Proof. Let be $P(\pi) = \{(y, f) \in M' \times P | \pi(y) = f(0),\ pf = f' \in P'\}$. If $(y, f) \in P(\pi)$, then $(p'(y), pf) \in P'(\pi')$ and, therefore, $C'(p,(y), pf) \in (N')^I$. Let $C'_1 : P'(\pi') \times I \to N'$ be the map $C'_1((y', f'), t) = C'((y', f'))(t)$ and let be the diagram

$$\begin{tikzcd} M' \arrow[r, "p'"] & N' \\ P(\pi) \arrow[u, "j"] & P'(\pi') \times I \arrow[u, "C'_1"'] \\ & P(\pi) \times I \arrow[u, "h"'] \end{tikzcd}$$

with $j(y, f) = y$, $h((y, f), t) = ((p'(y), pf), t)$. There exists a continous map $C_1 : P(\pi) \times I \to M'$ such that $C_1((y, f), 0) = y$ and $p'C_1((y, f), t) = C'_1 h((y, f), t) = C'_1((p'(y), pf), t) = C'((p'(y), pf)(t)$. We define $C : P(\pi) \to (M')^I$ by $C((y, f))(t) = C_1((y, f), t)$. Then, C is a continuous map and $C((y, f))(0) = C_1((y, f), 0) = y$, $p'C(y, f) = C'(p'(y), pf)$. We have $p\pi C(y, f)(t) = \pi' p' C(y, f)(t) = \pi' C'(p'(y), pf)(t) = pf(t)$. Moreover, $\pi C(y, f)(0) = \pi(y) = f(0)$. Because p is a covering map it results $\pi(C(y, f)) = f$, for $(y, f) \in P(\pi)$.

Let be $(y, f) \in P(\pi)$ and $y_1 = C(y, f)(\Phi(0))$, for $\Phi . I \to I$ a continuous piecewise linear map. We have $p'C(y_1, f \circ \Phi)(t) = C'(p'(y_1), p(f \circ \Phi))$, $p'C(y, f)(\Phi(t)) = C'(p'(p'(y), pf)(\Phi(t))$, that is, $p'C(y, f) \circ \Phi = C'(p'(y), pf) \circ \Phi$. Let be $y'_1 = p'C(y, f)(\Phi(0)) = C'(p'(y), pf)(\Phi(0))$; $C'(p'(y), pf) \circ \Phi = C'(y'_1, p(f \circ \Phi))$. We have $p'y_1 = p'C(y. f)(\Phi(0)) = y'_1$. It follows $C'(p'(y), pf) \circ \Phi = C'(p'(y_1), p(f \circ \Phi))$, that is $p'C(y_1, f \circ \Phi)(t) = p'(C(y, f) \circ \Phi))(t)$. Because $C(y_1, f \circ \Phi)(0) = y_1 = (C(y, f) \circ \Phi)(0)$ and p' is a covering map, we have $C(y_1, f \circ \Phi) = C(y, f) \circ \Phi$. Therefore C is a P-connection of π.

If $N'_* = \pi'^{-1}(*')$ has a structure of vector space, then $M'_* = \pi^{-1}(*)$ is also vector space. Moreover, we have the commutative diagram:

$$\begin{tikzcd} M'_* \times P_{**} \arrow[r, "\varphi"] \arrow[d, "p'_* \times p"'] & M'_* \arrow[d, "p'"] \\ N'_{*'} \times P'_{*'*'} \arrow[r, "\varphi'"] & N'_{*'} \end{tikzcd}$$

with $P_{**} = \{f \in P | f(0) = f(1) = *\}$, $P'_{*'*'} = \{f' \in P' | f'(0) = f'(1) = *'\}$, $\varphi(y, f) = C(y, f)(1)$ and $\varphi'(y', f') = C'(y', f')(1)$. In fact, $p'\varphi(y, f) = p' C(y, f)(1) = C'(p'(y), pf)(1) = \varphi'(p'(y), pf) = \varphi'(p'_* \times p)(y, f)$. It follows that if $\varphi''_f : N'_{*'} \to N'_{*'}$, $y' \to \varphi'(y', f')$, is a linear automorphism, for every $f' \in P'_{*'*'}$, then $\varphi_f : M'_* \to M'_*$ is also a linear automorphism, for every $f \in P_{**}$.

REFERENCES

1. G r o m o v M. L. — *Stabil'nye otobraženija sloenij v mnogobrazija.* Izvestija Akademii Nauk SSSR, Serija Matematičeskaja 33 (1969), 704—734.
2. T e l e m a n C. — *Connections and bundles I—II*, Koninkl. Neder]. Akademie Van Wetenschapten-Amsterdam, Reprinted from Proceedings Series A, 72, Nol. Indag. Math., 31, 1, 89—112 (1969).
3. R a y m o n d F. — *Local triviality for Hurewicz fiberings of manifolds*, Topology 3, 1 (1965).

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COHOMOLOGY OF LIE ALGEBROIDS

BY

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Let V be a manifold of C^∞ class, $A(V)$ be the algebra of the real differential mappings on V, (M, π, V) be a vector bundle with the base space V and $\Gamma(\pi)$ be the $A(V)$-module of the differential cross sections of π.

If the module $\Gamma(\pi)$ has a structure of Lie pseudoalgebra then (M, π, V) is called a *Lie algebroid* [5]. A structure of *Lie pseudoalgebra* on $\Gamma(\pi)$ is given [5] by:

1) a mapping $p^{\Gamma(\pi)} \in \mathrm{Hom}_{A(V)}(\Gamma(\pi), \mathcal{D}(A(V)))$ where $\mathcal{D}(A(V))$ is the $A(V)$-module of the derivations of $A(V)$;

2) a bracket operation $\lambda^{\Gamma(\pi)} : \Gamma(\pi) \times \Gamma(\pi) \to \Gamma(\pi)$ such that

$$p^{\Gamma(\pi)} \lambda^{\Gamma(\pi)}(X, Y) = [p^{\Gamma(\pi)} X, p^{\Gamma(\pi)} Y] \text{ and} \tag{1}$$

$$\lambda^{\Gamma(\pi)}(X, uY) - u\lambda^{\Gamma(\pi)}(X, Y) = (p^{\Gamma(\pi)} X . u) Y, \tag{2}$$

for all $u \in A(V)$ and $X, Y \in \Gamma(\pi)$.

$$J^{\Gamma(\pi)}(X, Y, Z) = \sum_{Cyrc} \lambda^{\Gamma(\pi)}(\lambda^{\Gamma(\pi)}(X, Y), Z) = 0 \quad \forall X, Y, Z \in \Gamma(\pi). \tag{3}$$

If a vector bundle (M, π, V) has the property that $\pi^{-1}(x)$ is a Lie algebra for all $x \in V$ then obviously it is a Lie algebroid with $p^{\Gamma(\pi)} = 0$; particularly, the trivial vector bundle $V \times g$ where g is a Lie algebra, is also a Lie algebroid.

Let (M, π, V) be a Lie algebroid and $Q_p(\Gamma(\pi), A(V))$ be the set of all alternating p-linear mappings of $\Gamma(\pi)$ into $A(V)$. Obviously, $Q_p(\Gamma(\pi), A(V))$ is a $A(V)$-module for each p and $Q_0(\Gamma(\pi), A(V)) = A(V)$. We define on the set

$$Q(\Gamma(\pi), A(V)) = \sum_{p \in z} Q_p(\Gamma(\pi), A(V)) \tag{4}$$

a structure of a differential graduate $\mathcal{D}(A(V))$-algebra ([2], [7]) in the following manner

$$\begin{aligned}(\theta^p_X \omega)(X_1, \dots X_p) &= p^{\Gamma(\pi)} X(\omega(X_1, \dots, X_p)) + \\ &+ \sum_{i=1}^{p} (-1)^i \omega(\lambda^{\Gamma(\pi)}(X, X_i), X_1, \dots, \check{X}_i, \dots, X_p)\end{aligned} \tag{5}$$

$$(i^p_X \omega)(X_1, \dots X_{p-1}) = \omega(X, X_1, \dots X_{p-1}); \tag{6}$$

$$\begin{aligned}d^p \omega(X_1, \dots X_{p+1}) &= \sum_{i=1}^{p+1} (-1)^{i+1} p^{\Gamma(\pi)} X_i \, \omega(X_1, \dots \check{X}_i, \dots X_{p+1}) + \\ &+ \sum_{i<j} (-1)^{i+j} \omega(\lambda^{\Gamma(\pi)}(X_i, X_j), \dots \check{X}_i, \dots \check{X}_j, \dots X_{p+1}).\end{aligned} \tag{7}$$

It is easy to prove that $0^p_X\omega$, $i^p_X\omega$, $d^p\omega$ are the multilinear mappings; moreover the operators 0^p_X, i^p_X, d^p satisfy the axioms of a differential graduate algebra over $\mathcal{D}(A(V))$. In fact, it is known that

$$(d^{p+1}\circ d^p)(\omega) = R\wedge\omega, \tag{8}$$

where R is the curvature of $p^{\Gamma(\pi)}$ defined by

$$R(X, Y) = [p^{\Gamma(\pi)}X,\ p^{\Gamma(\pi)}Y] - p^{\Gamma(\pi)}\lambda^{\Gamma(\pi)}(X,\ Y), \tag{9}$$

for all $X,\ Y\in\Gamma(\pi)$. From (1), it results that $R(X, Y) = 0$ and the left hand side of (8) vanishes. The other axioms can be verified by direct calculation.

So, we associate with such Lie algebroid a complex $Q(\Gamma(\pi), A(V))$ the cochains of which are all alternating p-linear maps and the coboundary operator defined by (7); the cohomology of this complex will be denoted $H(M\,;\,R)$.

An important example. Let Φ be an a-connected differential groupoid with the manifold 0_Φ as the base space and $(a,\ b):\Phi\to V\times V$ be its source-target map [1]; we denote by $L\Phi$ the Lie algebroid associated to Φ where L is the Lie functor defined by P r a d i n e s, [4], [5], [6]. Then $p^{L\Phi} = Tb|_{L\Phi}$.

In the particular case of a groupoid which is sum of groups ($a = b$), we obtain $p^{L\Phi} = 0$ such that from (7) it results

$$d^p\omega(X_1,\dots X_{p+1}) = \sum_{i<j}(-1)^{i+j}\,\omega(\lambda^{\Gamma(\pi)}(X_i\,X_j)X_1,\dots \check{X}_i,\dots \check{X}_j,\dots,X_{p+1})) \tag{10}$$

for all $X_1,\dots X_{p+1}\in\Gamma(L\Phi)$ and $\omega\in; Q_p(\Gamma(L\Phi), A(0_\Phi))$.

If $\Phi = V\times G$, where G is a Lie group, we find again the known results related to the real cohomology of a Lie algebra (see [2], [7]) taking in (7) the constant cross sections of $L\Phi = V\times g$ (here g is the Lie algebra of G).

A p-form ω on the groupoid Φ will be called a *right invariant p-form* if $\omega(X_1,\dots X_p)$ is an element from $A(\varepsilon_\Phi(0_\Phi))$ for all $X_1,\dots X_p\in\Gamma(L\Phi)$. Here $\varepsilon_\Phi: 0_\Phi\to\Phi$ is the canonical injection and by $A(\varepsilon_\Phi(0_\Phi))$ we denote the set of the real mappings on Φ which are constant on the fibers of the fiber bundle $(\Phi,\ a,\ 0_\Phi)$.

Let $Q^r_p(\Phi)$ be the $A(\varepsilon_\Phi(0_\Phi))$ — module of right invariant p-forms on Φ and $Q^r(\Phi)$ be the Grassman algebra of these forms. It is easy to see that the correspondence $\delta: Q^r(\Phi)\to Q(\Gamma(L\Phi),\ A(0_\Phi))$ given by

$$(\delta(\omega)(X_1,\dots X_p))(0) = \omega(b(0))(X_1(b(0)),\dots X_p(b(0))) \tag{11}$$

$X_1,\dots X_p\in\Gamma(L\Phi)$, $0\in\Phi$ is multiplicative and it establishes an isomorphism between the algebra of righ invariant differential forms on Φ and the Grassman algebra $Q(\Gamma(L\Phi), A(0_\Phi))$. Moreover we obtain that

$$d(\delta\omega) = \delta(d\omega). \tag{12}$$

Now, we suppose that the groupoid Φ is a sum of groups; then the Lie algebroid associated to it is a fiber bundle whose fibers are the Lie algebras of the fibers of Φ. A right invariant p-form ω on Φ will be called an *invariant p-form* if $L^p_X\omega = 0$ for all $X\in\Gamma(L\Phi)$ where L_X is the Lie derivative of ω on Φ.

In this case, from (13) it results that every invariant p-form on Φ is a cocycle. Moreover, if Φ is an a-connected (hence b-connected) groupoid, then, using the known results related to the cohomology of a compact connected Lie group [3], [7] we obtain that *the cohomology ring of Φ obtained using only invariant forms is isomorphic (under the natural inclusion homomorphism) with the cohomology ring of Φ obtained using all regular differential forms.*

This result admits also the following new interpretation:

The cohomology algebra $H(L\Phi\,;\,R)$ is isomorphic with the algebra of $L\Phi$-invariant elements from $Q(\Gamma(L\Phi),\ A(0_\Phi))$.

In fact, the elements which are associated by δ to the invariant forms on Φ are the $L\Phi$-invariant elements from the complex $Q(\Gamma(L\Phi)\,;\,A(0_\Phi))$.

BIBLIOGRAPHY

1. E c k e, P. V e r — *Introduccion a las conexiones de orden superior.* Facult. de Ciencias, Zaragoza, 1968 (Mimeo. Notes).
2. C a r t a n H. — *Notions d'algèbre différentielle; application aux groupes de Lie, et aux variétés où opère un groupe de Lie.* Colloque de Topologie. Bruxelles (1950), 15—27.
3. C h e v a l l e y C., E i l e n b e r g S. — *Cohomology theory of Lie groups and Lie algebras.* Trans. Amer. Math. Soc., T. 63, 1948, 85—124.
4. P r a d i n e s J. — *Théory de Lie pour les groupoïdes différentiables. Relations entre propriétés locales et globales.* C. R. Acad. Sci. Paris T. 263, 1966, P. 907.
5. P r a d i n e s J. — *Théory de Lie pour les groupoïdes différentiables. Calcul différentielle dans la catégorie des groupoïdes.* C. R. Acad. Sci. Paris, T. 264, 1967, p. 245.
6. P r a d i n e s J. — *Géométrie différentielle au dessus d'un groupoïde.* C. R. Acad. Sci. Paris, T. 266, 1968, 1194—1196.
7. V e r o n a A. — *Introducere în coomologia algebrelor Lie.* Ed. Acad., Bucureşti, 1974.

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