\tau_{12}}{\partial\xi_2} + \frac{\partial\tau_{31}}{\partial\xi_3} - (G+M)\frac{\partial H_1}{\partial\xi_1} - (T+N)\frac{\partial H_1}{\partial\xi_2} - N\Bigl[2\tau\xi_2 + \\ &2a\{\nu_2(\xi_1^2 - \xi_2^2) - \nu_3\xi_1\xi_2\} - a\xi_3\Bigl(2h - \frac{3}{2}\xi_3\Bigr)\Bigr] - 2M\bigl[\tau\xi_1 - a(\nu_1 + \\ &\nu_2)\xi_1\xi_2\bigr] - a\xi_3\Bigl\{(h-\xi_3)\bigl[H\nu_3 + Q(\nu_1-\nu_2) + (Q+R)\nu_2\bigr] - \\ &-T\Bigl(h - \frac{1}{2}\xi_3\Bigr)\Bigr\} = 0, \end{aligned}$$

$$
\left\{
\begin{aligned}
&\frac{\partial \tau_{12}}{\partial \xi_1}+\frac{\partial \tau_{22}}{\partial \xi_2}+\frac{\partial \tau_{23}}{\partial \xi_3}-(T+N)\frac{\partial H_1}{\partial \xi_1}-(F+L)\frac{\partial H_1}{\partial \xi_2}-L\Big[2\tau\xi_2+\\
&+2a\,\{\nu_2(\xi_1^2-\xi_2^2)-\nu_3\,\xi_1\,\xi_2\}-a\,\xi_3\Big(2-\frac{3}{2}\,\xi_3\Big)\Big]-2N\,[\tau\xi_1-\\
&-a\,(\nu_1+\nu_2)\,\xi_1\,\xi_2]-a\xi_3\Big\{(h-\xi_3)\,[R\nu_3+D\,(\nu_1-\nu_2)+(B+\\
&+H)\,\nu_2]-F\Big(h-\frac{1}{2}\,\xi_3\Big)\Big\}=0,\\
&\frac{\partial \tau_{31}}{\partial \xi_1}+\frac{\partial \tau_{23}}{\partial \xi_2}+\frac{\partial \tau_{33}}{\partial \xi_3}+2\,\xi_3\Big[(M-L)\,\frac{\partial^2 h_1}{\partial \xi_1\,\partial \xi_2}+N\Big(\frac{\partial^2 h_1}{\partial \xi_2^2}-\frac{\partial^2 h_1}{\partial \xi_1^2}\Big)-\\
&-\tau\,(F+G+M+L)\Big]+a\xi_3\,\{(M+L)\,\nu_2\,\xi_2+[L\nu_3+N\,(\nu_1-\\
&-\nu_2)]\,\xi_1\}-a\,(h-\xi_3)\,\{[L\nu_3+N\,(\nu_1-\nu_2)\}\,\xi_1+[M\,(\nu_1+\\
&+\nu_2)+2L\nu_2+N\nu_3-C-E]\,\xi_2\}-a\,(h-2\xi_3)\,[F\nu_3+\\
&+T\,(\nu_1-\nu_2)]\,\xi_1+(F+G)\,\nu_2\,\xi_2]=0.
\end{aligned}
\right.
\tag{16}
$$

The stresses τ_{ij}^* given by

$$
\left\{
\begin{aligned}
\tau_{11}^* &=\tau\,(A+H)\,\xi_3^2+GH_1+a\xi_3\,(h-\xi_3)\,[\{H\nu_3+Q\,(\nu_1-\nu_2)\}\,\xi_1+\\
&+(A+H)\,\nu_2\,\xi_2-aG\,\xi_2\,\xi_3\Big(h-\frac{1}{2}\,\xi_3\Big),\\
\tau_{22}^* &=\tau\,(B+H)\,\xi_3^2+FH_1+a\xi_3\,(h-\xi_3)\,[\{B\nu_3+R\,(\nu_1-\\
&-\nu_2)\}\,\xi_1+(B+H)\,\nu_2\,\xi_2]-aF\,\xi_2\,\xi_3\Big(h-\frac{1}{2}\,\xi_3\Big),\\
\tau_{12}^* &=\tau\,(Q+R)\,\xi_3^2+TH_1+a\,\xi_3\,(h-\xi_3)\,[\{R\nu_3+D\,(\nu_1-\\
&-\nu_2)\}\,\xi_1+(Q+R)\,\nu_2\,\xi_2]-aT\,\xi_2\,\xi_3\Big(h-\frac{1}{2}\,\xi_3\Big),\\
\tau_{33}^* &=-aE\,\xi_2\,\xi_3\,(h-\xi_3)+\tau\,(F+G)\,\xi_3^2+CH_1+a\xi_3\,(h-\\
&-\xi_3)\,[\{F\nu_3+T\,(\nu_1-\nu_2)\}\,\xi_1+(F+G)\,\nu_2\,\xi_2]-aC\,\xi_2\,\xi_3\Big(h-\\
&-\frac{1}{2}\,\xi_3\Big),
\end{aligned}
\right.
\tag{17}
$$

$$
\begin{aligned}
\tau_{23}^* &=L\Big\{\frac{\partial H_1}{\partial \xi_2}+2\tau\xi_2\xi_3+a\,(h-2\,\xi_3)[\nu_2\,(\xi_2^2-\xi_1^2)+\nu_3\,\xi_1\,\xi_2]-\\
&-a\xi_3^2\Big(h-\frac{1}{2}\,\xi_3\Big)\Big\}+N\Big[\Big(\frac{\partial H_1}{\partial \xi_1}+2\tau\xi_1\Big)\,\xi_3+a\,(h-2\,\xi_3)\,(\nu_1+\nu_2)\,\xi_1\,\xi_2\Big],\\
\tau_{31}^* &=N\Big\{\Big(\frac{\partial H_1}{\partial \xi_2}+2\tau\xi_2\Big)\,\xi_3+a\,(h-2\,\xi_3)[\nu_2\,(\xi_2^2-\xi_1^2)+\nu_3\,\xi_1\,\xi_2]-\\
&-a\xi_3^2\Big(h-\frac{1}{2}\,\xi_3\Big)\Big\}+M\Big[\Big(\frac{\partial H_1}{\partial \xi_1}+2\tau\xi_1\Big)\xi_3+a\,(h-2\,\xi_3)\,(\nu_1+\nu_2)\,\xi_1\,\xi_2\Big]
\end{aligned}
$$

verify the equations (16) and the corresponding strains γ_{ij}^* satisfy the conditions of compatibility.

If we put

$$
\tau_{ij}=\tau_{ij}^*+\tilde{\tau}_{ij}\,,
\tag{18}
$$

we find that the new stresses $\tilde{\tau}_{ij}$ satisfy the equations

$$
\tilde{\tau}_{ij,j}=0 \quad \text{in } \mathcal{D}_1\,.
\tag{19}
$$

Here the indices 1, 2, 3 after comma indicate partial differentiation with respect to ξ_1, ξ_2, ξ_3.

Taking into account (8), (15), (18) and (5), the boundary conditions for the stresses $\tilde{\tau}_{ij}$ can be written in the form

$$
\left\{
\begin{aligned}
&\tilde{\tau}_{11}\cos\alpha+\tilde{\tau}_{12}\cos\beta=p_1\,,\\
&\tilde{\tau}_{12}\cos\alpha+\tilde{\tau}_{22}\cos\beta=p_2\,,\\
&\tilde{\tau}_{31}\cos\alpha+\tilde{\tau}_{23}\cos\beta=-aE\,\xi_1\,(\xi_2\cos\alpha-\xi_1\cos\beta)\,(h-\\
&-\xi_3)+p_3\,\xi_3 \quad \text{on } \mathcal{F}_1,
\end{aligned}
\right.
\tag{20}
$$

where

$$
\begin{aligned}
p_1=\Big\{&N\Big[\frac{\partial h_1}{\partial \xi_2}+\tau\xi_1-a\Big(\nu_2\,\xi_1\,\xi_2+\frac{1}{2}\,\nu_2\,\xi_1^2\Big)\Big]+M\Big[\frac{\partial h_1}{\partial \xi_1}-\tau\xi_2-\\
&-\frac{1}{2}\,a(\nu_1\,\xi_1^2-\nu_2\,\xi_2^2)\Big]\Big\}(\xi_2\cos\alpha-\xi_1\cos\beta),\\
p_2=\Big\{&L\Big[\frac{\partial h_1}{\partial \xi_2}+\tau\xi_1-a\Big(\nu_2\,\xi_1\,\xi_2+\frac{1}{2}\,\nu_3\,\xi_1^2\Big)\Big]+N\Big[\frac{\partial h_1}{\partial \xi_1}-\tau\xi_2-
\end{aligned}
$$

$$-\frac{1}{2} a (\nu_1 \xi_1^2 - \nu_2 \xi_2^2)\Big]\Big\} (\xi_2 \cos \alpha - \xi_1 \cos \beta).$$

$$p_3 = \left[\frac{\partial h_1}{\partial \xi_2} + \tau \xi_1 - a\left(\nu_2 \xi_1 \xi_2 + \frac{1}{2} \nu_3 \xi_1^2\right)\right] (L \cos \alpha - N \cos \beta) +$$

$$+ \left[\frac{\partial h_1}{\partial \xi_1} - \tau \xi_2 - \frac{1}{2} a (\nu_1 \xi_1^2 - \nu_2 \xi_2^2)\right] (N \cos \alpha - M \cos \beta) - \left\{\frac{\partial H_1}{\partial \xi_2} - \frac{\partial h_1}{\partial \xi_1} +\right.$$

$$\left.+ \tau \xi_2 - a [\nu_2 (\xi_2^2 - \xi_1^2) + \nu_3 \xi_1 \xi_2]\right\} (N \cos \alpha + L \cos \beta) - \left[\frac{\partial H_1}{\partial \xi_1} + \frac{\partial \xi_1}{\partial \xi_2} +\right.$$

$$\left.+ \tau \xi_1 - a (\nu_1 + \nu_2) \xi_1 \xi_2\right] (M \cos \alpha + N \cos \beta).$$

In this way the determining of the stresses $\tilde{\tau}_{ij}$ is an Almansi problem [4].

Because the right hand sides of the relations (20) are linear functions of ξ_3 we can solve the obtained problem in an analogous way as that used for the Almansi-Michell's problem [3]. To that end let us put

$$
(21) \quad \begin{cases}
\tau_{11} = \dfrac{\partial^2 \Phi}{\partial \xi_2^2} - M (\omega_1 + \theta_1) - N\left(\dfrac{1}{3} a_1 \xi_1^3 + \dfrac{1}{2} c_1 \xi_1^2\right), \\
\tau_{22} = \dfrac{\partial^2 \Phi}{\partial \xi_1^2} - L (\omega_1 + \theta_2) - N\left(\dfrac{1}{3} b_1 \xi_2^3 - \dfrac{1}{2} c_1 \xi_2^2\right), \\
\tau_{12} = - \dfrac{\partial^2 \Phi}{\partial \xi_1 \partial \xi_2} - N \omega_1, \\
\tau_{33} = \Omega + \dfrac{1}{2} E (A_1 \xi_1 + A_2 \xi_2 + A_3) \xi_3^2, \\
\tau_{23} = L \dfrac{\partial}{\partial \xi_2} [(\omega_1 + \theta_2) \xi_3 + \omega_0] + N \dfrac{\partial}{\partial \xi_1} [(\omega_1 + \theta_1) \xi_3 + \omega_0], \\
\tau_{31} = N \dfrac{\partial}{\partial \xi_2} [(\omega_1 + \theta_2) \xi_3 + \omega_0] + M \dfrac{\partial}{\partial \xi_1} [(\omega_1 + \theta_1) \xi_3 + \omega_0],
\end{cases}
$$

where Φ, ω_1, ω_0 are unknown functions of ξ_1 and ξ_2 which must be determined and

$$\Omega = E \Big\{ \omega_1 + \frac{1}{2} [a_1 \xi_1^2 \xi_2 + b_1 \xi_1 \xi_2^2 + A_3 (\nu_1 \xi_1^2 + \nu_2 \xi_2^2 + \nu_3 \xi_1 \xi_2) +$$

$$+ \left(\frac{1}{3} \nu_1 \xi_1 + \frac{1}{2} \nu_3 \xi_2\right) A_1 \xi_1^2 + \left(\frac{1}{3} \nu_2 \xi_2 + \frac{1}{2} \nu_3 \xi_1\right) A_2 \xi_9^2 \Big\} +$$

$$+ \nu_1 \left[\frac{\partial^2 \Phi}{\partial \xi_2^2} - M (\omega_1 + \theta_1) - N\left(\frac{1}{3} a_1 \xi_1^3 + c_1 \xi_1^2\right)\right] + \nu_2 \left[\frac{\partial^2 \Phi}{\partial \xi_1^2} -\right.$$

$$\left.- L (\omega_1 + \theta_2) - N\left(\frac{1}{3} b_1 \xi_2^3 - \frac{1}{2} c_1 \xi_2^2\right)\right] + \nu_3 \left[- \frac{\partial^2 \Phi}{\partial \xi_1 \partial \xi_2} - N \omega_1\right].$$

$$\theta_1 = b_1 \xi_1 \xi_2^2 - c_1 \xi_1 \xi_2, \quad \theta_2 = a_1 \xi_1^2 \xi_2 + c_1 \xi_1 \xi_2,$$

$a_1, b_1, c_1, A_1, A_2, A_3$ being constants which will be determined from the conditions of compatibility and the conditions of existence of the functions ω_0, ω_1 and Φ.

The equations of equilibrium (19) will be satisfied if the functions ω_0 and ω_1 satisfy the equations

$$\Delta \omega_0 = 0 \quad \text{in } S,$$

$$\Delta \omega_1 + (EA_1 + 2Na_1) \xi_1 + (EA_2 + 2Nb_1) \xi_2 + EA_3 = 0 \quad \text{in } S$$

and from the third equation (20) we get for these functions the following boundary conditions :

$$\mathcal{D} \omega_0 = - aEh \xi_1 (\xi_2 \cos \alpha - \xi_1 \cos \beta) \quad \text{on } \Gamma,$$

$$\mathcal{D} \omega_1 = p_3 + aE \xi_1 (\xi_2 \cos \alpha - \xi_1 \cos \beta) - (a_1 \xi_1^2 + c_1 \xi_1) (N \cos \beta + L \cos \beta) -$$

$$- (b_1 \xi_2^2 - c_1 \xi_2) (M \cos \alpha + N \cos \beta) \quad \text{on } \Gamma,$$

where the operators Δ and $\mathcal{D}$ are given by (14).

The conditions of the existence of the function ω_0 is identically satisfied and the existence of the function ω_1 requires

$$A_3 = \frac{1}{ES} \int_\Gamma p_3 \, ds.$$

The stresses $\tilde{\tau}_{ij}$ given by (21) will satisfy the compatibility conditions if

$$a_1 = \nu_1 A_2 - \frac{1}{2} \nu_3 A_1, \quad b_1 = \nu_2 A_1 - \frac{1}{2} \nu_3 A_2$$

and the function Φ satisfies the equation

$$\beta_{22}\frac{\partial^4 \Phi}{\partial \xi_1^4} - 2\beta_{23}\frac{\partial^4 \Phi}{\partial \xi_1^3 \partial \xi_2} + (\beta_{33} + 2\beta_{12})\frac{\partial^4 \Phi}{\partial \xi_1^2 \partial \xi_2^2} - 2\beta_{13}\frac{\partial^4 \Phi}{\partial \xi_1 \partial \xi_2^3} + \beta_{11}\frac{\partial^4 \Phi}{\partial \xi_2^4} =$$
$$= (M\beta_{12} + L\beta_{22} + N\beta_{23} + \nu_2)\frac{\partial^2 \omega_1}{\partial \xi_1^2} - (M\beta_{13} + L\beta_{23} + N\beta_{33} +$$
$$+ \nu_3)\frac{\partial^2 \omega_1}{\partial \xi_1 \partial \xi_2} + (M\beta_{11} + L\beta_{12} + N\beta_{13} + \nu_1)\frac{\partial^2 \omega_1}{\partial \xi_2^2} + 2\,[(M\beta_{11} + \nu_1)\, b_1\, \xi_1 +$$
$$+ (L\beta_{22} + \nu_2)\, a_1\, \xi_2] + (N\beta_{12} - L\beta_{23})\,(2a_1\, \xi_1 + c_1) + (N\beta_{12} - M\beta_{13})\,(2b_1\, \xi_2 -$$
$$- c_1) + (2\nu_1\, \nu_2 - \frac{1}{2}\,\nu_3^2)\, A_3 \quad \text{in } S,$$

where $\beta_{ij} = 1/E\,(\nu_{ij} - \nu_i\,\nu_j)$. From the first two equations (20) we get

$$\frac{\partial^2 \Phi}{\partial \xi_1^2}\cos\alpha - \frac{\partial^2 \Phi}{\partial \xi_1 \partial \xi_2}\cos\beta = p_1 + \left[M\,(\omega_1 + \theta_1) + N\left(\frac{1}{3}\, a_1\, \xi_1^3 + \right.\right.$$
$$\left.\left. + \frac{1}{2}\, c_1\, \xi_1^2\right)\right]\cos\alpha + N\omega_1\cos\beta\,; \qquad -\frac{\partial^2 \Phi}{\partial \xi_1 \partial \xi_2}\cos\alpha + \frac{\partial^2 \Phi}{\partial \xi_1^2}\cos\beta =$$
$$= p_2 + N\omega_1\cos\alpha + \left[L\,(\theta_1 + \omega_1) + N\left(\frac{1}{3}\, b_1\, \xi_2^3 - \frac{1}{2}\, c_1\, \xi_2^2\right)\right]\cos\beta \quad \text{on } \Gamma$$

and the determination of the function Φ is analogous with the determination of the corresponding function in the problem of Almansi-Michell [3].

In consequence, from the existence conditions of the function Φ we can determine the constants A_1, A_2 and c_1 in the same way as in [3].

Generally, the stress components determined above will satisfy all equations and boundary conditions excepting the conditions at the end $x_3 = h$. Therefore, it remains to correct the boundary conditions at the end $x_3 = h$ by superposition of solutions of some adequate problems for the cylindrical beam (3), but we know to solve all these additional problems [2, 5].

We can particularize the above results in order to obtain the results regarding the orthotropic beam sor the isotropic beams. On the other hand, taking $a = 0$ we obtain the results concerning the problem of torsion.

REFERENCES

1. B o r ş, C. I. — *The bending of the naturally twisted and orthotropic beam by a transverse load.* Bulletin de l'Académie Polonaise des Sciences, Sér. sci. tech., XVII, No. 10, 1970.
2. B o r ş, C. I. — *Teoria elasticităţii corpurilor anizotrope.* Ed. by Acad. R. S. România, Bucureşti, 1970.
3. B o r ş, C. I. — *On Almansi-Michell's problem for anisotropic beams.* Rendic. Accad. dei Lincei, L, No. 3, 1971.
4. K h a t i a s h v i l i, G. M. — *On Almansi's problem for homogeneous anisotropic beams (Russian).* Trudy Vycislitelnogo Tsentra, Akad. Nauk Gruz. SSR, IV, 1964.
5. L e k h n i t z k i i S. G. — *Theory of elasticity of anisotropic body* (Russian). Moskva-Leningrad, 1950.

Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XX, s. I a, fasc. 2, 1974

BENDING OF ORTHOTROPIC MICROPOLAR ELASTIC BEAMS BY TERMINAL COUPLES

BY

D. IEŞAN

1. Introduction. The Saint-Venant's problem in the linear theory of micropolar elasticity, for isotropic bodies, was studied in [1—4].

In this paper we give the constitutive equations for orthotropic micropolar elastic solids and we consider the problem of bending of homogeneous and orthotropic beams by terminal couples in the linear theory of micropolar elasticity.

2. Basic equations. Let V be a region of space occupied by a micropolar elastic material whose boundary is S. Throughout this paper a rectangular coordinate system Ox_k $(k = 1, 2, 3)$ is employed. We assume that body forces and body couples are absent.

The constitutive equations of an anisotropic centrosymmetric solid in the linear theory of micropolar elasticity are given in [5]. From these equations, it is easy to derive the constitutive equations for an orthotropic elastic solid.

The basic equations in the static theory of homogeneous and orthotropic micropolar elastic solids are:
equilibrium equations

$$t_{ji,j} = 0, \qquad m_{ji,j} + e_{ijk}\, t_{jk} = 0, \tag{2.1}$$

constitutive equations

$$\begin{aligned}
&t_{11} = A_{11}\,\varepsilon_{11} + A_{12}\,\varepsilon_{22} + A_{13}\,\varepsilon_{33}, && t_{22} = A_{12}\,\varepsilon_{11} + A_{22}\,\varepsilon_{22} + A_{23}\varepsilon_{33},\\
&t_{33} = A_{13}\,\varepsilon_{11} + A_{23}\,\varepsilon_{22} + A_{33}\,\varepsilon_{33}, && t_{23} = A_{44}\,\varepsilon_{23} + A_{45}\,\varepsilon_{32},\\
&t_{32} = A_{45}\,\varepsilon_{23} + A_{55}\,\varepsilon_{32}, && t_{31} = A_{55}\,\varepsilon_{31} + A_{56}\,\varepsilon_{13},\\
&t_{13} = A_{56}\,\varepsilon_{31} + A_{66}\,\varepsilon_{13}, && t_{12} = A_{77}\,\varepsilon_{12} + A_{78}\,\varepsilon_{21},\\
&t_{21} = A_{78}\,\varepsilon_{12} + A_{88}\,\varepsilon_{21}, && m_{11} = B_{11}\,\varphi_{1,1} + B_{12}\,\varphi_{2,2} + B_{13}\varphi_{3,3},\\
&m_{22} = B_{12}\,\varphi_{1,1} + B_{22}\,\varphi_{2,2} + B_{23}\,\varphi_{3,3}, && m_{33} = B_{13}\,\varphi_{1,1} + B_{23}\,\varphi_{2,2} + B_{33}\varphi_{3,3},\\
&m_{23} = B_{44}\,\varphi_{3,2} + B_{45}\,\varphi_{2,3}, && m_{32} = B_{45}\,\varphi_{3,2} + B_{55}\,\varphi_{2,3},\\
&m_{31} = B_{55}\,\varphi_{1,3} + B_{56}\,\varphi_{3,1}, && m_{13} = B_{56}\,\varphi_{1,3} + B_{66}\,\varphi_{3,1};\\
&m_{12} = B_{77}\,\varphi_{2,1} + B_{78}\,\varphi_{1,2}, && m_{21} = B_{78}\,\varphi_{2,1} + B_{88}\,\varphi_{1,2}
\end{aligned} \tag{2.2}$$