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ON THE ALGEBRAIC STRUCTURE OF LINEAR RELATIONS

BY

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1. We recall that in a previous paper ([4]), we considered submodules in the direct product $\prod_{j \in I} M_j$ of an arbitrary family of left (right) A-modules and we called these submodules linear relations; A denotes a ring, not necessarily commutative, with an identity element. These linear relations represent the generalization of the so-called additive relations (see [2] or [3] p. 51—53), or homomorphical relations ([1], p. 9—15), i.e. submodules of the direct sum of two left (right) modules over the same ring.

Let $R \subseteq \prod_{j \in I} M_j$ be a linear relation, $p_i : \prod_{j \in I} M_j \to M_i$ the canonical projection onto M_i and $q_i : M_i \to \prod_{j \in I} M_j$ the canonical injection of M_i. Denote by $\mathrm{Ker}_i R$ the submodule $p_i (R \cap q_i (M_i))$ of $p_i(R)$; we call $\mathrm{Ker}_i R$ *the kernel of index i* of the linear relation R; clearly, $a_i \in \mathrm{Ker}_i R$ iff the family $(b_j)_{j \in I}$, whose i-th component is a_i and whose remaining components are zero, belongs to R. Denote also by $R(a_i)$, for $a_i \in p_i(R)$, the set of elements $(a_k)_{k \in I \setminus \{i\}}$ of $\prod_{k \in I \setminus \{i\}} M_k$, such that each of them, together with a_i, form a family $(a_j)_{j \in I}$ from R (from now on, we denote $(a_k)_{k \in I \setminus \{i\}}$ by $(a_k)_{k \neq i}$). It's natural to call $\mathrm{Im}_i R = \bigcup_{a_i \in p_i(R)} R(a_i)$ *the image of index i* of the linear relation R.

It may come as a surprise to the reader that it is sufficient to assume that for one $i \in I$ we have $\lambda R(a_i) + \mu R(b_i) = R(\lambda a_i + \mu b_i)$, for any $\lambda, \mu \in A$, $a_i, b_i \in p_i(R)$, so that the subset $R \subseteq \prod_{j \in I} M_j$ is a linear relation and then the above equality of sets holds for all $j \in I$:

Proposition 1. *If $R \subseteq \prod_{j \in I} M_j$ is a linear relation, then for all $j \in I$, $\lambda R(a_j) + \mu R(b_j) = R(\lambda a_j + \mu b_j)$, where $\lambda, \mu \in A$ and $a_j, b_j \in p_j(R)$; conversely, if for a subset $R \subseteq \prod_{i \in I} M_j$ there exists $i \in I$ such that we have $\lambda R(a_i) + \mu R(b_i) = R(\lambda a_i + \mu b_i)$, for any $\lambda, \mu \in A$, $a_i, b_i \in p_i(R)$, then R is a linear relation.*

Proof. It is easily verified that $\lambda R(a_j) + \mu R(b_j) \subseteq R(\lambda a_j + \mu b_j)$. To see that the converse inclusion is also true let us consider $z \in R(\lambda a_j + \mu b_j)$ and let x and y be elements from $R(a_j)$, respectively $R(b_j)$: clearly, $\lambda x \in R(\lambda a_j)$, $\mu y \in R(\mu b_j)$ and $z - \lambda x \in R(\mu b_j)$, so that $z = (z - \lambda x) + \lambda x \in R(\lambda a_j + \mu b_j)$.

It remains to be proved the last assertion. Let $(a_i)_{i \in I}$ and $(b_i)_{i \in I}$ be two families from R: for $i \in I$ given in the hypothesis, we have $x = (a_k)_{k \neq i} \in R(a_i)$ and $y = (b_k)_{k \neq i} \in R(b_i)$, hence $\lambda x + \mu y \in R(\lambda a_i + \mu b_i)$, for $\lambda, \mu \in A$. It follows then, as required, that $(\lambda a_i + \mu b_i)_{i \in I} \in R$.

In *the structure theorem* for linear relations given in our paper [1] mentioned above, we proved that for each $i \in I$ we have the commutative diagram in fig. 1 where R_i is an additive (binary linear) relation uniquely determined by R: $(a_k)_{k \neq i} \in R_i(a_i)$ iff $(a_i)_{i \in I} \in R$; m_i, n_i are the natural projections; t_i and s_i are the structure epimorphisms and α_i is the structure isomorphism.

$$\begin{tikzcd}
p_i(R) \arrow[rr, dashed, "t_i"] \arrow[d, dashed, "R_i"'] \arrow[drr, dashed, "n_i" near start] & & \dfrac{\mathrm{Im}_i R}{R(0_i)} \\
\mathrm{Im}_i R \arrow[rr, dashed, "s_i"'] \arrow[urr, dashed, "m_i" near start] & & \dfrac{p_i(R)}{\mathrm{Ker}_i R} \arrow[u, "\alpha_i"']
\end{tikzcd}$$

Fig. 1

We showed also that the disjoint equivalence classes from the quotient module $\mathrm{Im}_i R/R(0_i)$ are all distinct sets of form $R(a_i)$, with $a_i \in p_i(R)$, and that $R = \bigcup_{C \in p_i(R)/\mathrm{Ker}_i R} (C \times \alpha_i(C))$. The last statement is a very suggestive result which shows that *the set* $R \subseteq \prod_{j \in I} M_j$ *is the union of all disjoint cartesian products of form* $C \times \alpha_i(C)$, where C is a coset of $p_i(R)/\mathrm{Ker}_i R$ and $\alpha_i(C)$ is the corresponding coset $R(a_i)$ from $\mathrm{Im}_i R/R(0_i)$ ($a_i - b_i \in \mathrm{Ker}_i R$ for $a_i, b_i \in p_i(R)$ iff $R(a_i) = R(b_i)$).

The structure isomorphism $\alpha_i : p_i(R)/\mathrm{Ker}_i R \to \mathrm{Im}_i R/R(0_i)$ *is in fact the general form for the usual isomorphism between the cokernel and the image of a homomorphism.*

If $R \subseteq M_1 \oplus M_2$ is a binary linear relation (which may be regarded as a many-valued homomorphism defined on a part of M_1), we denote as in [2] or [3], $\mathrm{Ker}_1 R = \mathrm{Ker}\, R$, $\mathrm{Ker}_2 R = \mathrm{Ind}\, R$, $p_1(R) = \mathrm{Def}\, R$, $p_2(R) = \mathrm{Im}\, R$; obviously, $R(0_1) = \mathrm{Ind}\, R$, $R(0_2) = \mathrm{Ker}\, R$ and the structure diagram for $i = 1 \in I = \{1, 2\}$ coincides with that for $i = 2$ (fig. 2):

We have $R = \bigcup_{C \in \mathrm{Def}\, R/\mathrm{Ker}\, R} (C \times \alpha(C))$. It is clear that for R homomorphism from M_1 to M_2, $\mathrm{Def}\ R = M_1$ and $\mathrm{Ind}\ R = 0$ and so $R = \bigcup_{\alpha^{-1}(a_2) \in \mathrm{Def}\, R/\mathrm{Ker}\, R} (\alpha^{-1}(a_2) \times \{a_2\})$, α being in this situation the well-known isomorphism between the cokernel $M_1/\mathrm{Ker}\ R$ and the image $\mathrm{Im}\ R$.

$$\begin{tikzcd}
\mathrm{Def}\, R \arrow[rr, "t"] \arrow[drr, "n" near start] & & \dfrac{\mathrm{Im}\, R}{\mathrm{Ind}\, R} \\
\mathrm{Im}\, R \arrow[u, "R"'] \arrow[rr, "s"'] \arrow[urr, "m" near start] & & \dfrac{\mathrm{Def}\, R}{\mathrm{Ker}\, R} \arrow[u, "\alpha"']
\end{tikzcd}$$

Fig. 2

2. Consider a linear relation $R \subseteq \prod_{j \in I} M_j$ and denote by R_{ik} for $i \in I$, $k \in I$ the binary linear relation induced by R in $M_i \oplus M_k$, that is $R_{ik} = \{(a_i, a_k) \mid \exists (b_j)_{j \in I} \in R, \text{ with } b_i = a_i,\ b_k = a_k\}$. By means of these binary linear relations *we shall establish the connection between structure diagrams of* R *different indices.*

We assert first that $R_{ik} \subseteq p_i(R) \oplus p_k(R)$ and $u_{ik} \subseteq \mathrm{Im}_i R/R(0_i) \oplus \mathrm{Im}_k R/R(0_k)$ defined by $u_{ik}(R(a_i)) = \{R(a_k) / a_k \in R_{ik}(a_i)\}$ are two binary linear relations which establish the connection between the structure diagram of index i and the structure diagram of index k:

Proposition 2. a) $u_{ik} = \{(R(a_i), R(a_k)) / a_k \in R_{ik}(a_i),\ a_i \in p_i(R),\ a_k \in p_k(R)\}$ *is a binary linear relation which makes the following diagram commutative (fig. 3):*

$$\begin{tikzcd}
p_i(R) \arrow[rr, "t_i"] \arrow[d, dashed, "R_{ik}"'] & & \dfrac{\mathrm{Im}_i R}{R(0_i)} \arrow[d, dashed, "u_{ik}"] \\
p_k(R) \arrow[rr, "t_k"'] & & \dfrac{\mathrm{Im}_k R}{R(0_k)}
\end{tikzcd}$$

Fig. 3

b) *If* R_{ik} *is a homomorphism of* $p_i(R)$ *(on) to* $p_k(R)$, *then the structure epimorphism* t_k *coincides with the structure isomorphism* α_k *and* u_{ik} *is a homomorphism.*

Proof. a) We must verify that $u_{ik}(R(a_i)) = u_{ik}(R(b_i))$ in case that $R(a_i) = R(b_i)$; but, if $R(a_i) = R(b_i)$ then $a_k \in R_{ik}(a_i)$ iff $a_k \in R_{ik}(b_i)$. Thus, $R(a_k) \in u_{ik}(R(a_i))$ iff $R(a_k) \in u_{ik}(R(b_i))$, as required.

Consider two pairs $(R(a_i), R(a_k))$ and $(R(b_i), R(b_k))$ from u_{ik}, for which we have $a_k \in R_{ik}(a_i)$ and, respectively, $b_k \in R_{ik}(b_i)$; then $\lambda a_k + \mu b_k \in R_{ik}(\lambda a_i + \mu b_i)$, for $\lambda, \mu \in A$, hence $(R(\lambda a_i + \mu b_i), R(\lambda a_k + \mu b_k)) = \lambda(R(a_i), R(a_k)) + \mu(R(b_i), R(b_k))$ belongs to u_{ik}. We conclude that u_{ik} is a linear relation.

b) Obviously, for $a_i \in p_i(R)$ we have $u_{ik}(t_i(a_i)) = t_k(R_{ik}(a_i)) = \{R(a_k) \mid a_k \in R_{ik}(a_i)\}$. If R_{ik} is a homomorphism, $R_{ik}(a_i)$, for $a_i \in p_i(R)$, consists of a unique element $a_k \in p_k(R)$. Therefore $u_{ik}(R(a_i))$, for $R(a_i) \in \mathrm{Im}_i R/R(0_i)$, consists only of a unique coset $R(a_k)$, what proves that in this case u_{ik} is also a homomorphism.

In the same case, we show that $\mathrm{Ker}_k R = 0$. If not, there exists $a_k \neq 0_k$, $a_k \in R_{ik}(0_i)$; since $(0_j)_{j \in I} \in R$, we have also $0_k \in R_{ik}(0_i)$; it follows then $a_k = 0_k$, which is a contradiction. Thus $\mathrm{Ker}_k R = \mathrm{Ker}\, t_k = 0$ and t_k is isomorphism; by the commutativity of the structure diagram of index $k \in I$, we obtain that t_k coincides with $\alpha_k : p_k(R) \to \mathrm{Im}_k R/R(0_k)$. It follows also that $R_k = \bigcup\limits_{a_k \in p_k(R)} (\{a_k\} \times R(a_k))$.

Remark. We can establish homomorphisms between the structure diagram of one index $i \in I$ and all the other structure diagrams of indices $k \neq i$ when all R_{ik}, with $k \neq i$, $k \in I$, are homomorphisms. It is easy to check that this condition is equivalent to each of the following ones:

(a) $\mathrm{Ker}\, R_{ki} = 0$, for all $k \neq i$.
(b) R_i is homomorphism.
(c) $R(0_i) = 0$.

Since $\mathrm{Ker}_k R \subseteq \mathrm{Ker}\, R_{ik}$, we have also $\mathrm{Ker}_k R = 0$, for any $k \neq i$ (in the above case).

Next, we note that if we assume that both R_i and R_k are homomorphisms, for two different indices $i \neq k$ from I, we obtain a very special linear relation:

Proposition 3. *Let $R \subseteq \prod\limits_{j \in I} M_j$ be a linear relation such that both R_i and R_k homomorphisms, $i \neq k$. Then $p_i(R)$ and $p_k(R)$ are isomorphic, moreover R_i and R_k are even isomorphisms; thus, $p_i(R) \cong p_k(R) \cong \mathrm{Im}_i R \cong \mathrm{Im}_k R$.*

Proof. Indeed, R_{ih} with $h \neq i$, and R_{km} with $m \neq k$ are homomorphisms, in particular R_{ik} and R_{km} are homomorphisms; but $R_{ik}^{-1} = R_{ki}$; therefore R_{ik} is isomorphism from $p_i(R)$ onto $p_k(R)$. By applying the above Remark, we have that $\mathrm{Ker}_j R = 0$, $\forall j \in I$ and so $a_j \neq b_j$, $a_j, b_j \in p_j(R)$, implies $R(a_j) \cap R(b_j) = \varnothing$, $\forall j \in I$; we observe that $R(a_j) = R_j(a_j)$ and $R(b_j) = R_j(b_j)$; if we take $j = i$ or $j = k$, it follows that R_i and R_k are injective. Thus, R_i and R_k are even isomorphisms from $p_i(R)$ onto $\mathrm{Im}_i R$ and, respectively, from $p_k(R)$ onto $\mathrm{Im}_k R$.

We now turn our attention to a particular n-ary linear relation $R \subseteq \bigoplus\limits_{i=1}^{n} M_i$. Consider a sequence of homomorphisms

$$M \xrightarrow{f_1} M_2 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} M_n ,$$

where $M_1, M_2, \ldots, M_n$ are left A-modules and M is a submodule of M_1.

Definition 1. *We call functional linear relation the submodule $R_n = \{(a_1, a_2, a_3, \ldots, a_n) \mid a_2 = f_1(a_1),\ a_3 = f_2(a_2), \ldots, a_n = f_{n-1}(a_{n-1})\}$ of $\bigoplus\limits_{i=1}^{n} M_i$.* We denote R by $R(M, f_1, \ldots, f_{n-1})$.

In this case, we have $R_{i,i+k} = f_{i+k-1} \circ f_{i+k-2} \circ \cdots \circ f_{i+1} \circ (f_i)_{p_i(R)}$ for $i = 1, 2, \ldots, n-1$ and $k = 1, 2, \ldots, n-i$; $(f_i)_{p_i(R)}$ denotes the restriction of f_i to $p_i(R) = \mathrm{Im}(f_{i-1} \circ f_{i-2} \circ \cdots \circ f_1)$. All R_{ij} with $i < j$ are homomorphisms and one can construct commutative diagrams as in Proposition 2 in order to establish the homomorphical connexion between the structure diagram of a given index $i \in I$ and all the structure diagrams of indices $j > i$. In case $i = 1$, this means that R_{1k}, $k = 2, 3, \ldots, n$ are homomorphisms, that is R_1 is homomorphism.

Proposition 4. *Let $R(M, f_1, f_2, \ldots, f_{n-1}) \subseteq \bigoplus\limits_{i=1}^{n} M_i$ be a functional linear relation; R_k, for $k \neq 1$, is (also) a homomorphism iff $(f_1)_M, (f_2)_{p_2(R)}, \ldots, (f_{k-1})_{p_{k-1}(R)}$ are injective.*

Proof. R_k is homomorphism from $p_k(R)$ onto $\mathrm{Im}_k R$ iff any R_{ki}, $i \neq k$, is homomorphism from $p_k(R)$ to M_i. This holds always for R_{ki} with $k < i$. We are now left with the task of proving that the condition from Prop. 4 is valid iff all R_{ki} with $k > i$ are homomorphisms. We have $R_{ki} = R_{ik}^{-1} = R_{i,i+(k-i)}^{-1} = [f_{i+(k-i)-1} \circ f_{i+(k-i)-2} \circ \cdots \circ f_{i+1} \circ (f_i)_{p_i(R)}]^{-1} = f_i^{-1} \circ f_{i+1}^{-1} \circ \cdots \circ f_{k-2}^{-1} \circ (f_{k-1}^{-1})_{p_k(R)}$. Let us assume that f_1 is injective on $p_1(R) = M$, f_2 is injective on $p_2(R) = \mathrm{Im}\, f_1, \ldots, f_{k-1}$ is injective on $p_{k-1}(R) = \mathrm{Im}(f_{k-2} \circ f_{k-3} \circ \cdots \circ f_1)$; then $f_1^{-1}, f_2^{-1}, \ldots, f_{k-1}^{-1}$ are homomorphisms defined on $p_2(R)$, $p_3(R), \ldots$, respectively on $p_k(R)$; our supposition is sufficient in order to have R_{ki} homomorphism for any $i < k$. Conversely, this hypothesis is also a necessary condition; as we must have in particular $R_{k,k-1} = (f_{k-1}^{-1})_{p_k(R)}$ homomorphism, then also $R_{k,k-2} = (f_{k-2}^{-1} \circ f_{k-1}^{-1})_{p_k(R)}$ must be homomorphism .... and so until R_{k1}; we obtain step by step that all f_h^{-1} with $h \leqslant k-1$ are homomorphisms on $p_{h+1}(R)$, i.e. f_h, $h \leqslant k-1$ are injective on $p_h(R)$.

Proposition 5. *A linear relation $R \subseteq \bigoplus\limits_{j=1}^{n} M_j$ is functional iff, up to a permutation of the indices, $p_{j+1}(R(0_j)) = 0$.*

Proof. Of course, p_{j+1} is the canonical projection $p_{j+1} : \bigoplus\limits_{h \neq j} M_h \to M_{j+1}$. The linear relation R is functional iff, after an eventual permutation of the indices $1, 2, \ldots, n$, $R_{k,k+1}$ are homomorphisms, i.e. iff $\mathrm{Ker}\, R_{k+1,k} = 0$. The reader will have no difficulty in convincing himself that for any linear relation $R \subseteq \coprod_{j\in I} M_j$, $\operatorname{Ker} R_{ik} = p_i(R(0_k)) = \bigcup_{a_i \in p_i(R(0_k))} C_{a_i}$, where $C_{a_i} \in p_i(R)/\operatorname{Ker}_i R$ and $p_i : \coprod_{h\neq k} M_h \to M_i$. Thus our Proposition is obviously right.

3. Definition 2. *We call the submodule* $C_i(R) = \{(a_j)_{j\in I} \in R\,;\ a_i \in \operatorname{Ker}_i R\} \subseteq R$ *the center of index* $i \in I$ *of the linear relation* $R \subseteq \coprod_{j\in I} M_j$.

Definition 3. *We call the center of the linear relationi* $R \subseteq \coprod_{j\in I} M_j$ *the submodule* $C(R) = \bigcap_{j\in I} C_j(R) \subseteq R$.

We remark that if $R \subseteq M_1 \oplus M_2$ is a binary linear relation, $C_1(R) = C_2(R) = \operatorname{Ker} R \oplus \operatorname{Ind} R$.

Returning to the general case let us recall an interesting equality which we have proved in [4]:

Proposition 6. $C_i(R) = \{(a_j)_{j\in I} \in R : (a_k)_{k\neq i} \in R(0_i)\}$.

Now we shall show that the factor module of R by $C_i(R)$ is directly isomorphic to $p_i(R)/\operatorname{Ker}_i R$ and to $\operatorname{Im}_i R/R(0_i)$ by naturally defined maps:

Proposition 7. *The maps* $e_i : R/C_i(R) \to \operatorname{Im}_i R/R(0_i)$, $e_i((a_j)_{j\in I} + C_i(R)) = (a_k)_{k\neq i} + R(0_i)$ *and* $d_i : R/C_i(R) \to p_i(R)/\operatorname{Ker}_i R$, $d_i((a_j)_{j\in I} + C_i(R)) = a_i + \operatorname{Ker}_i R$ *are isomorphisms such that the diagram (fig. 4) is commutative.*

$$\begin{tikzcd}
\dfrac{\operatorname{Im}_i R}{R(0_i)} & & \\
& & \dfrac{R}{C_i(R)} \arrow[ull, dashed, "e_i"'] \arrow[dll, dashed, "d_i"] \\
\dfrac{p_i(R)}{\operatorname{Ker}_i R} \arrow[uu, "\alpha_i"] & &
\end{tikzcd}$$

Fig. 4

Proof. We must verify that $d_i((a_j)_{j\in I} + C_i(R))$ and $e_i((a_j)_{j\in I} + C_i(R))$ defined above depend only on the coset $(a_j)_{j\in I} + C_i(R)$ and not on its representative element $(a_j)_{j\in I}$: but $(a_j)_{j\in I} + C_i(R) = (b_j)_{j\in I} + C_i(R)$ for $(a_j)_{j\in I} \in R$, $(b_j)_{j\in I} \in R$ means $(a_j - b_j)_{j\in I} \in C_i(R)$, which implies $a_i - b_i \in \operatorname{Ker}_i R$ and, by Proposition 6, $(a_k - b_k)_{k\neq i} \in R(0_i)$: therefore $a_i + \operatorname{Ker}_i R = b_i + \operatorname{Ker}_i R$ and $(a_k)_{k\neq i} + R(0_i) = (b_k)_{k\neq i} + R(0_i)$. Since the above implications are valid in both senses, we have the equivalence between the equalities

$$(a_j)_{j\in I} + C_i(R) = (b_j)_{j\in I} + C_i(R),\quad a_i + \operatorname{Ker}_i R = b_i + \operatorname{Ker}_i R$$

and

$$(a_k)_{k\neq i} + R(0_i) = (b_k)_{k\neq i} + R(0_i)$$

and thus d_i and e_i are injective. It is easily verified that d_i and e_i are surjective and that they are homomorphisms.

To complete the proof, we observe that $(a_k)_{k\neq i} + R(0_i)$, with $(a_j)_{j\in I} \in$ $\in R$, is just $R(a_i)$, hence $(\alpha_i \circ d_i)((a_j)_{j\in I} + C_i(R)) = \alpha_i(a_i + \operatorname{Ker}_i R) = R(a_i) =$ $= e_i((a_j)_{j\in I} + R(0_i))$ and we can conclude that the given diagram is commutative.

In the case of a binary linear relation $R \subseteq M_1 \oplus M_2$, Proposition 7 gives us direct isomorphisms between $R/\operatorname{Ker} R \oplus \operatorname{Ind} R$, $\operatorname{Def} R/\operatorname{Ker} R$ and $\operatorname{Im} R/\operatorname{Ind} R$.

The center $C(R)$ of a linear relation $R \subseteq \coprod_{j\in I} M_j$ may be regarded as having the same significance as the kernel for a homomorphism (as a matter of fact, $C(R) = \operatorname{Ker} R \oplus \{0\}$, if R is a homomorphism). Indeed, for example, the congruence relation which defines $R/C(R)$ coincides with the equivalence relation defined on R by $(a_j)_{j\in I} \sim (b_j)_{j\in I}$ iff $R(a_j) = R(b_j)$, $\forall j \in I$. Moreover, *the wellknown fact that a homomorphism is injective iff it has null kernel is just a particular case of the following general property:*

Proposition 8. *The following conditions are equivalent:*

(1) $\operatorname{Ker}_j R = 0$, $\forall j \in I$;

(2) *Two different families from* R *differ at least by two corresponding components;*

(3) $C(R) = 0$.

Proof. (1) $\Rightarrow$ (2). Let $(a_j)_{j\in I}$ and $(b_j)_{j\in I}$ be two different families from R. Then, there exists an $i \in I$ such that $a_i \neq b_i$. Since $\operatorname{Ker}_i R = 0$, it follows that α_i establishes isomorphism between $\operatorname{Def} R$ and $\operatorname{Im}_i R/R(0_i)$ with $\alpha_i(a) = R(a_i) \neq \alpha_i(b)_i = R(b_i)$. Then $R(a_i) \cap R(b_i) = 0$ and so $(a_k)_{k\neq i} \neq$ $\neq (b_k)_{k\neq i}$, that is there exists at least another $a_h \neq b_h$, $h \neq i$.

(1) $\Rightarrow$ (3). It is obvious.

(2) $\Rightarrow$ (1). Suppose we have $a_i \neq 0_i$, $a_i \in \operatorname{Ker}_i R$ for $i \in I$. Then we have $(0_j)_{j\in I} \in R$ and also $(a_j)_{j\in I} \in R$, with $a_k = 0_k$ for $k \neq i$ and $a_i \neq 0_i$; but this is against the hypothesis, since we obtained two families $(0_j)_{j\in I}$ and $(a_j)_{j\in I}$ which belong to R and differ only by the component $a_i \neq 0_i$. Thus for all $a_j \in \operatorname{Ker}_j R$ we must have $a_j = 0_j$, hence for all $j \in I$, $\operatorname{Ker}_j R = \{0_j\}$.

(3) $\Rightarrow$ (2). Suppose we have $(a_j)_{j\in I} \in R$, $(b_j)_{j\in I} \in R$, $a_i \neq b_i$ and $a_k = b_k$ for all $k \neq i$. Then we have $(a_k)_{k\neq i} = (b_k)_{k\neq i} \in R(a_i) \cap R(b_i)$, hence $R(a_i) = R(b_i)$. Thus $R(a_j) = R(b_j)$ for all $j \in I$ and, by a property mentioned above (which we have proved in [4]), this means $(a_j)_{j\in I} - (b_j)_{j\in I} \in C(R)$. Since $C(R) = 0$, it follows that $a_j = b_j$, for all $j \in I$, which contradicts the hypothesis $a_i \neq b_i$. We conclude that if $(a_j)_{j\in I} \in R$, $(b_j)_{j\in I} \in R$ and $a_i \neq b_i$, we must have another $a_h \neq b_h$ with $h \neq i$.

Remarks. Obviously, in the case $R \subseteq M_1 \oplus M_2$, Proposition 8 shows that $\operatorname{Ker} R \oplus \operatorname{Ind} R = 0$ is equivalent to the assertion that for $(a_1, a_2) \in R$, $(b_1, b_2) \in R$, $a_1 \neq b_1$ iff $a_2 \neq b_2$ (i.e. R is an isomorphism of $\operatorname{Def} R$ onto $\operatorname{Im} R$).

Finally, let us remark that allthough in the binary case $C_1(R) = C_2(R) = C(R)$, however, generally, for $R \subseteq \prod_{j \in I} M_j$, $i \neq h$ we have not $C_i(R) = C_h(R)$. Let us consider $R(M, f_1, f_2, \ldots, f_{n-1}) \subseteq \bigoplus_{j=1}^{n} M_j$ a functional linear relation. Then it is easily seen that $C(R) = C_1(R) = C_2(R) \subseteq C_3(R) \subseteq \ldots \subseteq C_n(R)$ and $C_j(R) \subseteq f_1^{-1} \circ f_2^{-1} \circ \cdots \circ f_{j-2}^{-1}(\mathrm{Ker}\, f_{j-1}) \oplus \ldots \oplus f_{j-2}^{-1}(\mathrm{Ker}\, f_{j-1}) \oplus \mathrm{Ker}\, f_{j-1} \oplus \{0_j\} \oplus \ldots \oplus \{0_n\}$, $h = 2, 3, \ldots, n$.

If, for example $\mathrm{Im}\, f_1 \cap \mathrm{Ker}\, f_2 \neq 0$, i.e. there exists $a_2 \neq 0_2$ with $f_2(0_2) = 0_3$ and $a_2 = f_1(a_1)$, $a_1 \in M$, then we have the strict inclusion $C_2(R) = \mathrm{Ker}\, f_1 \oplus \{0_2\} \oplus \ldots \oplus \{0_n\} \cap C_3(R)$, because $(a_1, a_2, 0_3, \ldots, 0_n) \in C_3(R)$ and $(a_1, a_2, 0_3, \ldots, 0_n) \notin C_2(R)$.

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ASUPRA STRUCTURII ALGEBRICE A RELAŢIILOR LINIARE

Rezumat

Se continuă studiul relaţiilor liniare între module, definite de noi în [4] ca submodule ale produsului direct al unei familii oarecare de module, stabilindu-se legăturile dintre diagramele lor de structură de indici diferiţi. Ne oprim şi asupra unui submodul caracteristic al unei relaţii liniare, pe care-l denumim centrul acesteia, şi arătăm că el are aceeaşi semnificaţie ca nucleul pentru un homomorfism.

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ÜBER DIE DARSTELLUNG DER ALGEBRAISCHEN ZAHLEN ZWEITEN GRADES DURCH REGULÄRE KETTENBRÜCHE

VON

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In dieser Arbeit wird mit Hilfe einer rekursiven Relation zweiter Ordnung eine neue Charakterisierung der Kettenbrüche der algebraischen Zahlen zweiten Grades angegeben, und zugleich wird gezeigt, wie man diese Relation zur Bestimmung der enstprechenden Näherungsbrüche beliebiger Ordnung benützen kann. Unsere Methode besteht aus einer Verallgemeinerung der von S. Sanielevici für die periodischen Kettenbrüche der Periodenlänge eins benützten Methode (Siehe [2]). Diese Methode ist so verallgemeinert worden, dass man sie für die periodischen Kettenbrüche beliebiger Periodenlänge benutzen kann. Der Einfachheit halber geben wir hier einige Begriffe über Kettenbrüche, die von der von Felix Klein angegebenen Geometrisierungsmethode angeregt worden sind. Es bezeichnen N die Menge der natürlichen Zahlen (einschliessend Null), Z die Menge der ganzen Zahlen und Q die Menge der Rationalzahlen.

Definition 1. *Eine unendliche Folge*

$$q_0, q_1, q_2, \ldots, q_n, \ldots \quad (q_0 \in Z,\ q_n \in N \text{ für } n \geqslant 1) \tag{1}$$

mit der Eigenschaft, dass aus $q_n = 0$ die Beziehung $q_{n+k} = 0$ für jedes $k \in N$ folgt, heisst ein regulärer Kettenbruch.

Es ist wohlbekannt, dass sich jede reelle Zahl x in einen regulären Kettenbruch entwickeln lässt d.h. x kann unter folgender Form geschrieben werden

$$x = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \ddots}} \tag{2}$$

Definition 2. *Unter einer den Kettenbruch (1) zugeordneten R-Folge versteht man die Folge*