$$\beta_{22}\frac{\partial^4 \Phi}{\partial \xi_1^4} - 2\beta_{23}\frac{\partial^4 \Phi}{\partial \xi_1^3 \partial \xi_2} + (\beta_{33} + 2\beta_{12})\frac{\partial^4 \Phi}{\partial \xi_1^2 \partial \xi_2^2} - 2\beta_{13}\frac{\partial^4 \Phi}{\partial \xi_1 \partial \xi_2^3} + \beta_{11}\frac{\partial^4 \Phi}{\partial \xi_2^4} =$$

$$= (M\beta_{12} + L\beta_{22} + N\beta_{23} + \nu_2)\frac{\partial^2 \omega_1}{\partial \xi_1^2} - (M\beta_{13} + L\beta_{23} + N\beta_{33} +$$

$$+ \nu_3)\frac{\partial^2 \omega_1}{\partial \xi_1 \partial \xi_2} + (M\beta_{11} + L\beta_{12} + N\beta_{13} + \nu_1)\frac{\partial^2 \omega_1}{\partial \xi_2^2} + 2\,[(M\beta_{11} + \nu_1)\, b_1\, \xi_1 +$$

$$+ (L\beta_{22} + \nu_2)\, a_1\, \xi_2] + (N\beta_{12} - L\beta_{23})(2a_1\, \xi_1 + c_1) + (N\beta_{12} - M\beta_{13})(2b_1\, \xi_2 -$$

$$- c_1) + (2\nu_1\, \nu_2 - \frac{1}{2}\nu_3^2)\, A_3 \quad \text{in } S,$$

where $\beta_{ij} = 1/E\,(\nu_{ij} - \nu_i\, \nu_j)$. From the first two equations (20) we get

$$\frac{\partial^2 \Phi}{\partial \xi_1^2}\cos\alpha - \frac{\partial^2 \Phi}{\partial \xi_1 \partial \xi_2}\cos\beta = p_1 + \Big[M(\omega_1 + 0_1) + N\Big(\frac{1}{3}a_1\, \xi_1^3 +$$

$$+ \frac{1}{2}c_1\, \xi_1^2\Big)\Big]\cos\alpha + N\omega_1\cos\beta\,; \qquad -\frac{\partial^2 \Phi}{\partial \xi_1 \partial \xi_2}\cos\alpha + \frac{\partial^2 \Phi}{\partial \xi_1^2}\cos\beta =$$

$$= p_2 + N\omega_1\cos\alpha + \Big[L(0_1 + \omega_1) + N\Big(\frac{1}{3}b_1\, \xi_2^3 - \frac{1}{2}c_1\, \xi_2^2\Big)\Big]\cos\beta \quad \text{on } \Gamma$$

and the determination of the function Φ is analogous with the determination of the corresponding function in the problem of Almansi-Michell [3].

In consequence, from the existence conditions of the function Φ we can determine the constants A_1, A_2 and c_1 in the same way as in [3].

Generally, the stress components determined above will satisfy all equations and boundary conditions excepting the conditions at the end $x_3 = h$. Therefore, it remains to correct the boundary conditions at the end $x_3 = h$ by superposition of solutions of some adequate problems for the cylindrical beam (3), but we know to solve all these additional problems [2, 5].

We can particularize the above results in order to obtain the results regarding the orthotropic beam sor the isotropic beams. On the other hand, taking $a = 0$ we obtain the results concerning the problem of torsion.

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BENDING OF ORTHOTROPIC MICROPOLAR ELASTIC BEAMS BY TERMINAL COUPLES

BY

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1. Introduction. The Saint-Venant's problem in the linear theory of micropolar elasticity, for isotropic bodies, was studied in [1—4].

In this paper we give the constitutive equations for orthotropic micropolar elastic solids and we consider the problem of bending of homogeneous and orthotropic beams by terminal couples in the linear theory of micropolar elasticity.

2. Basic equations. Let V be a region of space occupied by a micropolar elastic material whose boundary is S. Throughout this paper a rectangular coordinate system Ox_k $(k = 1, 2, 3)$ is employed. We assume that body forces and body couples are absent.

The constitutive equations of an anisotropic centrosymmetric solid in the linear theory of micropolar elasticity are given in [5]. From these equations, it is easy to derive the constitutive equations for an orthotropic elastic solid.

The basic equations in the static theory of homogeneous and orthotropic micropolar elastic solids are:

equilibrium equations

$$t_{ji,j} = 0, \qquad m_{ji,j} + e_{ijk}\, t_{jk} = 0, \tag{2.1}$$

constitutive equations

$$\begin{aligned}
&t_{11} = A_{11}\,\varepsilon_{11} + A_{12}\,\varepsilon_{22} + A_{13}\,\varepsilon_{33}, && t_{22} = A_{12}\,\varepsilon_{11} + A_{22}\,\varepsilon_{22} + A_{23}\varepsilon_{33},\\
&t_{33} = A_{13}\,\varepsilon_{11} + A_{23}\,\varepsilon_{22} + A_{33}\,\varepsilon_{33}, && t_{23} = A_{44}\,\varepsilon_{23} + A_{45}\,\varepsilon_{32},\\
&t_{32} = A_{45}\,\varepsilon_{23} + A_{55}\,\varepsilon_{32}, && t_{31} = A_{55}\,\varepsilon_{31} + A_{56}\,\varepsilon_{13},\\
&t_{13} = A_{56}\,\varepsilon_{31} + A_{66}\,\varepsilon_{13}, && t_{12} = A_{77}\,\varepsilon_{12} + A_{78}\,\varepsilon_{21},\\
&t_{21} = A_{78}\,\varepsilon_{12} + A_{88}\,\varepsilon_{21}, && m_{11} = B_{11}\,\varphi_{1,1} + B_{12}\,\varphi_{2,2} + B_{13}\varphi_{3,3},\\
&m_{22} = B_{12}\,\varphi_{1,1} + B_{22}\,\varphi_{2,2} + B_{23}\,\varphi_{3,3}, && m_{33} = B_{13}\,\varphi_{1,1} + B_{23}\,\varphi_{2,2} + B_{33}\varphi_{3,3},\\
&m_{23} = B_{44}\,\varphi_{3,2} + B_{45}\,\varphi_{2,3}, && m_{32} = B_{45}\,\varphi_{3,2} + B_{55}\,\varphi_{2,3},\\
&m_{31} = B_{55}\,\varphi_{1,3} + B_{56}\,\varphi_{3,1}, && m_{13} = B_{56}\,\varphi_{1,3} + B_{66}\,\varphi_{3,1},\\
&m_{12} = B_{77}\,\varphi_{2,1} + B_{78}\,\varphi_{1,2}, && m_{21} = B_{78}\,\varphi_{2,1} + B_{88}\,\varphi_{1,2}
\end{aligned} \tag{2.2}$$

geometrical equations

$$\varepsilon_{ij} = u_{j,i} + e_{jim}\,\varphi_m . \tag{2.3}$$

In these relations we have used the following notations: t_{ij} — components of the stress tensor, m_{ij}-components of the couple-stress tensor, u_i-components of the displacement vector, φ_i-components of the microrotation vector, ε_{ij}-components of the micropolar strain tensor, e_{ijk}-alternating symbol, A_{rs}, B_{rs} $(r, s = 1, 2, \ldots, 8)$-characteristic constants of the material, the comma denotes partial derivation with respect to the variables x_j.

The surface tractions and surface couples acting at a point x on the surface S are given by

$$t_{ji} = t_i\, n_j , \qquad m_i = m_{ji}\, n_j , \tag{2.4}$$

where n_j are the direction cosines of the exterior normal to S at x.

3. Auxiliary plane strain problems. We consider a finite regular plane region Σ occupied by a micropolar elastic material whose boundary is L. The plane strain is characterized by

$$u_\alpha = u_\alpha(x_1, x_2), \quad \varphi_3 = \varphi(x_1, x_2), \quad u_3 = \varphi_\alpha = 0 \quad (\alpha = 1, 2).$$

The basic equations in the static theory of the plane strain of an homogeneous and orthotropic elastic solid are:

equilibrium equations

$$t_{\beta\alpha,\beta} = 0, \qquad m_{\alpha 3,\alpha} + e_{\alpha\beta 3}\, t_{\alpha\beta} = 0, \tag{3.1}$$

constitutive equations

$$\begin{aligned} t_{11} &= A_{11}\,\varepsilon_{11} + A_{12}\,\varepsilon_{22}, & t_{22} &= A_{12}\,\varepsilon_{11} + A_{22}\,\varepsilon_{22}, \\ t_{12} &= A_{77}\,\varepsilon_{12} + A_{78}\,\varepsilon_{21}, & t_{21} &= A_{78}\,\varepsilon_{12} + A_{88}\,\varepsilon_{21} \\ m_{13} &= B_{66}\,\varphi_{,1} & m_{23} &= B_{44}\,\varphi_{,2} \end{aligned} \tag{3.2}$$

geometrical equations

$$\varepsilon_{\alpha\beta} = u_{\beta,\alpha} + e_{\beta\alpha 3}\,\varphi . \tag{3.3}$$

We consider the following boundary conditions

$$t_{\beta\alpha}\, n_\beta = f_\alpha , \qquad m_{\alpha 3}\, n_\alpha = g \quad \text{on } L. \tag{3.4}$$

The necessary and sufficient conditions to solve the above problem are [7]:

$$\int_L f_\alpha\, ds = 0, \qquad \int_L (x_1 f_2 - x_2 f_1 + g)\, ds = 0. \tag{3.5}$$

In what follows we denote by $v_\alpha^{(1)}$, $w^{(1)}$, $e_{\alpha\beta}^{(1)}$, $\sigma_{\alpha\beta}^{(1)}$, $\mu_{\alpha 3}^{(1)}$ $(\alpha, \beta = 1, 2)$, the solution of the plane strain problem (3.1)—(3.4) corresponding to the boundary conditions

$$f_\alpha = 0, \qquad g = (B_{44}\,\nu_2 + B_{45})\, n_2 .$$

and by $v_\alpha^{(2)}$, $w^{(2)}$, $e_{\alpha\beta}^{(2)}$, $\sigma_{\alpha\beta}^{(2)}$, $\mu_{\alpha 3}^{(2)}$ the solution of the plane strain problem corresponding to the boundary conditions

$$f_\alpha = 0, \qquad g = -(B_{56} + B_{66}\,\nu_1)\, n_1 .$$

where

$$\nu_1 = \frac{A_{13} A_{22} - A_{23} A_{12}}{A_{11} A_{22} - A_{12}^2}, \qquad \nu_2 = \frac{A_{11} A_{23} - A_{12} A_{13}}{A_{11} A_{22} - A_{12}^2}. \tag{3.6}$$

It is known [7] that $A_{11} A_{22} - A_{12}^2 > 0$.

Thus, the functions $v_\alpha^{(\rho)}(x_1, x_2)$, $w^{(\rho)}(x_1, x_2)$, $e_{\alpha\beta}^{(\rho)}(x_1, x_2)$, $\sigma_{\alpha\beta}^{(\rho)}(x_1, x_2)$, $\mu_{\alpha 3}^{(\rho)}(x_1, x_2)$, $(\alpha, \beta, \rho = 1, 2)$ satisfy the equations

$$\sigma_{\alpha\beta,\alpha}^{(\rho)} = 0, \qquad \mu_{\alpha 3,\alpha}^{(\rho)} + e_{\alpha\beta 3}\,\sigma_{\alpha\beta}^{(\rho)} = 0, \tag{3.7}$$

$$\begin{aligned} \sigma_{11}^{(\rho)} &= A_{11}\, e_{11}^{(\rho)} + A_{12}\, e_{22}^{(\rho)}, & \sigma_{22}^{(\rho)} &= A_{12}\, e_{11}^{(\rho)} + A_{22}\, e_{22}^{(\rho)}, \\ \sigma_{12}^{(\rho)} &= A_{77}\, e_{12}^{(\rho)} + A_{78}\, e_{21}^{(\rho)}, & \sigma_{21}^{(\rho)} &= A_{78}\, e_{12}^{(\rho)} + A_{88}\, e_{21}^{(\rho)}, \\ \mu_{13}^{(\rho)} &= B_{66}\, w_{,1}^{(\rho)}, & \mu_{23}^{(\rho)} &= B_{44}\, w_{,2}^{(\rho)}, \end{aligned} \tag{3.8}$$

$$e_{\alpha\beta}^{(\rho)} = v_{\beta,\alpha}^{(\rho)} + e_{\beta\alpha 3}\, w^{(\rho)}, \quad \text{in } \Sigma, \tag{3.9}$$

and

$$\begin{aligned} &\sigma_{\alpha\beta}^{(\rho)}\, n_\alpha = 0, \qquad \mu_{\alpha 3}^{(1)}\, n_\alpha = (B_{44}\,\nu_2 + B_{45})\, n_2, \\ &\mu_{\alpha 3}^{(\rho)}\, n_\alpha = -(B_{56} + B_{66}\,\nu_1)\, n_1 \quad \text{on } L. \end{aligned} \tag{3.10}$$

Obviously, the necessary and sufficient conditions (3.5) to solve the above problems are satisfied. In what follows, we assume that the functions $v_\alpha^{(\rho)}$, $w^{(\rho)}$ are known.

4. Bending of beams by terminal couples. We consider a cylindrical beam of homogeneous and orthotropic micropolar elastic material bounded by plane ends perpendicular to the generators. The cross-section Σ is assumed to be a simply connected region bounded by a closed Liapunov curve L.

Throughout this paper the axis Ox_3 of our coordinate system will be directed parallel to the generators of the beam. The beam is assumed to be of length l, and one of its bases is taken to lie in the $x_1 Ox_2$-plane, while the other is in the plane $x_3 = l$.

We assume that the body forces and the body couples are absent, the lateral surface is free of applied forces and couples, and that the load of the beam is distributed over its bases, $x_3 = 0$ and $x_3 = l$, in a way which fulfils the equilibrium conditions of a rigid body.

Let the loading applied to the plane $x_3 = 0$ be statically equivalent to a force $P(0, 0, 0)$ and a couple $M(M_1, M_2, 0)$.

On the plane $x_3 = 0$ we have the following conditions

$$\int_\Sigma t_{31}\, d\sigma = 0, \quad \int_\Sigma t_{32}\, d\sigma = 0, \tag{4.1}$$

$$\int_\Sigma t_{33}\, d\sigma = 0, \tag{4.2}$$

$$\int_\Sigma (x_2 t_{33} + m_{31})\, d\sigma = -M_1, \quad \int_\Sigma (x_1 t_{33} - m_{32})\, d\sigma = M_2, \tag{4.3}$$

$$\int_\Sigma (x_1 t_{32} - x_2 t_{31} + m_{33})\, d\sigma = 0. \tag{4.4}$$

The resultant forces and moments calculated across each cross section satisfy the conditions of equilibrium, so that the conditions (4.1)–(4.4) must be satisfied for $x_3 = h\ (0 \leqslant h \leqslant l)$.

On the lateral surface of the beam we have the following conditions

$$t_{\alpha k}\, n_\alpha = 0, \quad m_{\alpha k}\, n_\alpha = 0, \quad (k = 1, 2, 3;\ \alpha = 1, 2). \tag{4.5}$$

The problem consists in solving of the equations (2.1)–(2.3) with the conditions (4.1)–(4.5).

We try to solve the problem assuming that

$$\begin{aligned} u_1 &= -\frac{1}{2} a_1 (x_3^2 + \nu_1 x_1^2 - \nu_2 x_2^2) - a_2 \nu_1 x_1 x_2 - b \nu_1 x_1 + a_1 v_1^{(1)} + a_2 v_1^{(2)}, \\ u_2 &= -a_1 \nu_2 x_1 x_2 - \frac{1}{2} a_2 (x_3^2 - \nu_1 x_1^2 + \nu_2 x_2^2) - b \nu_2 x_2 + a_1 v_2^{(1)} + a_2 v_2^{(2)}, \\ u_3 &= (a_1 x_1 + a_2 x_2 + b)\, x_3, \\ \varphi_1 &= a_2 x_3, \quad \varphi_2 = -a_1 x_3, \\ \varphi_3 &= a_2 \nu_1 x_1 - a_1 \nu_2 x_2 + a_1 w^{(1)} + a_2 w^{(2)}, \end{aligned} \tag{4.6}$$

where $v_\alpha^{(\rho)}, w^{(\rho)}\ (\rho, \alpha = 1, 2)$ are the solutions of the problems (3.7)–(3.10) and the constants a_1, a_2, b will be determined in the following.

From (2.3) and (4.6) we obtain

$$\begin{aligned} \varepsilon_{11} &= -\nu_1 (a_1 x_1 + a_2 x_2 + b) + a_1 e_{11}^{(1)} + a_2 e_{11}^{(2)}, \\ \varepsilon_{22} &= -\nu_2 (a_1 x_1 + a_2 x_2 + b) + a_1 e_{22}^{(1)} + a_2 e_{22}^{(2)}, \\ \varepsilon_{12} &= a_1 e_{12}^{(1)} + a_2 e_{12}^{(2)}, \quad \varepsilon_{21} = a_1 e_{21}^{(1)} + a_2 e_{21}^{(2)}, \\ \varepsilon_{33} &= a_1 x_1 + a_2 x_2 + b, \qquad \varepsilon_{\alpha 3} = \varepsilon_{3\alpha} = 0, \end{aligned} \tag{4.7}$$

where $e_{\alpha\beta}^{(\rho)}$ are given by (3.9).

Using (4.6), (4.7) from (2.2) we get

$$\begin{aligned} t_{\alpha\beta} &= a_1 \sigma_{\alpha\beta}^{(1)} + a_2 \sigma_{\alpha\beta}^{(2)}, \quad t_{\alpha 3} = t_{3\alpha} = 0, \qquad (\alpha, \beta = 1, 2), \\ t_{33} &= E (a_1 x_1 + a_2 x_2 + b) + A_{13} (a_1 e_{11}^{(1)} + a_2 e_{11}^{(2)}) + \\ &\quad + A_{23} (a_1 e_{22}^{(1)} + a_2 e_{22}^{(2)}), \\ m_{\alpha\beta} &= 0, \quad m_{33} = 0, \quad m_{23} = -a_1 (B_{44} \nu_2 + B_{45}) + a_1 \mu_{23}^{(1)} + a_2 \mu_{23}^{(2)}, \\ m_{32} &= -a_1 (B_{45} \nu_2 + B_{55}) + a_1 \mu_{32}^{(1)} + a_2 \mu_{32}^{(2)}, \\ m_{13} &= a_2 (B_{56} + B_{66} \nu_1) + a_1 \mu_{13}^{(1)} + a_2 \mu_{13}^{(2)}, \\ m_{31} &= a_2 (B_{55} + B_{56} \nu_1) + a_1 \mu_{31}^{(1)} + a_2 \mu_{31}^{(2)}, \end{aligned} \tag{4.8}$$

where $\sigma_{\alpha\beta}^{(\rho)}, \mu_{\alpha 3}^{(\rho)}$ are given by (3.8). We used the notations

$$E = A_{33} - A_{13} \nu_1 - A_{23} \nu_2, \qquad \mu_{32}^{(\rho)} = B_{45} w_{,2}^{(\rho)}, \ \mu_{31}^{(\rho)} = B_{55} w_{,1}^{(\rho)}. \tag{4.9}$$

Taking into account the relations (4.8), the equilibrium equations (2.1) are reduced to

$$t_{\alpha\beta,\alpha} = 0, \qquad m_{\alpha 3,\alpha} + e_{\alpha\beta 3}\, t_{\alpha\beta} = 0, \tag{4.10}$$

and the boundary conditions (4.5) become

$$t_{\alpha\beta}\, n_\alpha = 0, \qquad m_{\alpha 3}\, n_\alpha = 0 \ \text{on}\ L. \tag{4.11}$$

The equations (4.10) and the boundary conditions (4.11) are satisfied on the basis of the relations (3.7)–(3.10).

The conditions (4.1) and (4.4) are identically satisfied on the basis of the relations (4.8).

From (4.2) and (4.8) we get

$$b = -a_1 x_1^0 - a_2 x_2^0, \tag{4.12}$$

where

$$x_\alpha^0 = \frac{1}{EA} \int_\Sigma [E x_\alpha + A_{13} e_{11}^{(\alpha)} + A_{23} e_{22}^{(\alpha)}]\, d\sigma, \tag{4.13}$$

A being the area of the cross-section.

Therefore from (4.10) and (4.12) we obtain

$$\begin{aligned} t_{33} = a_1 [E (x_1 - x_1^0) + A_{13} e_{11}^{(1)} + A_{23} e_{22}^{(1)}] + a_2 [E (x_2 - x_2^0) + A_{13} e_{11}^{(2)} + \\ + A_{23} e_{22}^{(2)}]. \end{aligned} \tag{4.14}$$

From (4.3), (4.8) and (4.14) we get

$$L_{11} a_1 + L_{12} a_2 = M_2, \qquad L_{21} a_1 + L_{22} a_2 = -M_1, \tag{4.15}$$

where

$$
(4.16)\quad
\begin{aligned}
L_{11} &= \int_{\Sigma} \{x_1 [E(x_1 - x_1^0) + A_{13} e_{11}^{(1)} + A_{23} e_{22}^{(1)}] + B_{45} \nu_2 + B_{55} - \mu_{32}^{(1)}\} \, d\sigma, \\
L_{12} &= \int_{\Sigma} \{x_1 [E(x_2 - x_2^0) + A_{13} e_{11}^{(2)} + A_{23} e_{22}^{(2)}] - \mu_{32}^{(2)}\} \, d\sigma, \\
L_{21} &= \int_{\Sigma} \{x_2 [E(x_1 - x_1^0) + A_{13} e_{11}^{(1)} + A_{23} e_{22}^{(1)}] + \mu_{31}^{(1)}\} d\sigma, \\
L_{22} &= \int_{\Sigma} \{x_2 [E(x_2 - x_2^0) + A_{13} e_{11}^{(2)} + A_{23} e_{22}^{(2)}] + B_{55} + B_{56} \nu_1 + \mu_{31}^{(2)}\} \, d\sigma.
\end{aligned}
$$

Let us show that the linear equations (4.15) determine the constants a_1 and a_2.

We assume that the internal energy density

$$
(4.17)\qquad 2U(u, \varphi) = t_{ij} \varepsilon_{ij} + m_{ij} \varphi_{j,i}\,.
$$

is a positive definite quadratic form in ε_{ij} and $\varphi_{j,i}$.

Let us consider two elastic configurations $u_i', \varphi_i', \varepsilon_{ij}', t_{ij}', m_{ij}'$ and $u_i'', \varphi_i'', \varepsilon_{ij}'', t_{ij}'', m_{ij}''$. If we note

$$
(4.18)\qquad 2U(u', \varphi'; u'', \varphi'') = t_{ij}' \varepsilon_{ij}'' + m_{ij}' \varphi_{j,i}''
$$

we have

$$
(4.19)\qquad
\begin{aligned}
&U(u', \varphi'; u'', \varphi'') = U(u'', \varphi''; u', \varphi'), \\
&U(u, \varphi; u, \varphi) = U(u, \varphi).
\end{aligned}
$$

It is easy to show that

$$
(4.20)\qquad
\begin{aligned}
&\int_S (t_i' u_i'' + m_i' \varphi_i'') \, d\sigma = \int_S (t_i'' u_i' + m_i'' \varphi_i') \, d\sigma = 2 \int_V U(u', \varphi'; u'', \varphi'') \, d\sigma, \\
&\int_S (t_i u_i + m_i \varphi_i) \, d\sigma = 2 \int_V U(u, \varphi) \, d\sigma,
\end{aligned}
$$

where t_i and m_i are given by (2.4).

The relations (4.6), (4.8), (4.14) can be written in the form

$$
(4.21)\qquad
\begin{aligned}
&u_i = a_1 u_i^{(1)} + a_2 u_i^{(2)}, \quad \varphi_i = a_1 \varphi_i^{(1)} + a_2 \varphi_i^{(2)}, \\
&t_{ij} = a_1 t_{ij}^{(1)} + a_2 t_{ij}^{(2)}, \quad m_{ij} = a_1 m_{ij}^{(1)} + a_2 m_{ij}^{(2)}.
\end{aligned}
$$

It is easy to show that

$$
(4.22)\qquad
\begin{aligned}
2U(u, \varphi) = a_1^2 U(u^{(1)}, \varphi^{(1)}) &+ 2 a_1 a_2 U(u^{(1)}, \varphi^{(1)}; u^{(2)}, \varphi^{(2)}) + \\
&+ a_2^2 U(u^{(2)}, \varphi^{(2)}).
\end{aligned}
$$

The internal energy is

$$
(4.23)\qquad 2W = 2 \int_V U(u, \varphi) \, d\tau = a_1^2 W_{11} + 2a_1 a_2 W_{12} + a_2^2 W_{22},
$$

where

$$
(4.24)\qquad
\begin{aligned}
&W_{11} = \int_V U(u^{(1)}, \varphi^{(1)}) d\tau, \quad W_{12} = \int_V U(u^{(1)}, \varphi^{(1)}; u^{(2)}, \varphi^{(2)}) \, d\tau, \\
&W_{22} = \int_V U(u^{(2)}, \varphi^{(2)}) \, d\tau.
\end{aligned}
$$

The internal energy is a positive definite quadratic form, so that

$$
(4.25)\qquad W_{11} W_{22} - W_{12}^2 > 0.
$$

Let us apply the formula (4.20) for the functions $u_i^{(\alpha)}, \ldots, m_{ij}^{(\alpha)}, (\alpha = 1, 2)$. On the surface $x_3 = l$, we have

$$
(4.26)\qquad
\begin{aligned}
&t_1^{(\alpha)} = t_2^{(\alpha)} = 0, \ t_3^{(\alpha)} = t_{33}^{(\alpha)}, \ m_1^{(\alpha)} = m_{31}^{(\alpha)}, \ m_2^{(\alpha)} = m_{32}^{(\alpha)}, \ m_3^{(\alpha)} = 0, \\
&u_3^{(1)} = l(x_1 - x_1^0), \ \varphi_1^{(1)} = 0, \ \varphi_2^{(1)} = -l, \ u_3^{(2)} = l(x_2 - x_2^0), \ \varphi_1^{(2)} = l, \\
&\varphi_2^{(2)} = 0, \ (\alpha = 1, 2),
\end{aligned}
$$

and on $x_3 = 0$, we have

$$
(4.27)\qquad t_1^{(\alpha)} = t_2^{(\alpha)} = 0, \quad u_3^{(\alpha)} = 0, \quad \varphi_1^{(\alpha)} = \varphi_2^{(\alpha)} = 0, \quad m_3^{(\alpha)} = 0, \quad (\alpha = 1, 2).
$$

Taking into account the boundary conditions (4.5) and the relations (4.26), (4.27) from $(4.20)_2$, (4.24) we obtain

$$
(4.28)\qquad
\begin{aligned}
2W_{11} = l \int_{\Sigma} \{(x_1 - x_1^0)[E(x_1 - x_1^0) &+ A_{13} e_{11}^{(1)} + A_{23} e_{22}^{(1)}] + B_{45} \nu_2 + \\
&+ B_{55} - \mu_{32}^{(1)}\} \, d\sigma = l L_{11}, \\
2W_{22} = l \int_{\Sigma} \{(x_2 - x_2^0)[E(x_2 - x_2^0) &+ A_{13} e_{11}^{(2)} + A_{23} e_{22}^{(2)}] + B_{55} + B_{56} \nu_1 + \\
&+ \mu_{31}^{(2)}\} d\sigma = l L_{22},
\end{aligned}
$$

on the basis of relations (4.13).

If we apply the formula $(4.20)_1$ for $u_i' = u_i^{(1)}$, $\varphi_i' = \varphi_i^{(1)}$ and $u_i'' = u_i^{(2)}$ $\varphi_i'' = \varphi_i^{(2)}$ and use (4.24), (4.13), we get

$$2W_{12} = lL_{12} = lL_{21}\,. \tag{4.29}$$

From (4.25), (4.28), (4.29) it follows that

$$d = L_{11}L_{22} - L_{12}^2 \neq 0, \tag{4.30}$$

so that from (4.15) we obtain

$$a_1 = \frac{1}{d}(L_{22}M_2 + L_{12}M_1), \quad a_2 = -\frac{1}{d}(L_{11}M_1 + L_{12}M_2). \tag{4.31}$$

Thus, the solution of our problem has the form (4.6), where $v_\alpha^{(\rho)}, w^{(\rho)}$ $(\alpha, \rho = 1, 2)$ are the solutions of the problems (3.7)–(3.10) and the constants a_α are given by (4.31).

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ÎNCOVOIEREA DE CĂTRE CUPLURI A BARELOR ELASTICE MICROPOLARE ORTOTROPE

Rezumat

În această Notă sînt date ecuaţiile constitutive ale mediilor elastice micropolare ortotrope şi se rezolvă problema încovoierii de către cupluri a barelor omogene şi ortotrope în teoria liniară a elasticităţii micropolare.

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EXISTENCE THEOREMS IN THE THEORY OF CONTINUUM WITH DEFORMABLE DIRECTORS

BY

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1. Introduction. In the classical theory of finite elastostatics, the existence and the uniqueness of solution for a class of traction boundary valul problems was proved by F. Stoppelli [5]. The first boundary value problem for hyperelastic bodies, has been treated by I. Beju [1]. The main aim of this paper is to establish restrictions on the density of action L, in the theory of continuum with deformable directors, such as to assure the existence of the solution.

We reduce the boundary value problem to the problem of the minimum of a functional.

2. Formulation of the boundary value problem. Let $\mathcal{B}$ be an elastic body. We assume that the reference configuration of $\mathcal{B}$ is a bounded region $\Omega \subset R^3$, with boundary surface $\partial\Omega$ and that $\partial\Omega$ is sufficiently regular to assure the common laws of transformation of surface integrals and Friedrichs'inequality. We shall use Cartesian coordinates ξ^α $(\alpha = 1, 2, 3)$ for the place occupied by a particle of $\mathcal{B}$ in the reference configuration, and Cartesian coordinates x^i $(i = 1, 2, 3)$ for the place in other configuration. In the theory of continuum with deformable directors, it is considered that the motion of a generical particle of $\mathcal{B}$, at the time t, is given by

$$\begin{aligned} x^i &= x^i(\xi^\alpha, t), \\ d_{(a)}^i &= d_{(a)}^i(\xi^\alpha, t), \qquad \xi^\alpha \in \partial\Omega, \end{aligned} \tag{2.1}$$

where $d_{(a)}^i$ $(a = 1, 2, \ldots, n)$ is a set of n spatial vector fields.

Let $D_{(a)}^i$ be, the directors in the reference configuration. Throughout this paper the summation convention over the repeated indices is used. Using the Hamilton's principle, Toupin [6] has obtained
— the equilibrium equations