If we apply the formula $(4.20)_1$ for $u_i' = u_i^{(1)}$, $\varphi_i' = \varphi_i^{(1)}$ and $u_i'' = u_i^{(2)}$ $\varphi_i'' = \varphi_i^{(2)}$ and use (4.24), (4.13), we get

$$2W_{12} = lL_{12} = lL_{21}\,. \tag{4.29}$$

From (4.25), (4.28), (4.29) it follows that

$$d = L_{11}L_{22} - L_{12}^2 \neq 0, \tag{4.30}$$

so that from (4.15) we obtain

$$a_1 = \frac{1}{d}(L_{22}M_2 + L_{12}M_1), \quad a_2 = -\frac{1}{d}(L_{11}M_1 + L_{12}M_2). \tag{4.31}$$

Thus, the solution of our problem has the form (4.6), where $v_\alpha^{(\rho)}, w^{(\rho)}$ $(\alpha, \rho = 1, 2)$ are the solutions of the problems (3.7)—(3.10) and the constants a_α are given by (4.31).

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ÎNCOVOIEREA DE CĂTRE CUPLURI A BARELOR ELASTICE MICROPOLARE ORTOTROPE

Rezumat

În această Notă sînt date ecuaţiile constitutive ale mediilor elastice micropolare ortotrope şi se rezolvă problema încovoierii de către cupluri a barelor omogene şi ortotrope în teoria liniară a elasticităţii micropolare.

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EXISTENCE THEOREMS IN THE THEORY OF CONTINUUM WITH DEFORMABLE DIRECTORS

BY

S. CHIRIŢĂ

1. Introduction. In the classical theory of finite elastostatics, the existence and the uniqueness of solution for a class of traction boundary valul problems was proved by F. S t o p p e l l i [5]. The first boundary value problem for hyperelastic bodies, has been treated by I. B e j u [1]. The main aim of this paper is to establish restrictions on the density of action L, in the theory of continuum with deformable directors, such as to assure the existence of the solution.

We reduce the boundary value problem to the problem of the minimum of a functional.

2. Formulation of the boundary value problem. Let $\mathcal{B}$ be an elastic body. We assume that the reference configuration of $\mathcal{B}$ is a bounded region $\Omega \subset R^3$, with boundary surface $\partial\Omega$ and that $\partial\Omega$ is sufficiently regular to assure the common laws of transformation of surface integrals and Friedrichs'inequality. We shall use Cartesian coordinates ξ^α $(\alpha = 1, 2, 3)$ for the place occupied by a particle of $\mathcal{B}$ in the reference configuration, and Cartesian coordinates x^i $(i = 1, 2, 3)$ for the place in other configuration. In the theory of continuum with deformable directors, it is considered that the motion of a generical particle of $\mathcal{B}$, at the time t, is given by

$$\begin{aligned} x^i &= x^i(\xi^\alpha, t), \\ d^i_{(a)} &= d^i_{(a)}(\xi^\alpha, t), \qquad \xi^\alpha \in \partial\Omega, \end{aligned} \tag{2.1}$$

where $d^i_{(a)}$ $(a = 1, 2, \ldots, n)$ is a set of n spatial vector fields.

Let $D^i_{(a)}$ be, the directors in the reference configuration. Throughout this paper the summation convention over the repeated indices is used. Using the Hamilton's principle, T o u p i n [6] has obtained

— the equilibrium equations

$$T^{\alpha}_{i,\alpha} + F_i = -\frac{\partial L}{\partial x^i},$$

(2.2)

$$H^{(a)\alpha}_{i,\alpha} + G^{(a)}_i = -\frac{\partial L}{\partial d^i_{(a)}}, \quad \xi^{\alpha} \in \Omega;$$

— the constitutive equations

$$T^{\alpha}_i = -\frac{\partial L}{\partial x^i,\alpha},$$

(2.3)

$$H^{(a)\alpha}_i = -\frac{\partial L}{\partial d^i_{(a),\alpha}}, \quad \xi^{\alpha} \in \Omega;$$

— the Cauchy's formulae

(2.4)

$$T^{\alpha}_i N_{\alpha} - T_i = 0,$$
$$H^{(a)\alpha}_i N_{\alpha} - H^{(a)}_i = 0, \qquad \xi^{\alpha} \in \partial\Omega.$$

In these relations we have used the following notations: F_i and $G^{(a)}_i$, are certain generalized body forces, T_i and $H^{(a)}_i$ are certain generalized surface tractions, T^{α}_i and $H^{(a)\alpha}_i$ are certain generalized stress measured per face unit area of reference configuration, N_{α} is the unit normal to $\partial\Omega$, the comma denotes partial derivation with respect to the variable ξ^{α}, and L is the density of action

(2.5) $$L = L\,(x^i, d^i_{(a)}, x^i_{,\alpha}, d^i_{(a),\alpha}).$$

We have assumed that L does not depend explicitly on ξ^{α}. It is convenient to introduce the functions

(2.6) $$u^i = x^i - \xi^i; \quad u^i_{(a)} = d^i_{(a)} - D^i_{(a)}.$$

In what follows we consider the boundary value problem which consists in solving the equations (2.2), (2.3), (2.4), (2.5) with boundary conditions

(2.7)

$$u^i(\xi^{\alpha}) = \tilde{u}^i(\xi^{\alpha}),$$
$$u^i_{(a)}(\xi^{\alpha}) = \tilde{u}^i_{(a)}(\xi^{\alpha}), \qquad \xi^{\alpha} \in \partial\Omega,$$

where $\tilde{u}^i, \tilde{u}^i_{(a)}$ are prescribed functions.

3. Langenbach's theorems. Let Ω be a bounded region of R^n, with the boundary surface $\partial\Omega$ and let $\mathcal{H}(\Omega)$ be a Hilbert space on Ω. The boundary surface $\partial\Omega$ is assumed sufficiently regular to assure the validity of the common laws of transformation of surface integrals. Let P be an operator with

domain $D(P)$ and range $\mathcal{H}(\Omega)$, $D(P) \subset \mathcal{H}(\Omega)$, $D(P)$ being a linear subset, dense in $\mathcal{H}(\Omega)$.

Definition 1 [1]. *The operator P has a linear Gâteaux differential on $\omega \subset D(P)$ for $h \in D(P)$, if there exists an operator (DP) such that $(DP): \omega \to L(D(P), \mathcal{H}(\Omega))$, and*

(3.1) $$\lim_{t \to 0} \frac{P(x + th) - P(x)}{t} = (DP)(x)\,h, \quad x \in \omega, \; h \in D(P),$$

where $L(D(P), \mathcal{H}(\Omega))$ is the set of all linear operators from $D(P)$ in $\mathcal{H}(\Omega)$.

Remark 1 [1]. The relation between P and (DP) is given by

(3.2) $$Px - Px_0 = \int_0^1 (DP)(x_0 + t(x - x_0))(x - x_0)\,dt.$$

Definition 2 [4]. *The operator P is said to be monotone if, for all $u, v \in D(P)$,*

(3.3) $$< Pu - Pv, u - v > \;\geq 0.$$

The operator P is said to be strictly monotone if it is monotone and $< Pu - Pv, u - v > = 0$, only for $u = v$.

A. Langenbach [3], has investigated the nonlinear equation

(3.4) $$Pu = f,$$

with linear and homogeneous boundary conditions

(3.5) $$L_i u = 0, \quad (i = 1, 2, \ldots, p).$$

Let be

(3.6) $$D_0(P) = \{u \in D(P);\; L_i u = 0\},$$

and let f be an element of $\mathcal{H}(\Omega)$.

Theorem 1. [3]. *If*

1°) *$D_0(P)$ and $D(P)$ are linear sets, and $D_0(P)$ is dense in $\mathcal{H}(\Omega)$,*
2°) *for all $u, h \in D(P)$, P has a linear Gâteaux differential, and $(DP)(u)h$ is continuous mapping of u in every two-dimensional hyperplane which contains the point u,*
3°) $\quad P(0) = 0,$
4°) *for all $u \in D(P)$, $h, g \in D_0(P)$ we have*

$$< (DP)(u)\,h, g > \;=\; < (DP)(u)\,g, h >,$$

5°) *for all $u \in D(P)$, $h \in D_0(P)$, $h \neq 0$,*

$$< (DP)(u)\,h, h > \;> 0,$$

then:

a) *if there exists a solution $u_0 \in D_0(P)$ of the equation (3.4), it is unique, and it attains on $D_0(P)$ the minimum of the functional*

(3.7) $$F(u) = \int_0^1 < P(tu), u > dt - < f, u > \quad ;$$

b) *conversely, if an element of $D_0(P)$ attains on $D_0(P)$ the minimum of the functional (3.7), then it is a solution of (3.4).*

Theorem 2 [3]. *If the condition 5°) of theorem 1 is changed into the stronger one,*

(3.8) $$< (DP)(u)h, h > \geq C\|h\|^2, \; u \in D(P), \; h \in D_0(P), \; C > 0, \quad C = \text{const.},$$

then:
a) *the functional (3.7) is bounded below on $D_0(P)$,*
b) *the functional (3.7) is strictly convex on $D_0(P)$, that is, for $u, v \in D_0(P)$, $u \neq v$, $t \in (0, 1)$,*

$$F(tu + (1-t)v) < tF(u) + (1-t)F(v),$$

c) *any minimizing sequence of the functional (3.7) is convergent in $\mathcal{H}(\Omega)$.*

Definition 3 [3]. *We call a generalized solution of (3.4) (3.5) the limit of a minimizing sequence of the functional (3.7).*

Theorem 3 [3]. *If there exists a $u_0 \in D_0(P)$ such that the condition (3.8) assumes the form*

(3.9) $$< (DP)(u)h, h > \geq C_1 < (DP)(u_0)h, h > \geq C_2\|h\|^2, \; C_1, C_2 > 0,$$

then the generalized solution of (3.4), (3.5) is an element of the energetic space of the linear operator $(DP)(u_0)$.

4. The first boundary value problem. We consider the space consisting of all vectors with $(3n+3)$ dimensions in Ω. Let us denote by u the vector $(u^i, u^i_{(a)})$ and by T the vector $(T_i, H_i^{(a)})$ in this space. Let $\tilde{u}$ be the vector $(\tilde{u}^i, \tilde{u}^i_{(a)})$.

If v is a vector in Ω, so that $v(\xi^\alpha) = \tilde{u}(\xi^\alpha)$, $\xi^\alpha \in \partial\Omega$, then, introducing the unknown vector w through the relation $w = u - v$, we can make the boundary condition (2.7) to be homogeneous.

Taking into account (2.3), the equations (2.2) can be written in the form

(4.1) $$-\left[\frac{\partial L}{\partial x^i_{,\alpha}}\right]_{,\alpha} + \frac{\partial L}{\partial x^i} = -F_i, \qquad -\left[\frac{\partial L}{\partial d^i_{(a),\alpha}}\right]_{,\alpha} + \frac{\partial L}{\partial d^i_{(a)}} = -G_i^{(a)}, \; \xi^\alpha \in \Omega.$$

We introduce the operators

(4.2) $$\mathcal{A}_i u = -\left[\frac{\partial L}{\partial x^i_{,\alpha}}\right]_{,\alpha} + \frac{\partial L}{\partial x^i}, \qquad \mathcal{A}_i^{(a)} u = -\left[\frac{\partial L}{\partial d^i_{(a),\alpha}}\right]_{,\alpha} + \frac{\partial L}{\partial d^i_{(a)}}.$$

We note

(4.3) $$\mathcal{A}u = \begin{pmatrix} A_i u \\ A_i^{(a)} u \end{pmatrix}; \quad F = \begin{pmatrix} F_i \\ G_i^{(a)} \end{pmatrix},$$

so that. the equations (4.1) becomes

(4.4) $$\mathcal{A}u = -F, \; \xi^\alpha \in \Omega.$$

Introducing the operator $\mathcal{E}$ by

(4.5) $$\mathcal{E}w = \mathcal{A}u - \mathcal{A}v = \mathcal{A}(v + w) - \mathcal{A}(v),$$

we obtain the first boundary value problem in the form

(4.6) $$\mathcal{E}w = f, \; \xi^\alpha \in \Omega.$$

(4.7) $$w(\xi^\alpha) = 0, \quad \xi^\alpha \in \partial\Omega,$$

where

(4.8) $$f = -\mathcal{A}v - F.$$

Definition 1. *The operator $\mathcal{E}$ will be called the operator of finite elastostatics with deformable directors for the first boundary value problem.*

Let $L_2(\Omega)$ be the Hilbert space of vectorial functions with $(3n+3)$-components square-integrable on Ω. The norm of this space is generated by the scalar product

(4.9) $$< u, v > = \int_\Omega (u^i v^i + u^i_{(a)} v^i_{(a)})\, d\Omega,$$

where $u = (u^i, u^i_{(a)})$, $v = (v^i, v^i_{(a)})$.

Let $W_0^2(\Omega)$ denote the set of elements of $L_2(\Omega)$ belonging to $C^2(\Omega)$ and satisfying the condition (4.7). It is known that $W_0^2(\Omega)$ is a linear set, dense in $L_2(\Omega)$.

Let us denote $\mathcal{E}: W_0^2(\Omega) \to L_2(\Omega)$. We assume that $f \in L_2(\Omega)$.

Theorem 1. *If the function L has continuous derivatives of second order and satisfies the following inequality:*

$$
\begin{aligned}
(4.10)\quad & \int_{\Omega} \Big\{ \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial x^r_{,\beta}} \cdot g^i_{,\alpha}\, g^r_{,\beta} + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial d^r_{(b),\beta}} \cdot g^i_{(a),\alpha}\, g^r_{(b),\beta} + \frac{\partial^2 L}{\partial x^i\,\partial x^r} \cdot g^i g^r + \\
& + \frac{\partial^2 L}{\partial d^i_{(a)}\,\partial d^r_{(b)}} \cdot g^i_{(a)}\, g^r_{(b)} + 2 \Big[\frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial x^r} \cdot g^i_{,\alpha}\, g^r + \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial d^r_{(a)}} \cdot g^i_{,\alpha}\, g^r_{(a)} + \\
& + \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial d^r_{(a),\beta}} \cdot g^i_{,\alpha}\, g^r_{(a),\beta} + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial x^r} \cdot g^i_{(a),\alpha}\, g^r + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial d^r_{(b)}} \cdot g^i_{(a),\alpha}\, g^r_{(b)} + \\
& + \frac{\partial^2 L}{\partial d^r_{(a)}\,\partial x^i} \cdot g^r_{(a)}\, g^i \Big] \Big\} d\Omega > 0,
\end{aligned}
$$

for all $w, g \in W^2_0(\Omega)$, $g = (g^i, g^i_{(a)}) \neq 0$, then
a) *if a solution $w_0 \in W^2_0(\Omega)$ of the equation (4.6) exists, it is unique and attains on $W^2_0(\Omega)$ the minimum of the functional*

$$
(4.11)\qquad \mathcal{F}(w) = \int_0^1 < \mathcal{E}(tw), w > dt - < f, w > ;
$$

b) *conversely, if an element $w_0 \in W^2_0(\Omega)$ attains on $W^2_0(\Omega)$ the minimum of the functional (4.11), then it is a solution of (4.6).*

Proof. It is easy to show that the conditions of theorem 1 (sect. 3) are satisfied. Thus

1°) $W^2_0(\Omega)$ is linear set and dense in $L_2(\Omega)$,
2°) for all $w, g \in W^2_0(\Omega)$, $\mathcal{E}$ has the linear Gâteaux differential

$$
\begin{aligned}
(4.12)\quad & (\mathcal{D}\mathcal{E}_i)(w)\, g = -\Big\{ \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial x^r} \cdot g^r + \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial d^r_{(a)}} \cdot g^r_{(a)} + \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial x^r_{,\beta}} \cdot g^r_{,\beta} + \\
& + \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial d^r_{(a),\beta}} \cdot g^r_{(a),\beta} \Big\}_{,\alpha} + \frac{\partial^2 L}{\partial x^i\,\partial x^r} \cdot g^r + \frac{\partial^2 L}{\partial x^i\,\partial d^r_{(a)}} \cdot g^r_{(a)} + \\
& + \frac{\partial^2 L}{\partial x^i\,\partial x^r_{,\beta}} \cdot g^r_{,\beta} + \frac{\partial^2 L}{\partial x^i\,\partial d^r_{(a),\beta}} \cdot g^r_{(a),\beta}, \\
& (\mathcal{D}\mathcal{E}^{(a)}_i)(w)\, g = -\Big\{ \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial x^r} \cdot g^r + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial d^r_{(b)}} \cdot g^r_{(b)} + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial x^r_{,\beta}} \cdot g^r_{,\beta} + \\
& + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial d^r_{(b),\beta}} \cdot g^r_{(b),\beta} \Big\}_{,\alpha} + \frac{\partial^2 L}{\partial d^i_{(a)}\,\partial x^r} \cdot g^r + \frac{\partial^2 L}{\partial d^i_{(a)}\,\partial x^r_{(b)}} \cdot g^r_{(b)} + \\
& + \frac{\partial^2 L}{\partial d^i_{(a)}\,\partial x^r_{,\beta}} \cdot g^r_{,\beta} + \frac{\partial^2 L}{\partial d^i_{(a)}\,\partial d^r_{(b),\beta}} \cdot g^r_{(b),\beta},
\end{aligned}
$$

3°) from (4.5), $\mathcal{E}(0) = 0$,
4°) for all $w, g, h \in W^2_0(\Omega)$, $g = (g^i, g^i_{(a)})$, $h = (h^i, h^i_{(a)})$, we have

$$
\begin{aligned}
(4.13)\quad & < (\mathcal{D}\mathcal{E})(w)\, g, h > = \int_{\Omega} \Big\{ \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial x^r_{,\beta}} \cdot g^r_{,\beta}\, h^i_{,\alpha} + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial d^r_{(b),\beta}} \cdot g^r_{(b),\beta}\, h^i_{(a),\alpha} + \\
& + \frac{\partial^2 L}{\partial x^r\,\partial x^i} \cdot g^r h^i + \frac{\partial^2 L}{\partial d^i_{(a)}\,\partial d^r_{(b)}} \cdot g^r_{(b)}\, h^i_{(a)} + \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial x^r} (g^r h^i_{,\alpha} + h^r g^i_{,\alpha}) + \\
& + \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial d^r_{(a)}} (g^r_{(a)}\, h^i_{,\alpha} + h^r_{(a)}\, g^i_{,\alpha}) + \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial d^r_{(a),\beta}} (g^r_{(a),\beta}\, h^i_{,\alpha} + h^r_{(a),\beta}\, g^i_{,\alpha}) + \\
& + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial x^r} (g^r h^i_{(a),\alpha} + h^r g^i_{(a),\alpha}) + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial d^r_{(b)}} (g^r_{(b)}\, h^i_{(a),\alpha} + h^r_{(b)}\, g^i_{(a),\alpha}) + \\
& + \frac{\partial^2 L}{\partial d^r_{(a)}\,\partial x^i} (g^r_{(a)}\, h^i + h^r_{(a)}\, g^i) \Big\} d\Omega = < (\mathcal{D}\mathcal{E})(w)\, h, g >,
\end{aligned}
$$

5°) for all $w, g \in W^2_0(\Omega)$, $g \neq 0$, using (4.10), (4.13) we obtain

$$
< (\mathcal{D}\mathcal{E})(w)\, g, g > \; > 0.
$$

The proof results, then, from theorem 1 (sect. 3).

Lemma 1. *If there exists a solution $u \in C^2(\Omega)$ for (2.2), (2.3), (2.7) then this solution is unique.*

Proof. Let u_1 and u_2 be, two solutions of (2.2), (2.3), (2.7). If $\mathcal{A}$ from (4.4) is strictly monotone on $C^2(\Omega)$, then $< \mathcal{A}u_1 - \mathcal{A}u_2, u_1 - u_2 > = 0$ and $u_1 = u_2$. To prove that $\mathcal{A}$ is strictly monotone on $C^2(\Omega)$, we use the theorem of Minty [4]: *If $\mathcal{D}(\mathcal{A})$ is convex, then a sufficient condition for $\mathcal{A}$ to be strictly monotone on $\mathcal{D}(\mathcal{A})$ is that the derivative*

$$
\frac{d}{dt} [< \mathcal{A}(u + tg), g >]_{t=0}
$$

exists and is strictly positive for $u, v \in \mathcal{D}(\mathcal{A})$, $g = v - u$, $g \neq 0$.

In our case one can show that $\mathcal{D}(\mathcal{A})$ is a convex set. The directional derivative exists for all $u, v \in \mathcal{D}(\mathcal{A})$, $g = v - u$, $g|_{\partial\Omega} = 0$, and from (4.10)

$$
\frac{d}{dt} [< \mathcal{A}(u + tg), g >]_{t=0} = < (\mathcal{D}\mathcal{E})(u)\, g, g > \; > 0.
$$

On the basis of theorem 2 (sect. 3) we have

Theorem 2. *If the hypotheses of theorem 1, are satisfied, and if*

$$\mathcal{L} \equiv \int_\Omega \Big\{ \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial x^r_{,\beta}} \cdot g^i_{,\alpha}\, g^r_{,\beta} + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial d^r_{(b),\beta}} \cdot g^i_{(a),\alpha}\, g^r_{(b),\beta} + \frac{\partial^2 L}{\partial x^i\,\partial x^r} \cdot g^i g^r +$$

$$+ \frac{\partial^2 L}{\partial d^i_{(a)}\,\partial d^r_{(b)}} \cdot g^i_{(a)}\, g^r_{(b)} + 2\Big[\frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial x^r} \cdot g^i_{,\alpha}\, g^r + \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial d^r_{(a)}} \cdot g^i_{,\alpha}\, g^r_{(a)} +$$

$$+ \frac{\partial^2 L}{\partial x^i_{,\alpha}\,\partial d^r_{(a),\beta}} \cdot g^i_{,\alpha}\, g^r_{(a),\beta} + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial x^r} \cdot g^i_{(a),\alpha}\, g^r + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\,\partial d^r_{(b)}} \cdot g^i_{(a),\alpha}\, g^r_{(b)} +$$

$$+ \frac{\partial^2 L}{\partial d^r_{(a)}\,\partial x^i} \cdot g^r_{(a)}\, g^i \Big]\Big\} d\Omega \geqslant C \int_\Omega \{g^i g^i + g^i_{(a)} g^i_{(a)}\}\, d\Omega,$$

for all $g = (g^i, g^i_{(a)}) \in W^2_0(\Omega)$, $C = \text{const.}$, $C > 0$, *then*: a) *the functional* (4.11) *is bounded below on* $W^2_0(\Omega)$, b) *the functional* (4.11) *is strictly convex on* $W^2_0(\Omega)$, c) *any minimizing sequence of the functional* (4.11) *is convergent in* $L_2(\Omega)$, *and its limit is the generalized solution of equation* (4.6) *with the condition* (4.7), d) *the generalized solution is unique.*

From theorem 3 (sect. 3) we get:

Theorem 3. *If there exists an element* $w_0 \in W^2_0(\Omega)$ *and* $C_1 > 0$, $C_2 > 0$, *such that, for all* $w, g \in W^2_0(\Omega)$,

$$L(w) \geqslant C_1\, L(w_0) \geqslant C_2 \int_\Omega \{g^i g^i + g^i_{(a)} g^i_{(a)}\}\, d\Omega,$$

then the generalized solution of (4.6) *belongs to the energetic space of the linear operator* $(\mathcal{D}\mathcal{E})(w_0)$.

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A THEOREM OF EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE EQUATIONS OF NON-LINEAR THERMOELASTICITY

BY

I. NISTOR

§ 1. Preliminaries. In this paper we shall consider the equations of non-linear thermoelasticity with boundary conditions of place and temperature. We shall prove that there exists a unique solution of the system of the equations of finite thermoelasticity and that this solutions has Hölder-continuous second order derivatives. For other boundary conditions see [3].

Let $P(z)$ be an operator with domain in a real Banach space S and range in a real Banach space S' and let $V(z_0, R)$ be the ball $\| z_0 - z \| \leqslant R$ in the space S.

Let us consider the equation

$$P(z) = 0. \tag{1.1}$$

As in ([2], ch. 2) we can prove

Theorem 1.1. *Let the following conditions be satisfied:*

1. *The Fréchet derivative* $P'(z)$ *exists in the ball* $V(z_0, R)$ *and is uniformly continuous for* $z \in V(z_0, R)$ *i.e. for any* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $\| z_1 - z_2 \| < \delta$ *implies* $\| P'(z_1) - P'(z_2) \| < \varepsilon$.
2. *There exists* $[P'(z_0)]^{-1}$ *and* $\mu \| P(z_0) \| < (1 - K) R_0$, *where*

$$\mu = \| [P'(z_0)]^{-1} \|, \quad 0 < K < 1, \quad R_0 = \min\left\{\frac{R}{2}, \frac{\delta}{2}\right\}.$$

Then there exists a unique solution of the equation (1.1) *in the ball* $V(z_0, R)$.

§2. Fundamental hypotheses and auxiliary space of functions. We consider an elastic body and denote by C_0 a homogeneous reference configuration of the body. Let ρ be the density of the body in C_0 at temperature T_0, Σ_0 the boundary of C_0 and let x^i, y^i be the material and spatial coordinates respectively.

The equations of thermoelasticity are ([1], ch. VIII).