$$
\mathscr{L} \equiv \int_{\Omega} \Bigg\{ \frac{\partial^2 L}{\partial x^i_{,\alpha}\, \partial x^r_{,\beta}} \cdot g^i_{,\alpha}\, g^r_{,\beta} + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\, \partial d^r_{(b),\beta}} \cdot g^i_{(a),\alpha}\, g^r_{(b),\beta} + \frac{\partial^2 L}{\partial x^i\, \partial x^r} \cdot g^i g^r +
$$

$$
+ \frac{\partial^2 L}{\partial d^i_{(a)}\, \partial d^r_{(b)}} \cdot g^i_{(a)}\, g^r_{(b)} + 2 \Bigg[\frac{\partial^2 L}{\partial x^i_{,\alpha}\, \partial x^r} \cdot g^i_{,\alpha}\, g^r + \frac{\partial^2 L}{\partial x^i_{,\alpha}\, \partial d^r_{(a)}} \cdot g^i_{,\alpha}\, g^r_{(a)} +
$$

$$
+ \frac{\partial^2 L}{\partial x^i_{,\alpha}\, \partial d^r_{(a),\beta}} \cdot g^i_{,\alpha}\, g^r_{(a),\beta} + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\, \partial x^r} \cdot g^i_{(a),\alpha}\, g^r + \frac{\partial^2 L}{\partial d^i_{(a),\alpha}\, \partial d^r_{(b)}} \cdot g^i_{(a),\alpha}\, g^r_{(b)} +
$$

$$
+ \frac{\partial^2 L}{\partial d^r_{(a)}\, \partial x^i} \cdot g^r_{(a)}\, g^i \Bigg] \Bigg\} d\Omega \geqslant C \int_{\Omega} \{ g^i g^i + g^i_{(a)}\, g^i_{(a)} \}\, d\Omega ,
$$

for all $g = (g^i, g^i_{(a)}) \in W^2_0(\Omega)$, $C = \text{const.}$, $C > 0$, *then :* a) *the functional* (4.11) *is bounded below on* $W^2_0(\Omega)$, b) *the functional* (4.11) *is strictly convex on* $W^2_0(\Omega)$. c) *any minimizing sequence of the functional* (4.11) *is convergent in* $L_2(\Omega)$, *and its limit is the generalized solution of equation* (4.6) *with the condition* (4.7), d) *the generalized solution is unique.*

From theorem 3 (sect. 3) we get :

Theorem 3. *If there exists an element* $w_0 \in W^2_0(\Omega)$ *and* $C_1 > 0$, $C_2 > 0$, *such that, for all* $w, g \in W^2_0(\Omega)$.

$$
L(w) \geqslant C_1\ L(w_0) \geqslant C_2 \int_{\Omega} \{ g^i g^i + g^i_{(a)}\, g^i_{(a)} \}\, d\Omega,
$$

then the generalized solution of (4.6) *belongs to the energetic space of the linear operator* $(\mathcal{D}\,\mathcal{E})\,(w_0)$.

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A THEOREM OF EXISTENCE AND UNIQUENESS OF THE SOLUTION OF THE EQUATIONS OF NON-LINEAR THERMOELASTICITY

BY

I. NISTOR

§ 1. Preliminaries. In this paper we shall consider the equations of non-linear thermoelasticity with boundary conditions of place and temperature. We shall prove that there exists a unique solution of the system of the equations of finite thermoelasticity and that this solutions has Hölder-continuous second order derivatives. For other boundary conditions see [3].

Let $P(z)$ be an operator with domain in a real Banach space S and range in a real Banach space S' and let $V(z_0, R)$ be the ball $\| z_0 - z \| \leqslant R$ in the space S.

Let us consider the equation

$$
P(z) = 0. \tag{1.1}
$$

As in ([2], ch. 2) we can prove

Theorem 1.1. *Let the following conditions be satisfied :*
1. *The Fréchet derivative* $P'(z)$ *exists in the ball* $V(z_0, R)$ *and is uniformly continuous for* $z \in V(z_0, R)$ *i.e. for any* $\varepsilon > 0$ *there is a* $\delta > 0$ *such that* $\| z_1 - z_2 \| < \delta$ *implies* $\| P'(z_1) - P'(z_2) \| < \varepsilon$.
2. *There exists* $[P'(z_0)]^{-1}$ *and* $\mu \| P(z_0) \| < (1 - K) R_0$,
where

$$
\mu = \| [P'(z_0)]^{-1} \|, \quad 0 < K < 1, \quad R_0 = \min \left\{ \frac{R}{2}, \frac{\delta}{2} \right\}.
$$

Then there exists a unique solution of the equation (1.1) *in the ball* $V(z_0, R)$.

§2. Fundamental hypotheses and auxiliary space of functions. We consider an elastic body and denote by C_0 a homogeneous reference configuration of the body. Let ρ be the density of the body in C_0 at temperature T_0, Σ_0 the boundary of C_0 and let x^i, y^i be the material and spatial coordinates respectively.

The equations of thermoelasticity are ([1], ch. VIII).

$$\frac{\partial}{\partial x^j}(y^k_{,i}\, t^{ij}) + \rho F^k = 0, \qquad q^k_{,k} + W = 0, \qquad x \in C_0,$$

$$(2.1) \quad u^i(x^j) - a^i(x^j) = 0, \qquad T(x^j) - \theta(x^j) = 0, \qquad x \in \Sigma_0$$

$$t^{ij} = \frac{\partial}{\partial e_{ij}} A(e_{pq}, T), \; q^k = M^k(u^i_{,j}, T_{,j}, T), \quad 2e_{ij} = u^i_{,j} + u^j_{,i} + u^k_{,i}\, u^k_{,j}.$$

where we have used the following notations:
t^{ij} — the second Piola-Kirchhoff stress tensor, F^i — the components of body force vector F, q^k — the components of heat flux vector q per unit area of Σ_0, W — heat source density per unit volume in C_0, M^k are given functions which satisfy the condition ([5], p. 362)

$$M^k(u^i_{,j}, 0, T) = 0,$$

u^i — the components of displacement vector u, e_{ij} — the components of strain tensor, A — the free energy per unit volume in C_0, T — the temperature of the body, $a^i(x)$, $\theta(x)$ are given functions; the comma denotes partial differentiation with respect to x^k. Repeated indices are assumed to take values over the range (1, 2, 3).

We assume that:

1) the reference configuration is a configuration of natural equilibrium, the body is isotropic in C_0 and the region occupied by C_0 is compact,

2) the boundary Σ_0 has Hölder-continuous principal curvatures with exponent λ and it is possible to cover Σ_0 by a finite number of regions Σ_0^p each of which being parametrically representable on a disc R_p and every point of Σ_0 being interior at least to one Σ_0^p,

3) F, W are Hölder-continuous in C_0 with exponent λ; $a^i(x^j)$, $\theta(x^j)$ and their second order derivatives with respect to the parameters of some parametric representation of Σ. (as in No. 2) are Hölder-continuous with exponent λ,

4) the free energy function A is continuous together with its derivatives up to the fourth order with respect to $p_{ij} = u_{i,j}$ and T, when they vary in the ball E_1 of the 10-dimensional Euclidian space. having its center at the point $(0, 0, \ldots, 0, T_0)$ and radius R_1,

5) the functions M^k and their derivatives up to the third order are continuous in the ball E_2 of 13-dimensional Euclidian space p_{ij}, $p_k = T_{,k}$ and T, having its center at the point $(0, 0, \ldots, 0, T_0)$ and radius R_2.

Let u be a vector-valued function and T a scalar-valued function defined on C_0. We denote by S the set of elements $z = (u, T)$ having Hölder-continuous second order derivatives with exponent λ. We put

$$(2.2) \quad \begin{aligned} \|z\| = \sum_{i=1}^{3} (\max_{x\in C_0} |u^i| + \max_{x\in C_0} |T_{,i}|) + \sum_{i,j=1}^{3} (\max_{x\in C_0} |u^i_{,j}| + \max_{x\in C_0} |T_{,ij}| + \\ B_{ij}) + \max_{x\in C_0} |T| + \sum_{i,j,k=1}^{3} (\max_{x\in C_0} |u^i_{,jk}| + B^i_{jk}). \end{aligned}$$

where B^i_{jk} and B_{ij} are the Hölder-coefficients of $u^i_{,jk}$ and $T_{,ij}$ respectively. With the norm (2.2), S becomes a Banach space.

Let Φ, φ be vector-valued functions and let Θ, ψ be scalar-valued functions. We denote by S' the class of elements

$$Z = (\Phi, \Theta, \varphi, \psi) = (\Phi_1, \Phi_2, \Phi_3, \Theta, \varphi_1, \varphi_2, \varphi_3, \psi)$$

with following properties:
a) the functions Φ, Θ are defined on C_0 and here Hölder-continuous with exponent λ in C_0.

b) φ and ψ are defined on Σ_0; φ, ψ and their second order derivatives with respect to the parameters of some parametric representation of Σ_0 are Hölder-continuous with exponent λ.

We put

$$(2.3) \quad \begin{aligned} \|Z\| = \sum_{i=1}^{3} (\max_{x\in C_0} |\Phi_i| + \max_{x\in \Sigma_0} |\varphi_i| + B_i + D_i) + \max_{x\in C_0} |\Theta| + \max_{x\in \Sigma_0} |\psi| + \\ + D + B. \end{aligned}$$

where D_i, D are the Hölder-coefficients of Φ_i, Θ and B_i, B are defined as in [4]. It can be verified that S' with the norm (2.3) is a Banach space.

From hypotheses 4), 5) it follows that the free energy function with its derivatives up to the fourth order and the functions M^k toghether with their derivatives up to the third order are continuous in the ball $V(z_0, R)$ of the space S, where $z_0 = (0, 0, 0, T_0)$, $R = \min(R_1, R_2)$.

It is obvious that the equations (2.1) can be written

$$(2.4) \qquad P(z) = 0,$$

where $P(z)$ is an operator defined in the $V(z_0, R)$ of the space S and having its range in S'.

§3. Theorem of existence and uniqueness. In order to prove a theorem of existence and uniqueness of the solution to the problem (2.1) it is sufficient to show that $P(z)$ defined by (2.4) satisfies all the conditions of theorem 1.1.

We use the notations

$$z = (v^1, v^2, v^3, T'), \qquad h = (u^1, u^2, u^3, T).$$

where, $z \in V(z_0, R)$ and h is an arbitrary point of S.

$$M_l(z) = \frac{\partial}{\partial x^i}\left(\frac{\partial y^l}{\partial x^j} t^{ij}\right), \quad Q(z) = \frac{\partial}{\partial x^i} M^i(v^i_{,j})\, T'_{,j}, T')$$

$$\varphi^l(z) = v^l - a^l, \qquad \psi(z) = T' - \theta.$$

From conditions 4), 5) it follows that $P(z)$ has Gâteaux differential

$$\delta M_l(z, h) = \frac{\partial^2 u^l}{\partial x^i \partial x^j} t^{ij}(z) + \frac{\partial^2 v^l}{\partial x^i \partial x^j}\left(\frac{\partial t^{ij}(z)}{\partial v^k_{,p}} u^k_{,p} + \frac{\partial t^{ij}(z)}{\partial T'} T\right) +$$

(3.1)
$$+ \frac{\partial u^l}{\partial x^j} \frac{\partial t^{ij}(z)}{\partial x^j} + \left(\delta^l_j + \frac{\partial v^l}{\partial x^j}\right) \frac{\partial}{\partial x^i}\left(\frac{\partial t^{ij}(z)}{\partial v^k_{,p}} u^k_{,p} + \frac{\partial t^{ij}(z)}{\partial T'} T\right),$$

$$\delta\varphi^l(z, h) = u^l, \qquad \partial\psi(z, h) = T,$$

$$\delta Q(z, h) = \frac{\partial}{\partial x^i}\left(\frac{\partial M^i(z)}{\partial v^i_{,j}} u^i_{,j} + \frac{\partial M^i(z)}{\partial T'_{,i}} T_{,i} + \frac{\partial M^i}{\partial T'} T\right).$$

We denote by H_0, H_1 the largest of maxima of the absolute values in the ball $V(z_0, R)$ of the functions $\frac{\partial A}{\partial e_{ij}}$, M^k and their derivatives of the first, second and third order with respect to their arguments.

If $z \in V(z_0, R)$ one obtains

(3.2)
$$\left|\frac{\partial}{\partial x^j}\left(\frac{\partial t^{ij}(z)}{\partial u^k_{,l}}\right)\right| < RH_0 \left|\frac{\partial}{\partial x^j}\left(\frac{\partial t^{ij}(z)}{\partial T'}\right)\right| < RH_0;$$
$$\left|\frac{\partial}{\partial x^k}\left(\frac{\partial M^k(z)}{\partial u^i_{,j}}\right)\right| < RH_1, \left|\frac{\partial}{\partial x^k}\left(\frac{\partial M^k(z)}{\partial T'_{,i}}\right)\right| < RH_1, \left|\frac{\partial}{\partial x^k}\left(\frac{\partial M^k(z)}{\partial T'}\right)\right| < RH_1.$$

From (3.2) it follows

(3.3) $$|\delta M_l(z, h)| < (2 + 3R + R^2) H_0 \|h\|, \quad |\delta Q(z, h)| < (1 + R) H_1 \|h\|.$$

It can be easily shown that the Hölder-coefficients of the functions $\delta M_l(z, h)$, $\delta Q(z, h)$ and the functions $\delta\psi(z, h)$, $\delta\varphi(z, h)$ satisfy similar relations, therefore one may determine a positive number H_2 such that

(3.4) $$\|\delta P(z, h)\| < H_2 \|h\|,$$

which shows that Gâteaux differential $\delta P(z, h)$ is continuous in h at the point $h = 0$.

We consider the functions

$$g(t) = \delta M_l(z_2 + t(z_1 - z_2), h), \quad f(t) = \delta Q(z_2 + t(z_1 - z_2), h),$$

defined for $t \in [0, 1]$. We have

$$\delta M_l(z_1, h) - \delta M_l(z_2, h) = g(1) - g(0) = g'(\xi_1), \quad (0 < \xi_1 < 1),$$
$$\delta Q(z_1, h) - \delta Q(z_2, h) = f(1) - f(0) = f'(\xi_2), \quad (0 < \xi_2 < 1).$$

Taking into account (3.2) and the fact that $z_2 + t(z_1 - z_2) \in V(z_0, R)$ we obtain

(3.5)
$$|\delta M_l(z_1, h) - \delta M_l(z_2, h)| < [(1 + R) H_0 + (3 + R) H'] \cdot \|z_1 - z_2\| \cdot \|h\|,$$
$$|\delta Q(z_1, h) - \delta Q(z_2, h)| < (2H_1 + RH_1) \|z_1 - z_2\| \|h\|,$$

where

$$H' = \max[H_0, RH_0].$$

It can be proved that the Hölder-coefficients of the functions

$$\delta M_l(z_1, h) - \delta M_l(z_2, h), \qquad \delta Q(z_1, h) - \delta Q(z_2, h)$$

satisfy similar relations, hence one may determine a positive number H such that

(3.6) $$\|\delta P(z_1, h) - \delta P(z_2, h)\| < H \|z_1 - z_2\| \cdot \|h\|.$$

The relation (3.6) shows that the Gâteaux differential $\delta P(z, h)$ is uniformly continuous in z for $z \in V(z_0, R)$.

The conditions for the existence of the Fréchet derivative are satisfied. From (3.6) it follows that the Fréchet derivative $P'(z)$ is uniformly continuous in z for $z \in V(z_0, R)$.

The vanishing of the Fréchet differential of the operator $P(z)$ at the point z_0 is equivalent with the system of the equations of uncoupled problem of linear thermoelasticity, which is a linear elliptic system, hence it admits solution having Hölder-continuous second order derivatives with exponent λ. It is obvious that

$$P(z_0) = (\rho F^1, \rho F^2, \rho F^3, W, -a^1, -a^2, -a^3, T_0 - \theta$$

Thus we have proved the following theorem of existence and uniqueness of the solution to the equations of thermoelasticity:

Theorem 3.1. *If hypotheses 1)—5) are satisfied and if*

$$\mu \|P(z_0)\| < (1 - K) R_0,$$

where

$$\mu = \|[P'(z_0)]^{-1}\|, \quad 0 < K < 1, \quad R_0 = \min\left[\frac{R}{2}, \frac{\delta}{2}\right], \quad \delta = \frac{\varepsilon}{H},$$

then the system (2.1) has a unique solution in the ball $V(z_0, R_0)$ and the solution has Hölder-continuous second order derivatives.

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O TEOREMĂ DE EXISTENȚĂ ȘI UNICITATE PENTRU ECUAȚIILE TERMOELASTICITĂȚII NELINIARE

Rezumat

În acest articol se demonstrează o teoremă de existență și unicitate pentru ecuațiile termoelasticității neliniare în cazul static, cînd pe frontieră se dau deplasările și temperatura. Se arată că soluția admite derivate secunde hölderiene.

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ON THE CARDINAL SEQUENTIAL OUTER MEASURES

BY

DAN A. SIMOVICI

1. Introduction. In our paper [6] we introduced the cardinal sequential outer measure (CSOM) induced by a semiautomaton $A = (I, S, f; s_0)$ on its free semigroup. Here I is the alphabet of inputs, S is the set of states, $f: S \times I \to S$ is the transition function and s_0 is the initial state of the semiautomaton A. We shall use the same letter to denote the extension of f to $S \times I^*$, I^* being the free semigroup generated by the input alpghabet I. This extension is obtained by induction on the length of the words:

$$f(s, pi) = f(f(s, p), i), \quad \forall p \in I^*, \ i \in I.$$

The cardinal outer measure $\mu_A : \mathcal{P}(I^*) \to \overline{R}_+$ is given by

$$\mu_A (E) = |\{s \mid f(s_0, p) = s, \ p \in E\}|, \qquad \forall E \in \mathcal{P}(I^*),$$

where $|M|$ is the cardinal number of the set M.

It was shown [6] that the set of μ_A-measurable sets is an atomic Boolean algebra whose set of atoms $\mathcal{A}$ is composed by the languages fro mI^* which are representable by only one state in the semiautomaton A. Moreover, the languages from $\mathcal{A}$ form a partition of I^*. The closure operator $\zeta_\mu : \mathcal{P}(I^*) \to \mathcal{P}(I^*)$ related to the measure is:

$$\zeta_\mu (E) = \cup \{H \mid H \in \mathcal{A}, \ H \cap E \neq \emptyset\}.$$

We have proved that an outer measure $\mu : \mathcal{P}(I^*) \to R_+$ is induced by a semiautomaton iff the following three conditions are fulfiled:

i) the set of all μ-measurable sets is an atomic Boolean algebra, whose set of atoms is $\mathcal{A}$;

ii) for every $E \in \mathcal{A}$, and $p \in I^*$ there exists $E_1 \in \mathcal{A}$ so that $Ep \subseteq E'$;

iii) μ is invariant with respect to the closure ζ_μ, which means that

$$\mu(\xi_\mu (E)) = \mu (E), \qquad \forall E \in \mathcal{P}(I^*).$$

The *CSOM* induced by the semiautomaton A is sensitive enough to record the isomorphism type of the semiautomaton. It will be of concern to extend these ideas to partial semiautomata [3] and to automata.