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O TEOREMĂ DE EXISTENŢĂ ŞI UNICITATE PENTRU ECUAŢIILE TERMOELASTICITĂŢII NELINIARE

Rezumat

În acest articol se demonstrează o teoremă de existenţă şi unicitate pentru ecuaţiile termoelasticităţii neliniare în cazul static, cînd pe frontieră se dau deplasările şi temperatura. Se arată că soluţia admite derivate secunde hölderiene.

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ON THE CARDINAL SEQUENTIAL OUTER MEASURES

BY

DAN A. SIMOVICI

I. Introduction. In our paper [6] we introduced the cardinal sequential outer measure (CSOM) induced by a semiautomaton $A = (I, S, f; s_0)$ on its free semigroup. Here I is the alphabet of inputs, S is the set of states, $f: S \times I \to S$ is the transition function and s_0 is the initial state of the semiautomaton A. We shall use the same letter to denote the extension of f to $S \times I^*$, I^* being the free semigroup generated by the input alpghabet I. This extension is obtained by induction on the length of the words:

$$f(s, pi) = f(f(s, p), i), \ \forall p \in I^*, \ i \in I.$$

The cardinal outer measure $\mu_A : \mathcal{P}(I^*) \to \overline{R}_+$ is given by

$$\mu_A(E) = |\{s \mid f(s_0, p) = s, \ p \in E\}|, \qquad \forall E \in \mathcal{P}(I^*),$$

where $|M|$ is the cardinal number of the set M.

It was shown [6] that the set of μ_A-measurable sets is an atomic Boolean algebra whose set of atoms $\mathcal{A}$ is composed by the languages fro mI^* which are representable by only one state in the semiautomaton A. Moreover, the languages from $\mathcal{A}$ form a partition of I^*. The closure operator $\zeta_\mu : \mathcal{P}(I^*) \to \mathcal{P}(I^*)$ related to the measure is:

$$\zeta_\mu(E) = \cup \{H \mid H \in \mathcal{A}, \ H \cap E \neq \emptyset\}.$$

We have proved that an outer measure $\mu : \mathcal{P}(I^*) \to R_+$ is induced by a semiautomaton iff the following three conditions are fulfiled:

i) the set of all μ-measurable sets is an atomic Boolean algebra, whose set of atoms is $\mathcal{A}$;

ii) for every $E \in \mathcal{A}$, and $p \in I^*$ there exists $E_1 \in \mathcal{A}$ so that $Ep \subseteq E'$;

iii) μ is invariant with respect to the closure ζ_μ, which means that

$$\mu(\xi_\mu(E)) = \mu(E), \qquad \forall E \in \mathcal{P}(I^*).$$

The *CSOM* induced by the semiautomaton A is sensitive enough to record the isomorphism type of the semiautomaton. It will be of concern to extend these ideas to partial semiautomata [3] and to automata.

Further we shall investigate the behaviour of *CSOM* with respect to free semigroup homomorphisms, the properties of measurable functions between two sequential measure spaces a.s.o.

II. Partial semiautomata. A *partial semiautomaton* is a 5-uple $A = (\rho, I, S, f; s_0)$ where $\rho \subseteq S \times I$ is the applicability relation, I, S and s_0 have the previous significances and $f: \rho \to S$ is the next-state function.

The relation ρ can be extended to a part of $S \times I^*$ by taking inductively on the length of the words

$$(s, pi) \in \hat{\rho}, \quad s \in S, \quad p \in I^*, \text{ iff } (s, p) \in \hat{\rho} \text{ and } (f(s, p), i) \in \rho.$$

The extension of f is given by

$$f(s, pi) = f(f(s, p), i) \text{ iff } (s, p) \in \hat{\rho} \text{ and } (f(s, p), i) \in \rho.$$

Let now $A_j = (\rho_j, I, S_j, f_j; s_{0j})$, $j = 1, 2$ two partial semiautomata. A homomorphism of partial semiautomata is a mapping $h: S_1 \to S_2$ which satisfies the following three conditions:

i) $h(s_{01}) = s_{02}$,

ii) if $(s_1, p) \in \hat{\rho}_1$ then $(h(s_1), p) \in \hat{\rho}_2$, $\forall s_1 \in S_1$, $p \in I^*$,

iii) if $(s, p) \in \hat{\rho}_1$ then $h(f(s_1, p)) = f_2(h(s_1), p)$.

The homomorphism h is an isomorphism of partial semiautomata iff h is a bijection.

A partial semiautomaton $A = (\rho, I, S, f; s_0)$ induces an outer measure on its free semigroup of inputs by taking

$$\mu_A(E) = |\{s \mid \exists p \in E \cap \alpha, \quad f(s_0, p) = s\}|.$$

Here $\alpha = \{p \mid (s_0, p) \in \rho\}$.

In [6] we have established the following

Theorem 1. *Let $A_j = (I, S_j, f_j; s_{0j})$, $j = 1, 2$ be two semiautomata both of which being connected by their initial states. The following three assertions are equivalent:*

i) *There exists a surjective semiautomata homomorphism $h: S_1 \to S_2$.*

ii) $\mu_{A_1}(E) \geqslant \mu_{A_2}(E)$, $\forall E \in \mathcal{P}(I^*)$,

iii) $\mu_{A_1}(E) \geqslant \mu_{A_2}(E)$, $\forall E \in \mathcal{P}(I^*)$, $|E| \leqslant 2$.

This result can be extended to partial semiautomata. Let us consider $A_j = (\rho_j, I, S_j, f_j : s_{0j})$, $j = 1, 2$, two partial semiautomata and $\alpha_j = \{p \mid (s_{0j}, p) \in \check{\rho}_j, j = 1, 2$ the sets of the words which are applicable to the initial states of the two semiautomata.

Theorem 2. *The following three assertions are equivalent for partial automata connected by their initial states:*

i) *There exists a surjective partial automata homomorphism $h: S_1 \to S_2$.*

ii) $\mu_{A_1}(E) \geqslant \mu_{A_2}(E)$, $\forall E \in \mathcal{P}(I^*)$, $\alpha_1 \subseteq \alpha_2$,

iii) $\mu_{A_1}(E) \geqslant \mu_{A_2}(E)$, $\forall E \in \mathcal{P}(I^*)$, $|E| \leqslant 2$, $\alpha_1 \subseteq \alpha_2$.

Proof. Only the implications i) $\to$ ii) and iii) $\to$ i) need a proof since ii) $\to$ iii) is obvious.

i) $\to$ ii) Suppose that there exists a surjective homomorphism $h: S_1 \to S_2$ and let $S' = \{s \mid \exists p \in E\ (s_{01}, p) \in \hat{\rho}_1,\ f_1(s_{01}, p) = s_1\}$, $S'' = \{s_2 \mid \exists p \in E, (s_{02}, p) \in \hat{\rho}_2,\ f_2(s_{02}, p) = s_2\}$. The mapping $\delta: S' \to S''$ given by $\delta(s_1) = f_2(s_{02}, p)$, where $f_1(s_{01}, p) = s_1$ is well defined for, if $f_1(s_{01}, q) = f_1(s_{01}, r)$ it follows that $f_2(s_{02}, q) = f_2(h(s_{01}), q) = h(f_1(s_{01}, p)) = h(f_1(s_{01}, q)) = f_2(s_{02}, p)$. Since δ is a surjection we obtain $|S'| \geqslant |S''|$ and it follows that $\mu_{A_1}(E) \geqslant \mu_{A_2}(E)$.

It is clear, that, if $p \in \alpha_1$, in view of the conditions i) and ii) from the definition of homomorphisms of partial automata it follows that $p \in \alpha_2$. Moreover, we can prove that i) implies a stronger property. If $p \in \alpha_2$, then $\mu_{A_1}(\{p\}) = 1 \leqslant \mu_{A_1}(\{p\}) \leqslant 1$ from which we obtain that p is applicable to the state s_{01}, hence $\alpha_1 = \alpha_2$.

iii) $\to$ i). Let us consider the mapping $h: S_1 \to S_2$ given by $h(s) = f_2(s_{02}, p)$, where $s = f(s_{01}, p)$. This mapping is well defined. Indeed, since A_1 is connected by its initial state for every $s \in S_1$, there exists p so that $s = f_1(s_{01}, p)$ hence $p \in \alpha_1$ and therefore $p \in \alpha_2$. Moreover, if $f_1(s_{01}, p) = f_1(s_{01}, q)$ it follows $\mu_{A_1}(\{p, q\}) = 1 \geqslant \mu_{A_2}(\{p, q\}) \geqslant 1$, from which we obtain $f_2(s_{02}, p) = f_2(s_{02}, q)$.

Since A_2 is connected by its initial state s_{02}, the mapping h is a surjective one. This mapping is also transition preserving, for, if $s = f_1(s_{01}, q)$ and $(s, p) \in \rho_1$ we obtain $h(f_1(s_1, p)) = h(f_1(f_1(s_{01}, q), p)) = h(f_1(s_{01}, qp)) = f_2(s_{02}, qp) = f_2(f_2(s_{02}, q), p) = f_2(h(s), p)$. Hence h is a surjective homomorphism.

Corollary 1. *If $A_j = (\rho_j, I, S_j, f_j, s_{0j})$, $j = 1, 2$ are two partial semiautomata connected by their initial states and $\mu_{A_1}(E) = \mu_{A_2}(E)$, $\forall E \in \mathcal{P}(I^*)$, $|E| \leqslant 2$, $\alpha_1 = \alpha_2$ then A_1 and A_2 are isomorphic partial semiautomata.*

If we drop the condition of connexity for A_2, for instance, and $\mu_{A_1} = \mu_{A_2}$, $\alpha_1 = \alpha_2$ then A_1 is isomorphic to a subsemiautomaton of A_2.

Let $A = (I, S, O, f, g; s_0)$ be an automaton, where I, S, f, s_0 have the previous significances, O is the output alphabet and $g: S \times I \to O$ is the output function.

This function can be extended to $S \times I^*$ by taking inductively on the length of the words

$$g(s, pi) = g(s, p)\, g(f(s, p), i), \quad \forall s \in S,\ p \in I^*,\ i \in I.$$

We shall relate to an automaton two semiautomata : the trunk of the automaton $\mathbf{Tr}(A) = (I, S, f; s_0)$ and the partial semiautomaton $\mathbf{Pa}(A) = (\rho, I \times O, S, \varphi; s_0)$ where the applicability relation is given by $\rho = \{(s, (i, O)) \mid g(s, i) = O\}$ and $\varphi(s, (i, o)) = f(s, i)$. By definition, an automaton induces on the free semigroup of its inputs the same outer measure as $\mathbf{Tr}(A)$ does.

Let $\mathrm{pr}_1 : I \times O \to I$, $\mathrm{pr}_2 : I \times O \to O$ the projections given by $\mathrm{pr}_1((i,O)) = i$, $\mathrm{pr}_2((i,O)) = O$. We shall use the same symbols to denote their extensions to the homomorphisms $\mathrm{pr}_1 : (I \times O)^* \to I^*$ and $\mathrm{pr}_2 : (I \times O)^* \to O^*$. A word from $(I \times O)^*$, $r = (i_1, o_1)(i_2, o_2) \dots (i_n, o_n)$ will be denoted further $r = (\mathrm{pr}_1 r, \mathrm{pr}_2 r)$.

The extension of the applicability relation for $\mathbf{Pa}(A)$ is given by

$$\hat{\rho} = \{(s, (p, q)) \mid g(s, p) = q\},$$

where $l(p) = l(q)$. Here $l(p)$ is the length of the word p.

As a matter of fact we remark that the applicability relation of $\mathbf{Pa}(A)$ is, after all, the sequential relation generated by the automaton A.

Let $A_j = (I, S_j, O, f_j, g_j; s_{0j})$, $j = 1, 2$ be two automata having the same input and output alphabet. A homomorphism from A_1 to A_2 is a mapping $h : S_1 \to S_2$ so that

$$h(f_1(s, p)) = f_2(h(s), p), \tag{1}$$

$$g_1(s, p) = g_2(h(s), p), \tag{2}$$

for every $s \in S$, $p \in I^*$.

Theorem 3. *The following assertions are equivalent for automata $A_j = (I, S_j, O, f_j, g_j; s_{0j})$ connected by their initial states:*

i) *there exists a surjective homomorphism from $\mathbf{Tr}(A_1)$ to $\mathbf{Tr}(A_2)$ and $g_1(s_{01}, p) = g_2(s_{02}, p)$, $\forall p \in I^*$;*

ii) *there exists a surjective homomorphism from A_1 to A_2;*

iii) *there exists a surjective homomorphism from $\mathbf{Pa}(A_1)$ to $\mathbf{Pa}(A_2)$;*

iv) $\mu_1(E) \geqslant \mu_2(E)$, $\forall E \in \mathcal{P}(I^*)$ *and* $g_1(s_{01}, p) = g_2(s_{02}, p)$, $\forall p \in I^*$;

v) $\mu_1(E) \geqslant \mu_2(E)$, $\forall E \in \mathcal{P}(I^*)$, $|E| \leqslant 2$ *and* $g_1(s_{01}, p) = g_2(s_{02}, p)$, $\forall p \in I^*$.

Proof. i) $\to$ ii) : Suppose that there exists the mapping $h : S_1 \to S_2$, so that $f_1(s, p) = f_2(h(s), p)$, $\forall s \in S_1$, $p \in I^*$ and $h(s_{01}) = s_{02}$. We have to prove that $g_1(s, p) = g_2(h(s), p)$, $\forall p \in I^*$, $s \in S_1$. Let us consider the state $s \in S_1$. Since A_1 is connected by its initial state, there exists $q \in I^*$ so that $f_1(s_{01}, q) = s$, hence $f(s_{02}, q) = h(s)$. Now $g_1(s, p) = g_1(f_1(s_{01}, q), p)$ and $g_1(s, p)$ is the suffix represented by the last $l(p)$ symbols of the word $g_1(s_{01}, qp)$. On the other kanv $g_2(h(s), p) = g_2(f_2(s_{02}, q), p)$, and $g_2(h(s), p)$ consists of the last $l(p)$ symbols of the word $g_2(s_{02}, qp)$. Since $g_1(s_{01}, qp) = g_2(s_{02}, qp)$ we have obtained the implication we looked for.

ii) $\to$ iii) : Let $h : S_1 \to S_2$ be a homomorphism of automata. We have

$$h(f_1(s, p)) = f_2(h(s), p), \quad \forall s \in S_1, \ p \in I^*, \ h(s_{01}) = s_{02}. \tag{3}$$

$$g_1(s, p) = g_2(h(s), p), \quad \forall s \in S_1, \ p \in I^*, \tag{4}$$

In view of (3) the first and the third condition from the definition of homomorphisms of partial semiautomata is satisfied.

Suppose now that $r \in (I \times O)^*$ and $(s_1, r) \in \hat{\rho}_1$, where $\hat{\rho}_1$ is the extended applicability relation of $\mathbf{Pa}(A_1)$. Then $g_1(s_1, \mathrm{pr}_1 r) = \mathrm{pr}_2 r$ and it follows, taking into account (4), that $g_2(h(s), \mathrm{pr}_1 r) = \mathrm{pr}_2 r$ which implies that $(h(s), r) \in \hat{\rho}_2$.

iii) $\to$ iv) : Taking into account Theorem 2 we obtain $\mu_1(E) \geqslant \mu_2(E)$, $\forall E \in \mathcal{P}(I^*)$ and $\alpha_1 \subseteq \alpha_2$. Suppose now that $g_1(s_{01}, p) = q$. It follows that $(s_{01}, (p, q)) \in \alpha_1$ and therefore $(s_{02}, (p, q)) \in \alpha_2$. Hence $g(s_{02}, p) = q = g(s_{01}, p)$, $\forall p \in I^*$.

iv) $\to$ v) and v) $\to$ i) are obvious implications in view of the same Theorem.

III. **Partitions and outer measures.** Let π be a partition of the set M. The outer measure induced by this partition $\mu_\pi : \mathcal{P}(M) \to R_+$ is given by

$$\mu_\pi(K) = |\{H \mid H \in \pi, \ H \cap K \neq \emptyset\}|.$$

Let us remark that $\mu_\pi(K) = 0$ implies $K = \emptyset$ and if $|\pi| < \infty$, it follows that $\mu_\pi(K) < \infty$, $\forall K \in \mathcal{P}(M)$.

Theorem 4. *A subset $M_1 \subseteq M$ is μ_π-measurable iff it is an union of blocks belonging to π.*

Proof. Let M_1 be a μ_π-measurable set. We have

$$\mu_\pi(K) = \mu_\pi(K \cap M_1) + \mu_\pi(K \cap M_1^c), \quad \forall K \in \pi.$$

Choosing K as a block of π, we obtain

$$1 = \mu_\pi(K \cap M_1) + \mu_\pi(K \cap M_1^c).$$

We conclude that one of the two sets $K \cap M_1$ and $K \cap M_1^c$, must be void, hence every block is included either in M_1 or in M_1^c. Therefore M_1 is an union of blocks belonging to π.

Conversely, suppose that $M_1 = \cup \{H_j \mid j \in J, \ H_j \in \pi\}$. It follows that $M_1^c = \cup \{H_{j'} \mid j' \in J'\}$ and

$$\begin{aligned} \mu_\pi(K) &= |\{H \mid H \in \pi, \ H \cap K \neq \emptyset\}| = \\ &= |\{H_j \mid H_j \in \pi, \ j \in J, \ H_j \cap K \neq \emptyset\} \cup \\ &\quad \{H_{j'} \mid H_{j'} \in \pi, \ j' \in J', \ H_{j'} \cap K \neq \emptyset\}| = \\ &= \mu_\pi(K \cap M_1) + \mu_\pi(K \cap M_1^c), \quad \forall K \in \mathcal{P}(M), \end{aligned}$$

which implies that M is a μ_π-measurable set.

Corollary 2. *The Boolean algebra of the μ_π-measurable sets is always an atomic Boolean algebra. The set of atoms coincides with the members of π.*

If $\zeta_\pi : \mathcal{P}(M) \to \mathcal{P}(M)$ is the closure operator defined by

$$\zeta_\pi(K) = \cup \{H \mid H \in \pi,\ H \cap K \neq 0\},$$

hen μ_π is invariant with respect to ζ_π which means that

$$\mu_\pi(\zeta(K)) = \mu_\pi(K),\ \forall K \subseteq M.$$

Corollary 3. *Suppose that I^* is a free semigroup and $\alpha \subseteq I^* \times I^*$ is a right congruence on I^*. Then the measure μ_α induced by the partition π_α related to the right congruence α is a CSOM, for the first and the third conditions from the definition of CSOMs are fulfiled and if $E \in \pi_\alpha$ and $p \in I^*$ it follows immediately the existence of an atom E' so that $Ep \subseteq E'$.*

Let now $h : A \to B$ be a mapping and μ an outer measure on B. Then the mapping $\mu_1 : \mathcal{P}(A) \to \bar{R}_+$ given by $\mu_1(L) = \mu(h(L))$ is an outer measure.

Moreover, if E is μ-measurable $h^{-1}(E)$ is μ_1-measurable. Indeed since $(h^{-1}(K))^c = h^{-1}(K^c)$ for every $K \in \mathcal{P}(B)$ we have:

$$h(H \cap h^{-1}(K)^c) = h(H \cap h^{-1}(K^c)) \subseteq h(H) \cap h(h^{-1}(K^c)) =$$

$$= h(H) \cap K^c,\ \forall H \in \mathcal{P}(A),$$

and

$$h(H \cap h^{-1}(K)) \subseteq h(K) \cap K,\ \forall H \in \mathcal{P}(A).$$

Therefore

$$\mu_1(H) = \mu(h(H)) = \mu(h(H) \cap E) + \mu(h(H) \cap E^c) \geqslant$$

$$\geqslant \mu(h(H \cap h^{-1}(E))) + \mu(h(H \cap (h^{-1}(E))^c)) =$$

$$= \mu_1(H \cap h^{-1}(E)) + \mu_1(H \cap (h^{-1}(E))^c),$$

which implies that $h^{-1}(E)$ is μ_1-measurable.

If h is an injective mapping, since $L = h^{-1}(h(L))$, a subset $L \in \mathcal{P}(A)$ will be μ_1-measurable if $h(L)$ is μ-measurable.

Let μ be a cardinal sequential outer measure on the free semigroup I^*, $M(\mu)$ the Boolean algebra of all μ-measurable sets and $\mathcal{A}_\mu$ the set of atoms of this algebra.

Theorem 5. *If $h : I_1^* \to I^*$ is a free semigroup homomorphism and μ is a cardinal sequential outer measure then the measure previously defined is a sequential outer measure; moreover μ_1 is equal to the measure induced by the partition $\pi_1 = \{h^{-1}(K) \mid K \in \mathcal{A}_\mu\}$ on I^*.*

Proof. Let us consider firstly the second part of the theorem. We have $\mu_1(L) = \mu(h(L)) = |\{H \mid H \cap h(L) \neq 0,\ H \in \mathcal{A}_\mu\}|$. But $\mu_{\pi_1}(L) = |\{h^{-1}(K) \mid h^{-1}(K) \cap L \neq 0,\ K \in \mathcal{A}_\mu\}|$. The mapping $\alpha : \{H \mid H \cap h(L) \neq 0,\ H \in \mathcal{A}_\mu\} \to \{h^{-1}(K) \mid h^{-1}(K) \cap L \neq 0,\ K \in \mathcal{A}_\mu\}$ given by $\alpha(H) = h^{-1}(H)$ is a bijection for, it is obvious surjective and if $h^{-1}(H_1) = h^{-1}(H_2)$ it follows immediately $H_1 = H_2$. Therefore $\mu_1(L) = \mu_{\pi_1}(L)$.

In view of the corollary 2 it follows that $M(\mu_1)$ is an atomic Boolean algebra having the blocks of π_1, as its atoms.

Let $E \in \pi_1$ and $p \in I_1^*$. Since $E \in \pi_1$ there exists $K \in \mathcal{A}_\mu$ so that $E = h^{-1}(K)$. We have $Ep = h^{-1}(K)p$. Taking into account that μ is a sequential measure, there exists $K_1 \in \mathcal{A}_\mu$ for which $Kh(p) \subseteq K_1$. Therefore $Ep \subseteq h^{-1}(Kh(p)) \subseteq h^{-1}(K_1)$ and the existence of the atom $h^{-1}(K_1) = E_1$ indicates that μ_1 satisfies the second characterization condition of sequential outer measures.

Let now $H \subseteq I_1^*$ be a language. In view of the definition of the closure ζ_{μ_1}, we obtain

$$\mu_1(\zeta_{\mu_1}(H)) = \mu(\cup \{E \mid E \in \pi_1,\ E \cap H \neq 0\}) =$$

$$= |\{E \mid E \in \pi_1,\ E \cap H \neq 0\}| = \mu_1(H),$$

hence μ_1 is invariant with respect to its related closure ζ_{μ_1}. We conclude that μ_1 is a cardinal sequential outer measure. We shall denote μ_1 by ${}^h\mu$.

Let us now consider the mapping $h : A \to B$ and μ_1 an outer measure on A. It is easy to check that the mapping $\mu_2 : \mathcal{P}(B) \to \bar{R}_+$ given by $\mu_2(H) = \mu_1(h^{-1}(H))$, $\forall H \in B$ is an outer measure on B. We shall denote it by μ_1^h. By $\varkappa_h$ we shall denote the nuclear closure operator $\varkappa_h : \mathcal{P}(A) \to \mathcal{P}(A)$ defined by

$$\varkappa_h(K) = \{a \mid \exists k \in K,\ h(a) = h(k)\}.$$

If $\varkappa_h(E)$ is μ_1-measurable then $h(E)$ is μ_1^h-measurable. Indeed, suppose that for $E \subseteq A$, $\varkappa_h(E)$ is μ_1-measurable and let $K \subseteq B$. We have

$$\mu_2(K) = \mu_1(h^{-1}(K)) \geqslant \mu_1(h^{-1}(K) \cap \varkappa_h(E)) + \mu_1(h^{-1}(K) \cap \varkappa_h(E)^c)$$

since $\varkappa_h(E)$ is μ_1-measurable.

Afterwards

$$\mu_1(h^{-1}(K) \cap h^{-1}h(E)) + \mu_1(h^{-1}(K) \cap (h^{-1}h(E))^c) =$$

$$= \mu_1(h^{-1}(K \cap h(E)) + \mu_1(h^{-1}(K) \cap h^{-1}(H(E)^c)) =$$

$$= \mu_1(h^{-1}(K \cap h(E))) + \mu_1(h^{-1}(K \cap h(E)^c)) = \mu_2(K \cap h(E)) +$$

$$+ \mu_2(K \cap h(E)^c)$$

which points that $h(E)$ is μ_2-measurable.

Let us remark that if h is an injective mapping, then $\varkappa_h(E) = E$ for every $E \in \mathcal{P}(A)$ and, in this special case, the μ_1-measurability of E implies the μ_2-measurability of $h(E)$.

Theorem 6. *Let $h : I_1^* \to I^*$ be an injective homomorphism of free semigroups and let μ_1 be a CSOM on I_1^*. Then μ_1^h is a CSOM on the subsemigroup $h(I_1^*)$ so that $K \in M(\mu_1)$ implies $h(K) \in M(\mu_1^h)$.*

Proof. Since h is injective the family of sets $\mathcal{A}' = \{h(E) \mid E \in \mathcal{A}_{\mu_1}\}$ is a partition of the subsemigroup $h(I_1^*)$. Here $\mathcal{A}_{\mu_1}$ is the set of atoms of the Boolean algebra $M(\mu_1)$. In view of the previous considerations, these sets are μ_1^h-measurable. Let now $K \subseteq h(I_1^*)$. Then $\mu_1^h(K) = \mu_1(h^{-1}(K)) = |\{L \mid L \in \mathcal{A}_{\mu_1},\ L \cap h^{-1}(K) \neq 0\}| = |\{h(L) \mid h(L) \cap K \neq 0\}|$ since $J \cap h^{-1}(K) \neq 0$ iff $K \cap h(J) \neq 0$ (because $K \subseteq h(I_1^*)$). Therefore $M(\mu_1^h)$ is an atomic Boolean algebra, the measure μ_1^h being induced by the partition $\mathcal{A}'$.

Let us now consider $h(E) \in \mathcal{A}'$ and the word $p \in h(I_1^*)$. Then $p = h(q)$ and q is the unique word having this property. Therefore $h(E)p = h(E)h(q) = h(Eq)$. There exists E' so that $Eq \subseteq E'$, from which it follows $h(E)p \subseteq h(E') \in \mathcal{A}'$. It is obvious that the third condition from the definition of CSOMs is also satisfied.

Corollary 4. *Let $h : I_1^* \to I^*$ be an isomorphism. Then for every CSOM $\mu_1 : \mathrm{P}(I_1^*) \to R_+$ the measure μ_1^h is a CSOM on I_2^*.*

We observe that, if μ_A is a measure on A, μ_B a measure on B, $h : A \to B$ and $\varkappa_h$ is the closure related to the nuclear equivalence of h, we obtain immediately that

$$^h(\mu_A^h) = \mu_A \circ \varkappa_h\,,$$

$$(^h\mu_B)^h = \mu_B\,.$$

IV. The states of a sequential outer measure. In this paragraph we shall use again the background of our paper [6], namely the measure semiautomata.

Let μ be a sequential outer measure $\mu : \mathcal{P}(I^*) \to R_+$. The state of the measure μ in the word $p \in I^*$ is the mapping $\mu_p : \mathcal{P}(I^*) \to R_+$ given by

$$\mu_p(E) = \mu(pE), \quad \forall E \in \mathcal{P}(I^*).$$

Theorem 7. *Every state of a sequential outer measure is again a sequential outer measure. Moreover, if $H \subseteq I^*$ is a μ-measurable language then the left derivative $p \mid H = \{r \mid r \in I^*,\ pr \in H\}$ is μ_p-measurable.*

Proof. Since μ is a sequential outer measure there exists a measure semiautomaton $A = (I, S, f, m; s_0)$ inducing μ on the input semigroup. We claim that the semiautomaton $A' = (I, S, f, m; f(s_0, p))$ induces the measure μ_p. Indeed, let μ' be the measure induced by A'. We have:

$$\mu'(E) = m(\{s \mid f(f(s_0, p), q) = s,\ q \in E\}) =$$

$$= m(\{s \mid f(s_0, pq) = s,\ q \in E\}) =$$

$$= m(\{s \mid f(s_0, r) = s,\ r \in pE\}) = \mu(pE) = \mu_p(E).$$

Let now H be a μ-measurable language. For every language $L \subseteq I^*$ it holds:

$$p(p \mid L) \subseteq L \text{ and } p(p \mid L)^c \subseteq L^c.$$

From the equality

$$\mu(pK) = \mu(pK \cap H) + \mu(pK \cap H^c), \quad \forall K \subseteq I^*,$$

implied by the μ-measurability of H, it follows that

$$\mu(pK) \geqslant \mu(pK \cap p(p \mid H)) + \mu(pK \cap p(p \mid K)^c), \quad \forall K \subseteq I^*.$$

Therefore

$$\mu(pK) \geqslant \mu(p[K \cap (p \mid H)]) + \mu(p[K \cap (p \mid H)^c]),$$

which comes to

$$\mu_p(K) \geqslant \mu_p(K \cap (p \mid H)) + \mu_p(K \cap (p \mid H)^c),$$

hence pH is μ_p-measurable.

Let $A = (I, S, f)$ be a semiautomaton and let $A_j = (I, S, f; s_j)$ be the cyclic semiautomata generated by s_j, $j = 1, 2$ [2]. If A_1 and A_2 induce the same CSOM on the free semigroup I^*, then s_1 and s_2 are called *equivalent in measure*. Let ε be this equivalence. If $f_s : I^* \to S$ is the mapping given by $f_s(p) = f(s, p)$, $\forall p \in I^*$ it is obvious that $(s_1, s_2) \in \varepsilon$ iff $\ker f_{s_1} = \ker f_{s_2}$. If $\varepsilon = \iota_s$, A is called a *measure-reduced semiautomaton*, ι_s being the identity on S.

Let $\Omega(\mu)$ be the set of states of the CSOM μ induced on I^* by $A = (I, S, f)$ and let $A(\mu) = (I, \Omega(\mu), f_\mu; \mu)$ the semiautomaton of the CSOM μ, where $f_\mu(\mu_p, q) = \mu_{pq}$.

Theorem 8. *If $\hat{\mu}$ is the measure induced by the semiautomaton $A(\mu)$ we have $\hat{\mu} \leqslant \mu$. If $A = (I, S, f; s_0)$ is a semiautomaton inducing μ, the equality $\hat{\mu} = \mu$ holds iff A is s_0-connected and measure-reduced.*

Proof. Let us consider the equivalences $\alpha = \{(p, q) \mid f(s_0, p) = f(s_0, q)\}$, $\alpha_\mu = \{(p, q) \mid f_\mu(\mu, p) = f_\mu(\mu, q)\}$. We claim that $\alpha \leqslant \alpha_\mu$. We remark that $\alpha_\mu = \{(p, q) \mid \mu_p = \mu_q\} = \{(p, q) \mid \mu(pE) = \mu(qE),\ \forall E \in \mathcal{P}(I^*)\} = \{(p, q) \mid |f(s_0, pE)| = |f(s_0, qE)|,\ \forall E \in \mathcal{P}(I^*)\} = \{(p, q) \mid |f(f(s_0, p), E)| = |f(f(s_0, q), E)|\}$. Now it is obvious that, if $(p, q) \in \alpha$, we have $(p, q) \in \alpha_\mu$. Taking into account that the measures $\hat{\mu}$ and μ are induced respectively by the partitions related to the equivalences α_μ and α, we obtain the first part of the theorem. For the second part, if the semiautomaton is measure-reduced, from $|f(f(s_0, p), E)| = |f(f(s_0, q), E)|$, $\forall E \in \mathcal{P}(I^*)$, it follows that $f(s_0, p) = f(s_0, q)$, hence $(p, q) \in \alpha$. Therefore $\alpha = \alpha_\mu$ and the proof is completed.

Let us now remark that the set of the outer measures related to the partitions can be ordered in the following manner: if μ_1, μ_2 are two outer measures induced respectively by π_1 and π_2, then $\mu_1 \leqslant \mu_2$ iff the partition π_2 is finer than π_1. Using this ordering the poset of outer measures which can be induced by partitions becomes a semilattice where $\mu_1 \vee \mu_2$ is the outer measure induced by the intersection $\pi_1 \wedge \pi_2$. We shall denote by SOM the semilattice of outer measures.

In the sequel, we shall use often the *nuclear measure* related to a language $\varkappa_E$, $E \subseteq I^*$ which corresponds to the partition $\{E, I^* \setminus E\}$ generated by the equivalence $\ker \varkappa_E$, where $\varkappa_E$ is the characteristic function of E. We have obviously $\varkappa_E(E) = \varkappa_E(E^c) = 1$, $\varkappa_E(I^*) = 2$.

Remark. It is clear that the set of partitions of a set endowed with the intersection of partitions, the set of equivalences with the intersection of equivalences and the set of outer measures with the ordering previously defined are isomorphic semilattices.

Restricting this isomorphism to the semilattice of CSOMs, we obtain that this semilattice is isomorphic with the meet semilattice of all right congruences on the semigroup I^*.

V. The semilattice of cardinal sequential outer measures. Let us consider a partial ordering „$\leqslant$" on the set of CSOMs on the free semigroup I^*, namely

$$\mu_1 \leqslant \mu_2 \text{ iff } \mu_1(E) \leqslant \mu_2(E), \quad \forall E \in \mathcal{P}(I^*)$$

This ordering is the same as the ordering we have introduced in the previous paragraph. Here we deal only with CSOMs.

Theorem 9. *The set of* CSOMs *on I^* with the partial ordering „$\leqslant$" is a* sup *semilattice* (SCSOM), *where* $\sup\{\mu_1, \mu_2\} = \mu_{A_1 \times A_2}$, *$A_1$ and A_2 being two semiautomata generating on I^* the measures μ_1 and μ_2 respectively.*

Proof. It is clear that $\mu_{A_1\times A_2} \geqslant \mu_1$ and $\mu_{A_1\times A_2} \geqslant \mu_2$. Suppose now that $\mu \geqslant \mu_1$ and $\mu \geqslant \mu_2$. Since μ is a CSOM there exists a semiautomaton $A = (I, S, f, s_0)$ so that $\mu_A = \mu$.

In view of Theorem 1 we have the homomorphisms $h: S \to S_1$, $h_2: S \to S_2$. In view of the properties of the direct product of sets, there exists $h: S \to S_1 \times S_2$ so that $\mathrm{pr}_1 h = h_1$ and $\mathrm{pr}_2 h = h_2$. We claim that h is a homomorphism. Indeed, from $h(s) = (s_1, s_2) \in S_1 \times S_2$, it follows that $f_{12}(h(s), p) = f_{12}((s_1, s_2), p)$, f_{12} being the transition function of the product semiautomaton $A_1 \times A_2$. From the definition of f_{12} we obtain $f_{12}((s_1, s_2), p) = f = (f_1(s_1, p), f_2(s_2, p)) = (f_1(\mathrm{pr}_1 h(s), p) f_2(\mathrm{pr}_2 h(s), p)) = (f_1(h_1(s), p), f_2(h_2(s), p)) = (h_1(f(s,p)) h_2(f(s,p)) = (\mathrm{pr}_1 h), h(f(s,p)), \mathrm{pr}_2 h(f(s,p)) = h(f(s,p))$. We remark also that h_1, h_2, h are surjective homomorphisms. Hence $\mu_A \geqslant \mu_{A_1\times A_2}$ and we have proved that $\mu_{A_1\times A_2}$ is the supremum of μ_1 and μ_2. In view of corollary 1 of the Theorem 2, the measure $\sup\{\mu_1, \mu_2\}$ is well definded.

$$\begin{tikzcd}
 & S_1 & \\
S_1 \times S_2 \arrow[ur, "pr_1"] \arrow[dr, "pr_2"'] & & S \arrow[ul, "h_1"'] \arrow[ll, "h"'] \arrow[dl, "h_2"] \\
 & S_2 &
\end{tikzcd}$$

The CSOM μ is bounded if $\mu(I^*) < \infty$. Let us remark that the set of all bounded CSOMs is an ideal R in the semilattice of all the CSOMs. We recall E is a regular language iff there exists a finite semiautomaton which recognizes E, hence a bounded CSOM μ for which E is μ-measurable. Therefore it appears to be of concern to us to study a certain extension of the notion of regularity.

The language $E \subseteq I^*$ is measurable with respect to the ideal H of the SCSOM if there exists a measure $\mu_1 \in H$ so that E is μ_1-measurable.

Hence a language E is a regular one iff it is measurable with respect to the ideal R of all bounded CSOMs.

It is easy to observe that, if E is μ-measurable then E is $(\mu]$-measurable. where $(\mu]$ is the principal ideal generated by μ.

Let $M(H)$ the set of all languages measurable with respect to the ideal H. Consider the equivalence $\nu_E = \{(p, q) \mid p, q \in I^*,\ p \mid E = q \mid E\} = \{(p, q) \mid \chi_E(pr) = \chi_E(qr),\ \forall r \in I^*\}$ and the outer measure μ_E related to the partition π_E generated by the equivalence ν_E.

Theorem 10. *For every $E \subseteq I^*$ the measure μ_E is a CSOM. Moreover, $E \in M(H)$ iff $\mu_E \in H$, for every ideal H of the SCSOM.*

Proof. Let us remark firstly that ν_E is a right congruence, for, if $(p, q) \in \nu_E$ it follows $(pt, qt) \in E$ for every $t \in I^*$. Indeed if $\chi_E(pr) = \chi_E(qr)$, $\forall r \in I^*$ then $\chi_E((pt)r) = \chi_E(p(tr)) = \chi_E(q(tr)) = \chi_E((qt)r)$. Therefore, due to the Corollary 3, μ_E is a CSOM. Suppose now that there exists a CSOM $\mu \in H$ so that E is μ-measurable. Then E is an union of atoms of the Boolean algebra $M(\mu)$ (in view of the Theorem 4) and $E = \cup \{H_\gamma \mid \gamma \in \Gamma_E, H_\gamma \in \mathcal{A}_\mu\}$. Let us consider the semiautomaton $A(\mu) = (I, \mathcal{A}_\mu, f_\mu; E_0)$, where E_0 is the atom containing the null word and $f_\mu(E, p) = E'$, E' being the unique atom including Ep. Then E is representable in the semiautomaton $A(\mu)$ by the set of states $\{H_\gamma \mid \gamma \in \Gamma_E H_\gamma \in \mathcal{A}_\mu\}$. Indeed, if $f_\mu(E_0, p) = H_\gamma$ since $e \in E_0$ we have $ep = p \in H_\gamma \subseteq E$ from which $p \in E$.

The semiautomaton $A(\mu)$ induces on the free semigroup I^* the measure μ. Suppose that μ' is the measure induced by $A(\mu)$. Then $\mu'(K) = |\{L_\delta \mid \delta \in \Delta, L_\delta \in \mathcal{A}_\mu, L_\delta \cap K \neq \varnothing\}| = \mu(K)$, for every $K \subseteq I^*$. We infer also that μ is generated by the partition related to the right congruence $\nu_A = \{(p, q) \mid f_\mu(E_0, p) = f_\mu(E_0, q)\}$. We have to prove that $\nu_A \leqslant \nu_E$. To this end, let us consider $(p, q) \in \nu_A$. Then, if $f_\mu(E_0, pr) \in \{H_\gamma \mid \gamma \in \Gamma_E\}$ we have $f_\mu(E_1, qr) = f_\mu(f_\mu(E_0, q), r) = f_\mu(f_\mu(E_0, p), r) = f_\mu(E_0, pr)$ and $f_\mu(E_0, qr) \in \{H_\gamma \mid \gamma \in \Gamma_E\}$. Hence $\nu_A \leqslant \nu_E$ and it follows that $\mu_E \leqslant \mu$, since the partition corresponding to ν_A is finer then the partition corresponding to ν_E.

The importance of the measure μ_E is emphasized by:

Theorem 11. *A language $E \subseteq I^*$ is representable in the semiautomaton A iff $\mu_E \leqslant \mu_A$.*

Proof. Suppose $\mu_E \leqslant \mu_A$, where $A = (I, S, f; s_0)$. Since μ_A is induced by the partition π_A related to the right congruence $\nu_A = \{(p, q) \mid f(s_0, pr) = f(s_0, qr),\ \forall r \in I^*\}$, we have $\nu_A \leqslant \nu_E$. Therefore E is the union of some blocks of the partition π_A because E is the union of some blocks of the partition related to ν_E.

Conversely, suppose that E is represented in A by a subset of the set of states $S_1 \subseteq S$ and $(p, q) \in \nu_A$. Then $f(s_0, pr) = f(s_0, qr)$. If $pr \in E$ we have $f(s_0, pr) \in S_1$ which implies $f(s_0, qr) \in S_1$ or $qr \in E$. Therefore $(p, q) \in \nu_E$ and $\nu_A \leqslant \nu_E$.

Let us remark that for a certain language $E \subseteq I^*$, the measure μ_E is the CSOM induced by the semiautomaton $A_E = (I, I^* \mid \nu_E, f_E; [e])$, hence from Theorem 1 we obtain:

Corollary 5. *A language $E \subseteq I^*$ is representable in a semiautomaton iff there exists a surjective semiautomata homomorphism from A to A_E.*

In order to prove closure properties of the family of languages $M(H)$ we need some preliminary results. Let us consider the equivalence

$$\delta_E = \{(p, q) \mid E \mid p = E \mid q\}$$

for every $E \subseteq I^*$, where $E \mid p = \{r \mid r \in I^*, rp \in E\}$; δ_E is a left congruence on the free semigroup I^*. Let σ_E be the outer measure related to this equivalence.

Theorem 12. *We have $\mu_E = \cup \{\varkappa_{E\mid r} \mid r \in I^*\}$ and $\sigma_E = \cup \{\varkappa_{r\mid E} \mid r \in I^*\}$ in the SOMs.*

Proof. It is more convenient to prove these equalities using the isomorphism between the join semilattice of outer measure and the meet semilattice of equivalences, namely, to prove instead of the previous equalities, that $\nu_E = \cap \{\ker \chi_{E\mid r} \mid r \in I^*\}$ and $\delta_E = \cap \{\ker \chi_{r\mid E} \mid r \in I^*\}$.

Suppose that $(p, q) \in \cap \{\ker \chi_{E\mid r} \mid r \in I^*\}$. Then $p \in E \mid r$ iff $q \in E \mid r$ This implies that $pr \in E$ iff $qr \in E$ or $r \in p \mid E$ iff $r \in q \mid E$ for every $r \in I^*$. Hence $p \mid E = q \mid E$ from which we obtain $(p, q) \in \nu_E$. Therefore $\cap \{\ker \chi_{E\mid r} \mid r \in I^*\} \subseteq \nu_E$. The converse can be proved immediately; a similar argument works for the second equality.

Theorem 13. *Let E be a language, $E \subseteq I^*$. The greatest left (right) congruence contained by $\ker \chi_E$ is $\delta_E\,(\nu_E)$.*

Proof. We have $\delta_E \subseteq \ker \chi_E$ for, if $(p, q) \in \delta_E$ then $\chi_E(rp) = \chi_E(rq)$, $\forall r \in I^*$ and, taking $r = e$, we obtain $(p, q) \in \ker \chi_E$. Afterwards, let $\beta \leqslant \ker \chi_E$ a left congruence and $(p, q) \in \beta$, hence $(pr, qr) \in \beta$, $\forall r \in I^*$. Since $\beta \leqslant \ker \chi_E$ we have $(pr, qr) \in \ker \chi_E$ for every $r \in I^*$; therefore $(p, q) \in \delta_E$. The same proof can be used for ν_E.

Remark. The last result implies that the least CSOM greater than $\varkappa_E$ is μ_E.

Let now H be an ideal in the SCSOMs and $\hat{H}$ the ideal in the SOM, generated by H:

$$\hat{H} = \{\tau \mid \exists \mu \in H, \tau \leqslant \mu\}.$$

Theorem 14. *We have $D \in M(H)$ iff $\varkappa_D \in \hat{H}$.*

Proof. Let us consider a language $D \in M(H)$. Then $\mu_D \in H$ in view of theorem 10 and, since $\varkappa_E \leqslant \mu_E$, we have $\varkappa_E \in \hat{H}$ in view of the definition of $\hat{H}$.

Conversely, let $\varkappa_D \in \hat{H}$. There exists a right congruence $\alpha \in H$ so that $\varkappa_D \leqslant \alpha$. On the other hand $\varkappa_D \leqslant \mu_D$ and taking into account that μ_D is the least congruence having this property we obtain $\varkappa_D \leqslant \mu_D \leqslant \alpha$. Then $\mu_D \in H$, in view of the definition of ideals and $D \in M(H)$.

Corollary 6. *Since $\mu_E \geqslant \varkappa_{E\mid r}$, $\forall r \in I^*$, we obtain $\varkappa_{E\mid r} \in \hat{H}$ and using Theorem 14 we get $E/\Gamma \in M(H)$, $\forall r \in I^*$. We conclude that $M(H)$ is closed with respect to left derivatives.*

Corollary 7. *We have proved up to now that the following three assertions are equivalent for an ideal H of the SCSOMs:*

$$\text{i) } D \in M(H). \quad \text{ii) } \mu_D \in H. \quad \text{iii) } \varkappa_D \in \hat{H}.$$

Theorem 15. *The class of sets $M(H)$ is closed with respect to set theoretical union, intersection and complementation.*

Proof. It is obvious, in view of the definition of μ-measurable sets that, if $E \in M(H)$ then $E^C \in M(H)$.

Afterwards, suppose that $E_1, E_2 \in M(H)$. Then E_1 is μ_1-measurable and E_2 is μ_2-measurable. Hence, using Theorem 2 from [6], there exists two semiautomata $A_1 = (I, S_1, f_1; s_{01})$ and $A_2 = (I, S_2, f_2; f_{02})$ generating the measures μ_1 and μ_2 in which E_1 and E_2 are representable by some subsets $S_1' \subseteq S_1$ and $S_2' \subseteq S_2$. The semiautomaton $A_1 \times A_2$ generates the measure $\mu_1 \vee \mu_2 \in H$ and since $E_1 \cup E_2$ and $E_1 \cap E_2$ are representable by the subset $S_1' \times S_2 \cup S_1 \times S_2'$ and $S_1' \times S_2'$ respectively we infer that both $E_1 \cup E_2$ and $E_1 \cap E_2$ are $\mu_1 \vee \mu_2$-measurable.

Theorem 16. *Let $D \subseteq I^*$ and $k = \mu_D(E)$ be the number of distinct lef derivatives of the language D with respect to words belonging to E. Then the number of distinct right derivatives of the language D with respect to words belonging to E, $\sigma_D(E)$ is bounded by:*

$$\log_2 \mu_D(E) \leqslant \sigma_D(E) \leqslant 2^{\mu_D(E)}.$$

Proof. Following theorem 12 we have $\sigma_D = \cup \{\varkappa_{r\mid D} \mid r \in I^*\} = \cup \{\varkappa_{r\mid E} \mid r \in \{r_1, \dots, r_k\}\}$, which implies that $\sigma_D(E) \leqslant \prod_{j=1}^{k} \varkappa_{r_j\mid D}(E) \leqslant 2^k = 2^{\mu_D(E)}$.

Since a similar argument gives $\mu_D(E) \leqslant 2^{\sigma_D(E)}$, or $\log_2 \mu_D(E) \leqslant \sigma_D(E)$, the theorem follows.

Corollary 8. *Taking $E = I^*$ we obtain*

$$\log_2 \mu_D(I^*) \leqslant \sigma_D(I^*) \leqslant 2^{\mu_D(I^*)},$$

hence the number of left derivatives is situated between $\log_2 k$ and 2^k, k being the number of right derivatives.

We intend now to reobtain some well known closure properties of regular languages. Let us denote by $EK = \{pq \mid p \in E, q \in K\}$ the complex product of E and K and by $E^* = \bigcup_{J=0}^{\infty} E^j$ the star-closure of the language

E. Here $E^0 = \{e\}$ and $E^{j+1} = E^j E$, e being the null word of the free semigroup I^*.

Theorem 17. *Suppose* $E, K \in M(R)$. *Then* $EK \in M(R)$ *and* $E^* \in M(R)$.

Proof. Let us prove firstly that $\mu_{EK} \leqslant \mu_E \cup \bigcup_{r \in I^*} \varkappa_{E(K|r)}$ and $\mu_{E^*} \leqslant \bigcup_{r \in I^*} \varkappa_{E^*(E|r)}$. In view of the isomorphism between SCSOMs and the meet semilattice of right congruences it will be sufficient to prove that $\nu_{EK} \geqslant \nu_E \cap \bigcap \{\ker \chi_{E(K|r)} \mid r \in I^*\} = \alpha$. Suppose that $(p, q) \in \alpha$. If $pr \in EK$ we have $pr = u_1 u_2$, $u_1 \in E$, $u_2 \in K$ and there are two possible situations: either $l(p) \leqslant l(u_1)$, or $l(p) > l(u_1)$. In the first case $u_1 = pw$ and $r = wu_2$, w being possibly the null word. Taking into account that $(p, q) \in \nu_E$ we obtain $qw \in E$, $qr = (qw) u_2 \in EK$. The second case gives $p = u_1 w$ and $wr = u_2$.

Then $w \in K \mid r$, $p = u_1 w \in E(K \mid r)$ and, since $(p, q) \in \cap \ker \chi_{E(K|r)}$ it follows $q \in E(K \mid r)$ or $q = v_1 t_2$, where $t_2 r \in K$. Therefore $qr \in EK$.

In order to prove the second inequality, we shall prove that $\nu_{E^*} \geqslant \cap \{\ker \chi_{E^*(E|r)} \mid r \in I^*\} = \beta$. Suppose now that $(p, q) \in \beta$ and $pr \in E$. There exist p_1, p_2, r_1, r_2 so that $p = p_1 p_2$, $r = r_1 r_2$, $p_1, r_2 \in E^*$, $p_2 r_1 \in E$. Therefore $p = p_1 p_2 \in E^*(E \mid r_1)$ which implies $q \in E^*(E \mid r_1)$ or $qr_1 \in E^* E \subseteq E^*$. Hence $qr = qr_1 r_2 \in E^* E \subseteq E^*$ and $(pr, qr) \in \nu_{E^*}$ for every $r \in I^*$.

If $E, K \in M(R)$, we have $\mu_E(I^*) < \infty$, $\mu_K(I^*) < \infty$ and using theorem 16 we obtain that there are $\sigma_E(I^*)$ and $\sigma_K(I^*)$ distinct right derivatives of the languages E and K respectively.

Hence

$\mu_{EK} \leqslant \mu_E \cup \bigcup \{\varkappa_{E(K|r_j)} \mid j = \overline{1, \sigma_K(I^*)}\}$ and $\mu_{E^*} \leqslant \cup \{\varkappa_{E^*(E|r_h)} \mid h = \overline{1, \sigma_E(I^*)}\}$, considering in the right members only the distinct measures. Now $\mu_{EK}(D) \leqslant \mu_E(D)\ \Pi\ \{\varkappa_{E(K|r_j)}(D) \mid j = \overline{1, \sigma_K(I^*)}\}$ and $\mu_{E^*}(D) \leqslant \cup \{\varkappa_{E^*(E|r_h)}(D) \mid h = \overline{1, \sigma_E(I^*)}\}$. Since $\varkappa_L(D) \leqslant 2$ for every L, $D \subseteq I^*$ it is obvious that μ_{EK} and μ_{E^*} are finite CSOMs, $\mu_{EK}, \mu_{E^*} \in R$, and $EK, E^* \in M(R)$.

The concept of measurability with respect to an ideal has also an algebraic interpretation. Let $A = (I, S, f; s_0)$ be a semiautomaton and let $U(A) = < \{f_i \mid i \in I\} \cup \{s_0\}; S >$ the unary algebra related to this semiautomaton, where $f_i(s) = f(s, i)$ and s_0 is a 0-ary operation.

Remark. If A is a connected semiautomaton the unary algebra $\mathcal{U}(A)$ has only trivial subalgebras. Indeed, since every $s \in S$ can be reached from s_0, $s = f(s_0, p)$ and $s = f_{i_m}(\ldots f_{i_1}(s)..)$ where $p = i_1 \ldots i_m \in I^*$, hence every s is an algebraic constant [see [4], p. 59]. Afterwards it is clear that the smallest subalgebra of $\mathcal{U}(A)$ is $\mathcal{U}(A)$ itself (see also [4], p. 45, Lemma 1).

A converse result also holds: for every algebra $B = <F, S>$ of the given type $\tau = < 1, 1, \ldots, 1, 0 >$ there exists a semiautomaton Aut (B which is a connected one iff S consists only of algebraic constants.

Suppose now that $A_j = (I, S^j, f^j, s_0^j)$, $j = 1, 2$ are two semiautomata and $h : S^1 \to S^2$ is a surjective homomorphism of semiautomata. Then $h(f^1(s, i)) = f^2(h(s), i)$ and $h(s_0^1) = s_0^2$. Using the operations of $\mathcal{U}(A)$, we have $h(f_i^1(s)) = f_i^2(h(s))$, $\forall i \in I$, hence h is also a surjective homomorphism of algebras.

Conversely, each surjective homomorphism of algebras is a surjective homomorphism of semiautomata.

Using the well known characterization of varieties of Birkhoff (see [5] p. 125, e. g.) we obtain

Theorem 18. *Let* $\mathfrak{A}$ *be a class of unary algebras having the same type* $\tau = < 1, 1, \ldots, 1, 0 >$ *consisting only of algebraic constants and* Aut ($\mathfrak{A}$) *their related semiautomata. Suppose that for every algebra* $B \in \mathfrak{A}$ *the operator domain is indexed by the same set* I, $B = < F^B, S^B >$, *with* $F^B = \{f_i^B \mid i \in I\}$.

Then $\mathfrak{A}$ *is a variety iff the set of CSOMs induced by the semiautomata from* Aut ($\mathfrak{A}$) *is a complete ideal in the SCSOMs on the free semigroup* I^*.

Proof. Let $\mathfrak{A}$ be a class of unary algebras satisfying the previous conditions and suppose that the set H of the measures induced on I^* by the semiautomata from Aut ($\mathfrak{A}$) is a complete ideal (closed with respect to arbitrary unions) of the SCSOMs. To prove that $\mathfrak{A}$ is a variety, we shall prove that $\mathfrak{A}$ is closed for taking subalgebras, arbitrary direct products and epimorphic images.

Taking into account the previous remark it is clear that $\mathfrak{A}$ is closed with respect to subalgebras.

Afterwards, let $B \in \mathfrak{A}$, $h : B \to C$ an epimorphism. Then Aut (C) is an epimorphic image of Aut (B) and $\mu_{\mathrm{Aut}(B)} \geqslant \mu_{\mathrm{Aut}(C)}$. Since $\mu_{\mathrm{Aut}(B)} \in H$ it follows that $\mu_{\mathrm{Aut}(C)} \in H$, hence Aut ($C$) $\in$ Aut $\mathfrak{A}$, or $C \in \mathfrak{A}$, which means that $\mathfrak{A}$ is closed with respect to epimorphisms.

Let $\{B_j \mid j \in J\} \subseteq \mathfrak{A}$ be a family of algebras, $\{\mathrm{Aut}(B_j) \mid j \in J\}$ their corresponding semiautomata inducing the family of CSOMs $\{\mu_j \mid j \in J\}$. Then $\mathrm{Aut}(\prod_{j \in J} B_j) = \prod_{j \in J} \mathrm{Aut}(B_j)$ and this last semiautomaton induces the measure $\cup \{\mu_j \mid j \in J\}$ which belongs to H since H is a complete ideal. Therefore $\mathrm{Aut}(\prod_{j \in J} B_j) \in \mathfrak{A}$, hence $\mathfrak{A}$ is a variety.

Conversely, let $\mathfrak{A}$ be a variety. Then $\mathfrak{A}$ is closed with respect to homomorphic images and arbitrary direct products, hence H is a complete ideal in the SCSOMs.

Indeed, let $\mu \in H$ and $\mu_1 \leqslant \mu$. Then μ is induced by a semiautomaton $A \in \mathrm{Aut}\,\mathfrak{A}$. If A_1 is the semiautomaton which induces μ_1 on I^* then, using Theorem 1 there exists a surjective homomorphism from A to A_1; $\mathcal{U}(A_1)$ is the homomorphic image of $\mathcal{U}$ (A), hence $\mathcal{U}(A_1) \in \mathfrak{A}$, $A_1 \in \mathrm{Aut}(\mathfrak{A})$ and $\mu_1 \in H$.

A similar argument shows that if $\{\mu_j \mid j \in J\} \subseteq H$ then $\cup \{\mu_j \mid j \in J\} \in H$, hence H is a complete ideal in the SCSOMs and the proof is completed.

VI. **Global extensions of csoms.** Let $A = (I, S, f)$ be a semiautomaton and $S_1 \subseteq S$. Se shall consider three outer measures on I^* which are derived from the CSOM induced by A, namely

$$\mu^s_{S_1}(E) = \sup\{\mu_s(E) \mid s \in S_1\},$$

$$\mu^\sigma_{S_1}(E) = \Sigma\{\mu_s(E) \mid s \in S_1\} \text{ and}$$

$$\overline{\mu}_{S_1}(E) = |\cup\{f(s_1, E) \mid s_1 \in S_1\}|$$

being the CSOM induced on I^* by the semiautomaton $A = (I, S, f; s)$.

Remark. It is immediate that $\mu^s_{S_1}$, $\mu^\sigma_{S_1}$ and $\overline{\mu}_{S_1}$ are outer measures on $\mathcal{P}(I^*)$ and $\mu^s_{S_1} \leqslant \overline{\mu}_{S_1} \leqslant \mu^\sigma_{S_1} \leqslant |S_1| \, \mu^s_{S_1}$. Moreover $\overline{\mu}_{S_1}$ is again a CSOM. Indeed, let us consider the semiautomaton $\overline{A} = (I, \mathcal{P}(S), F; S_1)$ where, $F(S', i) = \cup\{f(s', i) \mid s' \in S'\}$. Then $\mu_{\overline{A}}(E) = |\bigcup_{p \in E}\{f(s_1, p) \mid s_1 \in S_1\}| = |\cup\{f(s_1, E) \mid s_1 \in S_1\}| = \overline{\mu}_{S_1}(E)$.

We note also that, if $S_1 \subseteq S_2$ then $\mu^s_{S_1} \leqslant \mu^s_{S_2}$, $\mu^\sigma_{S_1} \leqslant \mu^\sigma_{S_2}$ and also $\overline{\mu}_{S_1} \leqslant \overline{\mu}_{S_2}$.

Suppose now that E is a $\mu^s_{S_1}$-measurable language and A is a finite semiautomaton. Then $\mu^s_{S_1}(K) = \mu^s_{S_1}(K \cap E) + \mu^s_{S_1}(K \cup E^c)$ or $\sup\{\mu_s(K) \mid s \in S_1\} = \sup\{\mu_s(K \cap E) \mid s \in S_1\} + \sup\{\mu_s(K \cap E^c) \mid \mid s \in S_1\}$, for every $K \subseteq I^*$. Since $\mu_s(K) \leqslant \mu_s(K \cap E) + \mu_s(K \cap E^c)$, it follows that there exists a state $s_1 \in S_1$ for which both suprema from the right member are reached, hence

$$\mu_{s_1}(K) = \mu_{s_1}(K \cap E) + \mu_{s_1}(K \cap E^c).$$

We obtain

Theorem 19. *If E is a $\mu^s_{S_1}$-measurable language, there exists a state $s_1 \in S_1$ so that E is μ_{A_1}-measurable, where $A_1 = (I, S, f; s_1)$.*

Suppose now that E is a $\mu^\sigma_{S_1}$-measurable language and A is a finite semiautomaton.

Then $\mu^\sigma_{S_1}(K) = \mu^\sigma_{S_1}(K \cap E) + \mu^\sigma_{S_1}(K \cap E^c)$ for every $K \subseteq I^*$. This comes to

$$\Sigma\{\mu_s(K) \mid s \in S_1\} = \Sigma\{\mu_s(K \cap E) \mid s \in S_1\} + \Sigma\{\mu_s(K \cap E^c) \mid s \in S_1\}$$

or

$$\sum_{s \in S_1}\{\mu_s(K) - \mu_s(K \cap E) - \mu_s(K \cap E^c)\} = 0.$$

Since $\mu_s(K) - \mu_s(K \cap E) - \mu_s(K \cap E^c) \leqslant 0$, $\forall E, K \in \mathcal{P}(I^*)$, we have $\mu_s(K) = \mu_s(K \cap E) + \mu_s(K \cap E^c)$ hence E is representable in every semiautomaton $A_s = (I, S, f; s)$ with $s \in S_1$.

Hence we have

Theorem 20. *If E is a $\mu^\sigma_{S_1}$-measurable language and $A = (I, S, f)$ is a finite semiautomaton then E is μ_{A_s}-measurable for every $s \in S_1$.*

VII. Sequential measure spaces. A sequential measure space (SSM) on the free semigroup I^* is a measure space $S = (I^*, M(\mu), \mu)$ where μ is a CSOM on I^*.

The class of SMSs on I^* can be partially ordered by taking $S_1 \leqslant S_2$, where $S_j = (I^*, M(\mu_j)\,\mu_j)$, $j = 1, 2$ iff $\mu_1 \leqslant \mu_2$. Since the partial ordering on the set of CSOMs on I^* is a semilatticial one, the class of SMS on I^* enjoys the same property. We shall use further the semilattice of SMS on I^* denoted by SSMS (I^*).

The mapping $h : I_1^* \to I_2^*$ is a S_1, S_2-measurable function if for every $E \in M(\mu_2)$ we have $h^{-1}(E) \in M(\mu_1)$.

Remark. If h is a S_1, S_2-measurable function, then h is S_1', S_2'-measurable for each pair (S_1', S_2') with $S_1' \geqslant S_1$ and $S_2 \geqslant S_2'$.

Theorem 21. *Let $h : I_1^* \to I_2^*$ be a mapping and $S_j = (I_j^*, M(\mu_j), \mu_j)$, $j = 1, 2$ two SMSs. Then ${}^h\mu_2 \leqslant \mu_1$ iff h is a S_1, S_2-measurable function.*

Proof. Let us suppose that ${}^h\mu_2 \leqslant \mu_1$. Then, from the argument preceding Theorem 5, it follows that for every $E \in M(\mu_2)$ the language $h^{-1}(E)$ belongs to $M({}^h\mu_2)$, hence $h^{-1}(E)$ is an exact union of atoms of the Boolean algebra $M({}^h\mu_2)$. Since ${}^h\mu_2 \leqslant \mu_1$ every atom of this algebra is an union of atoms of the Boolean algebra $M(\mu_1)$. Therefore $h^{-1}(E) \in M(\mu_1)$. Conversely, let $h : I_1^* \to I_2^*$ be a measurable function between $(I_1^*, M(\mu_1), \mu_1)$ and $(I_2^*, M(\mu_2), \mu_2)$. The set of atoms of the Boolean algebra $M({}^h\mu_2)$ consists of the set of languages $\{h^{-1}(H) \mid H$ an atom of $M(\mu_2)\}$. But, every language $h^{-1}(H)$ is also a μ_1-measurable set hence it is an union of atoms of the Boolean algebra $M(\mu_1)$. Since every atom of the Boolean algebra $M({}^h\mu_2)$ is an union of atoms of the algebra $M(\mu_1)$, it is immediate that ${}^h\mu_2 \leqslant \mu_1$.

Remark. Let $h : I_1^* \to I_2^*$ be a homomorphism of free semigroups. From the construction of the CSOM ${}^h\mu_2$ it is clear that if μ_2 is a bounded measure, then ${}^h\mu_2$ also is, hence, if E is a regular language then $h^{-1}(E)$ enjoys the same property.

A mapping $\varphi : I_1^* \to I_2^*$ is a sequential operator if for every $p, q \in I$ there exists $r \in I_2^*$ so that $\varphi(pq) = \varphi(p)\,r$. We shall denote $r = \varphi_p(q)$. It is easy to check (see [[2], p. 15]) that $\varphi_p : I_1^* \to I_2^*$ is again a sequential operator. The memory of the sequential operator φ is the set $\Omega_\varphi = \{\varphi_p \mid p \in I^*\}$.

We shall investigate the behaviour of CSOMs with respect to sequential operators having finite memory.

Let $\varphi : I_1^* \to I_2^*$ be a sequential operator and $A(\varphi) = (I, \Omega_\varphi, f_\varphi; \varphi)$ the canonical semiautomaton attached to φ, where $f_\varphi(\varphi_p, q) = \varphi_{pq}$. If β_φ is the right congruence on I_1^* given by $\beta_\varphi = \{(p, q) \mid p, q \in I_1^*, \varphi_p = \varphi_q\}$, then the measure induced by $A(\varphi)$ on I_1^* is given by $\mu_{A(\varphi)}(K) = |K/\beta_\varphi|$.

Theorem 22. *Let be $H \subseteq I_2^*$ and $E = \varphi^{-1}(H) \subseteq I_1^*$. Then $\mu_E \leqslant \mu_{A(\varphi)} \cdot {}^\varphi\mu_H$.*

Proof. Let $\delta_H = \{(p, q) \mid . \varphi(p) \mid H = \varphi(q) \mid H\}$. We have $r \in p \mid E$ iff $pr \in E$ which is equivalent to $\varphi(pr) \in \varphi(E) = H$, or $\varphi_p(r) \in \varphi(p) \mid H$. Hence $r \in p \mid E$ iff $r \in \varphi_p^{-1}(\varphi(p) \mid H)$ and we obtain $p \mid E = \varphi_p^{-1}(\varphi(p) \mid H)$.

Suppose that $(p, q) \in \delta_H \cap \beta_\varphi$. Then $\varphi_p^{-1} = \varphi_q^{-1}$, $\varphi(p) \mid H = \varphi(q) \mid H$ and it follows that $p \mid E = \varphi_p^{-1}(\varphi(p) \mid H) = \varphi_q^{-1}(\varphi(q) \mid H) = q \mid E$ which implies $(p, q) \in \nu_E$, hence $\beta_\varphi \cap \delta_H \leqslant \nu_E$.

Let now be $K \subseteq I_1^*$. From the definition of the measure μ_E it follows immediately that $\mu_E(K) = |K|\nu_E| \leqslant |K|\beta_\varphi \cap \delta_H| \leqslant |K|\beta_\varphi|.|K/\delta_H| = \mu_{A(\varphi)}(K)|K/\beta_H|$. Afterwards there exists a bijective mapping $h: K/\delta_H \to \varphi(K)/\nu_H$ given by $h([p]\delta_H) = [\varphi(p)]\nu_H$ since $(p, q) \in (K \times K) \cap \delta_H$ iff $\varphi(p)|H = \varphi(q)|H$ which is equivalent to $(\varphi(p), \varphi(q)) \in (\varphi(K) \times \varphi(K)) \cap \nu_H$. Therefore $|K/\delta_H| = |\varphi(K)/\nu_H| = \mu_H(\varphi(K)) = {}^\varphi\mu_H(K)$.

We conclude that $\mu_E(K) \leqslant \mu_{A(\varphi)}(K) . {}^\varphi\mu_H(K)$ which gives the inequality we looked for.

Corollary 9. *If φ is a sequential operator having finite memory and μ_H is a bounded measure then $\mu_{\varphi^{-1}(H)}$ is again a bounded measure.* for, if $|\Omega_\varphi| < \infty$ it follows $|\mu_{A(\varphi)}(K)| < |\Omega_\varphi|$ for every $K \subseteq I_1^*$. Hence we obtain a well known result, namely if H is a regular language and φ is a sequential operator then $\varphi^{-1}(H)$ is a regular language. We remark also that ${}^\varphi\mu_H$ is not necessarily a CSOM though μ_H is, since, Theorem 5 is valid only for homomorphisms (which are special sequential operators with $|\Omega_\varphi| = 1$).

Let now K be an ideal in the SCSOMs on I^*. The linear closure of the ideal K is the set:

$$\widetilde{K} = \{\mu \mid \mu \in \text{SCSOM}, \exists \mu_1 \in K, \alpha \in R_+, \mu \leqslant \alpha\mu_1\}.$$

$\widetilde{K}$ is an ideal for, if $\mu', \mu'' \in K$ then there exists $\alpha, \beta \in R_+$, $\mu_1', \mu_1'' \in \text{SCSOM}(I^*)$ so that $\mu' \leqslant \alpha\mu_1'$, $\mu'' \leqslant \beta\mu_1''$, hence $\mu' \vee \mu'' \leqslant \max(\alpha, \beta)(\mu_1' \vee \mu_1'')$. Afterwards, from $\mu' \leqslant \mu$ and $\mu \in \widetilde{K}$ it follows obviously $\mu' \in \widetilde{K}$.

We remark that the ideal of bounded measures is linearly closed.

Theorem 22 allows us to formulate:

Corollary 10. *If φ is a sequential operator $\varphi: I_1^* \to I_2^*$ which is a measurable mapping from $S_1 = (I_1^*, M(\mu_1), \mu_1)$ to $S_2 = (I_2^*, M(\mu_2)\, \mu_2)$ and hasa finite memory, then the preimage of each μ_2-measurable language is measurable with respect to the ideal $(\widetilde{\mu_1}]$.*

In view of Theorem 21 since φ is a measurable mapping we have ${}^\varphi\mu_2 \leqslant \mu_1$. Theorem 22 implies that $\mu_{\varphi^{-1}(H)} \leqslant \mu_{A(\varphi)} . {}^\varphi\mu_H$. Taking into account that H is a μ_2-measurable mapping, in view of Theorem 11 it follows that $\mu_H \leqslant \mu_2$ or ${}^\varphi\mu_H \leqslant {}^\varphi\mu_2$, hence $\mu_{\varphi^{-1}(H)} \leqslant \mu_{A(\varphi)}\, {}^\varphi\mu_H \leqslant \mu_{A(\varphi)}\, {}^\varphi\mu_2 \leqslant \mu_{A(\varphi)} . \mu_1$ Therefore $\mu_{\varphi^{-1}(H)} \in (\widetilde{\mu_1}]$ and $\varphi^{-1}(H) \in M((\widetilde{\mu_1}])$.

Let now consider $h: I_1^* \to I_2^*$ a measurable mapping from $S_1 = (I_1^*, M(\mu_1), \mu_1)$ to $S_2 = (I_2^*, M(\mu_2), \mu_2)$. $k: I_1^* \to I_2^*$ a mapping and let us denote by $\mathfrak{D}(h, k)$ the language:

$$\mathfrak{D}(h, k) = \{p \mid h(p) \neq k(p)\}.$$

Then we have

Theorem 23. *If $\mathfrak{D}(h, k)$ is a set so that every $L \subseteq \mathfrak{D}(h, k)$ is μ-measurable, then k is a measurable mapping from $S_1' = (I_1^*, M(\mu_1 \vee \mu), \mu_1 \vee \mu)$ to S_2.*

Proof. Suppose that $H \in M(\mu_2)$. Then $k^{-1}(H) = [k^{-1}(H) \cap \mathfrak{D}(h, k)] \cup [k^{-1}(H) \cap (\mathfrak{D}(h, k))^c] = [k^{-1}(H) \cup \mathfrak{D}(h, k)] \cup [h^{-1}(H) \cap (\mathfrak{D}(h, k))^c]$.

Since $h^{-1}(H)$ is μ_1-measurable and $\mathfrak{D}(h, k)$ is μ-measurable it follows firstly that $(\mathfrak{D}(h, k))^c$ is also μ-measurable and $h^{-1}(H) \cap (\mathfrak{D}(h, k))^c$ is $\mu_1 \vee \mu$-measurable.

Afterwards, since $k^{-1}(H) \cap \mathfrak{D}(h, k) \subseteq \mathfrak{D}(h, k)$, from the property of $\mathfrak{D}(h, k)$ we have $k^{-1}(H) \cap \mathfrak{D}(h, k) \in M(\mu)$. Therefore $k^{-1}(H) \in M(\mu_1 \vee \mu)$ hence k is measurable.

Remark. Let U be a finite language. Then there exists a finite semiautomaton $A = (I, S, f; s_0)$ so that U is representable in A by a subset $S_1 \subseteq S$ and each of the subsets of U is representable in the same semiautomaton by other subsets $S_1' \subseteq S_1 \subseteq S$. Hence every $L \subseteq U$ is a μ-measurable language where μ is the measure induced by μ on I^*.

Faculty of Mathematics and Mechanics
University of Iaşi—Romania

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ASUPRA MĂSURILOR EXTERIOARE CARDINAL-SECVENŢIALE

Rezumat

Se consideră o măsură indusă pe un semigrup liber de un semiautomat al cărui alfabet de intrare coincide cu mulţimea generatorilor semigrupului. Se studiază relaţia dintre măsurile induse şi homomorfismele de semiautomate pentru semiautomate parţiale, comportarea măsurilor exterioare cardinal-secvenţiale faţă de homomorfismele de semigrupuri.

Mulţimea măsurilor exterioare cardinal-secvenţiale se organizează ca o semilatice disjunctivă ceea ce permite introducerea şi studierea noţiunii de măsurabilitate a unui limbaj în raport cu un ideal H din această semilatice, o generalizare a noţiunii de regularitate. Se stabilesc caracterizări ale limbajelor H-măsurabile şi proprietăţi de închidere, regăsindu-se astfel proprietăţi de închidere ale clasei limbajelor regulate. Se arată că dacă o familie de măsuri exterioare cardinal-secvenţiale constituie un ideal complet, familia de algebre unare ataşate automatelor conexe ce induc aceste măsuri este o varietate.

În final, după prezentarea unor extensiuni globale ale măsurilor exterioare cardinal-secvenţiale se studiază unele proprietăţi ale funcţiilor măsurabile dintre spaţii cu măsuri exterioare cardinal-secvenţiale.

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NUMERICAL SOLUTIONS OF TWO NEURAL MODELS AND THE CORRESPONDING *FORTRAN* PROGRAMMING*)

BY

K. V. LEUNG, D. MANGERON, M. N. OGUZTORELI, R. B. STEIN

In the very ample and important framework of BIONICS, whose main lines of research are to explore the principles of computation in living organisms, to establish the structural and functional organization of such „biological computers", and to utilize this knowledge in the design and construction of cognitive systems, conversant systems, inductive inference machines, etc., the authors investigated the solutions of the neural equations considered in [17]. After studying the stability of the steady state solutions to these equations in general, they discussed finite systems with more details, because their importance in laboratory work and computer simulations.

In what follows the numerical analysis of the solutions is given and FORTRAN programs for the methods of variable step size and variable order of approximations is exposed.

A partial list of relevant references is given at the end of the present work.

I. Introduction. The infinite system of ordinary differential equations

$$\left(\tau \frac{d}{dt} + 1\right) \log \frac{x_i}{1 - x_i} = \gamma_i + \sum_{j=1}^{\infty} \beta_{ij} x_j \tag{I.1}$$

and

$$\left(\tau \frac{d}{dt} + \alpha_i\right) x_i = \frac{1}{1 + \exp\left\{-\left[\gamma_i + \sum_{j=1}^{\infty} \beta_{ij} x_j\right]\right\}}, \tag{I.2}$$

$i = 1, 2, 3, \ldots$, have been proposed by J. D. Cowan and R. B. Stein (cf. [17]) respectively to describe the neural activities in a neural net. In (I.1) and (I.2), $x_i = x_i(t)$ denotes Cowan's „sensitivity" of the i-th cell, $0 < x_i < 1$,

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