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GRASSMANNIANS OF HIGHER ORDER AND EQUATIONS OF MOVING FRAME

BY

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A problem of canonization of frames on submanifolds was been solved in a lot of particular cases. In the last ten years there appeared appers which contain some methodical treatments of this subject. In order to canonize a field of frames of the first order one needs commonly some contact elements of higher order. Therefore there has been developed a theory of prolongation of Lie groups and of fibre bundles. The use of infinitesimal equations of moving frame fields has been arranged by É. Cartan a long time ago. But only the theory of fibre bundles, especially of jet bundles, allows us to know in what spaces those equations are considered and solved. In the present contribution we consider bundles of contact elements over submanifolds. Generalized Grassmann manifolds appear there as standard fibres.

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We shall use the following notations:

1° $f^{\urcorner}$ will denote a mapping reciprocal to f, while a symbol^{-1} will be used for an inverse in groups.

2° If a smooth mapping, F, is defined in a product $\mathcal{G}\times\mathcal{D}$, $\mathcal{G}$ and $\mathcal{D}$ being real domains, then the symbols $j^r_{t|c}F(t, y)$ (and, respectively $j^r_{y|b}F(t, y)$) denote a holonomical jet of order r of a mapping $t\to F(t, y)$ with a source at the point $c\in\mathcal{G}$ (respectively, a holonomical jet of the mapping $y\to F(t, y)$ with a source at $b\in\mathcal{D}$).

3° $\mathcal{J}^r(A_{|x}, B_{|y})$ is a manifold of jets of order r from a manifold A to a manifold B, with sources at $x\in A$ and targets at $y\in B$:

$$\mathcal{J}^r(A_{|x}, B)=\bigcup_y\{\mathcal{J}^r(A_{|x}, B_y)\mid y\in B\},\quad \mathcal{J}^r(A, B_{|y})=\bigcup_x\{\mathcal{J}(A_{|x}, B_y)\mid x\in A\}.$$

4° $\mathrm{Reg}^r(A_{|x}, B_{|y})$ denotes $\mathcal{J}^r(A_{|x}, B_{|y})$ restricted to regular jets.

5° $\mathbf{R}^k$ will denote the topological vector space of k-tuples of real numbers.

6° L^k_r is a group structure on $\mathrm{Reg}^r(\mathbf{R}^k_{|0}, \mathbf{R}^k_{|0})$, $\mathbf{L}^k_r$ denotes its Lie algebra, 1_r is its unit element.

7° $T^n_p L^n_1$ is the group structure on $\mathcal{J}^r(\mathbf{R}^p_0, L^n_1)$, a group multiplication being defined as follows

$$(j^r_{y|0}\,g(y))\,(j^r_{y|0}\,h(y))=j^r_{y|0}\,(g(y)\,h(y)).$$

8° T assigns to any manifold M its tangent manifold TM; T_uM is a space tangent to M at u.

All the manifolds, mappings, etc. which we have to consider are assumed to be smooth

1. Grassmannians of higher order. We fix three integers, p, n, r, so that $r\geqslant 1$ and $1\leqslant p\leqslant n$. We assume that the indices i, j, k, h vary from 1 to n, a, b, c, g vary from 1 to p and u, v, w do from $p+1$ to n. We consider a manifold of p^r-frames on $\mathbf{R}^n$ at 0, i. e.

$$\mathcal{H}_r^{p,n} := \mathrm{Reg}^r\,(\mathbf{R}^p_{|0},\ \mathbf{R}^n_{|0}).$$

In the case $r = 1$ $\mathcal{H}_1^{p,n}$ is a Stiefel manifold. Thus we call $\mathcal{H}_1^{p,n}$ is a Stiefel manifold of order r. The group L_r^p acts on $\mathcal{H}_r^{p,n}$ by the following rule

$$\text{(1)} \qquad \mathcal{H}_r^{p,n} \times L_r^p \to \mathcal{H}_r^{p,n}, \quad (j_0^r x,\ j_0^r f) \to j_0^r x \circ f.$$

This is an associative right action. Then we define an action of $T_r^p L_1^n$ on $\mathcal{H}_r^{p,n}$ as follows

$$\text{(2)} \qquad T_r^p L_1^n \times \mathcal{H}_r^{p,n} \to \mathcal{H}_r^{p,n}, \quad (j_{s,o}^r \alpha(s),\ j_{s,o}^r x(s)) \to j_{t|0}^{r-1} j_{s|t}^1 (\alpha(t)\, x(s)),$$

where $\alpha(t)\,x(s)$ denotes the result of common matrix multiplication, since $\alpha(t) \in L_1^n$ and $x(s) \in \mathbf{R}^n$. Obviously the action defined by (2) is a left action. We remark that its result depends on $j_0^{r-1}\alpha$ and on the holonomical jet $j_0^r x$.

The group L_r^p acts on $T_r^p L_1^n$ by the following rule

$$\text{(3)} \qquad (j_0^r f, j_0^r \alpha) \to j_0^r \alpha \circ f.$$

This is a transitive right action. Both actions (1) and (3) may be turned into left actions if we replace $j_0^r f$ by $j_0^r f^{-1}$ which is equal to $(j_0^r f)^{-1}$.

Then we define an action of a cartesian (non-direct!) product $L_r^p \times T_r^p L_1^n$ on $\mathcal{H}_r^{p,n}$ as follows

$$\text{(4)} \qquad (<g,\ G>,\ X) \to (Gg^{-1})\,(Xg^{-1}),$$

where $Gg^{-1} \in T_r^p L_1^n$ is a result of the action which was defined by (3), Xg^{-1} is a result of (1), and $(Gg^{-1})(Xg^{-1})$ is computed in accordance with (2). The formula (4) defines an associative left action if we provide the product $L_r^p \times T_r^p L_1^n$ with the following inner composition rule

$$\text{(5)} \qquad <h,\ H><g,\ G> = <hg,\ (Hg)\,G>.$$

An inverse with respect to the above multiplication is given by

$$\text{(6)} \qquad <g,\ G>^{-1} = <g^{-1},\ (Gg^{-1})^{-1}>.$$

The following proposition may be examined directly.

1. Proposition : *The formulas* (5) *and* (6) *define on $L_r^p \times T_r^p L_1^n$ a group structure which is a semi-direct product, $L_r^p \times T_r^p L_1^n$ of the factors.*

We shall denote this group by $L_r^{p,n}$.

2. Proposition : *L_r^p is a Lie subgroup in $L_r^{p,n}$ and $T_r^p L_1^n$ is an invariant Lie subgroup in $L_r^{p,n}$.*

Let us denote by $\mathbf{L}_r^p$, $\mathbf{T}_r^p \mathbf{L}^n$, and by $\mathbf{L}_r^{p,n}$ respectively the Lie algebras of the groups L_r^p, $T_r^p L_1^n$ and of $L_r^{p,n}$. Vectors of the canonical basis of $\mathbf{L}_r^p$ will be denoted by $\mathbf{i}_a^b$, $\mathbf{i}_a^{b_1 b_2}$, ..., $\mathbf{i}_a^{b_1 b_2 \ldots b_r}$ and those in $\mathbf{T}_r^p \mathbf{L}_1^n$ will be denoted by $\mathbf{I}_k^i$, $\mathbf{I}_k^{ib}$, ..., $\mathbf{I}_k^{ib_1, \ldots, b_{r-1}}$ where we assume $b_1 \leqslant b_2 \leqslant \ldots$.

Let us denote by $\Omega^{(r)}$ and, respectively by $\Phi^{(r)}$ the canonical left-invariant forms on L_r^p and on $T_r^p L_1^n$ respectively. Then we write the decompositions

$$\Omega^{(r)} = \mathbf{i}_b^a \otimes \Omega_a^b + \mathbf{i}_b^{a_1 a_2} \otimes \Omega_{a_1 a_2}^b + \ldots + \mathbf{i}_b^{a_1, \ldots, a_r} \otimes \Omega_{a_1, \ldots, a_r}^b,$$

$$\Phi^{(r)} = \mathbf{I}_h^k \otimes \Phi_k^h + \mathbf{I}_h^{kb} \otimes \Phi_{kb}^h + \ldots + \mathbf{I}_h^{kb_1 \ldots b_{r-1}} \otimes \Phi_{kb_1 \ldots b_{r-1}}^h.$$

Now we are going to express an induced action of $\mathbf{L}_r^{p,n}$ on $\mathcal{H}_r^{p,n}$. Let $V = j_{t|0}^1 \gamma_t$ be a vector which is tangent to $L_0^{p,n}$ at the point γ_0. Then the form $\chi^{(r)} := \Omega^{(r)} \otimes \Phi^{(r)}$ sends V to a vector $V_0 := j_{t|0}^1 \gamma_0^{-1}\gamma_t$ which is tangent to $L_r^{p,n}$ at its unit element. Let us take some

$X \in \mathcal{H}_r^{p,n}$. Let $\gamma_0^{-1}\gamma_t X$ denote a result of the action (4) with $\gamma_0^{-1}\gamma_t$ on X. Then the vector $j_{t|0}^1 (\gamma_0^{-1}\gamma_t X) \in T\mathcal{H}_r^{p,n}$ is a result of the tangential action. On the other hand V represents some left-invariant field on $L_r^{p,n}$. Thus we define a linear mapping $X^\wedge \chi^{(r)}: TL_r^{p,n} \to T_x \mathcal{H}_r^{p,n}$ by the following commutative diagram

$$\begin{tikzcd}
T_{\gamma_0} L_r^{p,n} \arrow[rr, "X^\wedge \chi^{(r)}"] \arrow[dr, "\chi^{(r)}"'] & & T_x \mathcal{H}_r^{p,n} \\
& T_1 L_r^{p,n} \arrow[ur, "\text{tangential action}"'] &
\end{tikzcd}$$

Let $\mathbf{e}_i^a, \mathbf{e}_i^{a_1 a_2}, \ldots, \mathbf{e}_i^{a_1 \ldots a_r}$ be a canonical basis in $T\mathcal{H}_r^{p,n}$. We decompose $X^\wedge\chi^{(r)}$ with respect to this basis:

$$X\chi^{(r)} = \sum_{s=1}^{r} \left(X^\wedge \chi^{(r)}\right)^i_{a_1, \ldots a_s} \otimes \mathbf{e}_i^{a_1 \ldots a_s}$$

We introduce the following notations

$$\text{(7)} \qquad \begin{aligned}
\Phi^i_{j\{a_1 \ldots a_s} X^j_{.\}} &:= \sum_{l=1}^{s-1} \sum_{\pi} \Phi^i_{j\, a_{\pi 1} \ldots a_{\pi l}} X^j_{a\pi_{(l+1)} \ldots a_{\pi s}}, \\
X^i_{b\{a_1 \ldots a_s} \Omega^b_{.\}} &:= \sum_{l=1}^{s-1} \sum_{\pi} X^i_{b a_{\pi 1}, \ldots, a_{\pi l}} \Omega_{a\pi_{(l+1)} \ldots a_{\pi s}}
\end{aligned}$$

where $\sum_\pi$ is a sum over all permutations $(1, \ldots, s) \to (\pi 1, \ldots, \pi s)$ such that $a_{\pi 1} < a_{\pi 2} < \ldots < a_{\pi l}$ and $a_{\pi(l+1)} < \ldots < a_{\pi s}$. These and similar notations will be used throughout the paper.

3. Theorem : *The following equalities hold :*

$$\text{(8)} \qquad \begin{aligned}
\left(X^\wedge \chi^{(r)}\right)^i_a &= \Phi^i_j X^j_a - X^j_b \Omega^b_a, \\
\left(X^\wedge \chi^{(r)}\right)^i_{a_1 a_2} &= \Phi^i_j X^j_{a_1 a_2} + \Phi^i_{ja_1} X^j_{a_2} + \Phi^i_{ja_2} X^j_{a_1} - X^j_{ba_1} \Omega^b_{a_2} - X^j_{ba_2} \Omega^b_{a_1} - X^j_b \Omega^b_{a_1 a_2}, \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
\left(X^\wedge \chi^{(r)}\right)^i_{a_1, \ldots, a_r} &= \Phi^i_{j\{a_1 \ldots a_r} X^j_{.\}} - X^i_{b\{a_1 \ldots a_s} \Omega^b_{.\}},
\end{aligned}$$

Proof: Let us fix a vector $V \in T_r L_r^{p,n}$. Then we may write

$\Omega^{(r)}(V) = j_{t|0}^1 j_{x|0}^r [\alpha^1(t, x), \ldots, \alpha^p(t, x)]$, where $\alpha(t, -)$ is a diffeomorfism of a neighbourhood of $O \in \mathbf{R}^p$ and such that $\alpha(0, 0) = O$. Analogously we have

$$\Phi^{(r)}(V) = j_{t|0}^1 j_{x|0}^r [\beta_k^i(t, x)]_{i,k=1, \ldots, n},$$

where $\beta_k^i(0, 0) = \delta_k^i$. Then we have to compute components of

$$j_{t|0}^1 j_{x|0}^{r-1} \beta_k^i(t, x)\, (j_{y|x}^1 X^k \circ \alpha(t, y)).$$

If we simplify computations by changing the order of differentiations, then we obtain formulas (8) where all members are applied to the vector V. In view of the arbitrarity of this vector we obtain our theorem.

Notice that since the group L_r^p is a subgroup of $L_r^{p,n}$, it acts on the Stiefel manifold $\mathcal{H}_r^{p,n}$. Then we may take into consideration the space of orbits in $\mathcal{H}_r^{p,n}$ by the action of L_r^p. In a classical case, $r = 1$, an orbit of a 1-frame in $\mathcal{H}_1^{p,n}$ by the action of L_1^p is nothing but a p-dimensional plane in $\mathbf{R}^n$. Thus the orbit space $\mathcal{H}_1^{p,n} / L_1^p$ is a classical Grassmann space [13]. This gives a reason to state the following definition:

4. Definition: *A Grassmann space of order r is an orbit space $\mathcal{H}_r^{p,n} / L_r^p$. We shall denote this space by $\mathcal{G}_r^{p,n}$.*

In the following we shall briefly call $\mathcal{G}_r^{p,n}$ a Grassmannian. We shall show that $\mathcal{G}_r^{p,n}$ is a homogeneous space, $T_r^p L_1^n$ being its group of motions.

II. Infinitesimal equations in an associated bundle. Let F be a manifold, K be a left action of K on F,

$$\tau : (f, k) \to \tau_k f \in F.$$

Thus there exists an induced mapping, τ^+, which injects the Lie algebra of left-invariant forms on K into a Lie algebra of vector fields on F. In order to express τ and τ^+ in local coordinates we fix a local map $(\mathcal{U}, m)$ on F and a local map (O, μ) on K, so that the unity e of K is in O. Then we write

$$m(\tau_{\mu^{-1}(s)} m^{-1}(x)) = [T^1(x,s), \dots, T^d(s, x)],$$

where $d = \dim F$, $s \in \mu(O)$, $x \in m\mathcal{U}$. Then we consider a matrix $[t_\alpha^\lambda(x)]_{\lambda=1,\dots,d,\ \alpha=1,\dots,k}$ where $k = \dim K$,

$$t_\alpha^\lambda(x) = \frac{\partial}{\partial s^\alpha}\Big|_{\mu(e)} T^\lambda(s, x).$$

Let $\psi = \sum u_\alpha \otimes \psi^\alpha$ be a left-invariant form on K. A mapping μ defines a holonomic field of frames on O. Thus $u^1, \dots, u_k$ are vectors of this field which are tangent to K at e. Respectively, let $m_1, \dots, m_d$ be components of a holonomical field of frames which corresponds to the local map $(\mathcal{U}, m)$. Then the vector-valued form

$$\sum m_\alpha \otimes t_\lambda^\alpha(x)\, \psi^\lambda \tag{9}$$

is a local expression of τ^+.

5. Definition: *A system of equations*

$$dx^\alpha + \sum t_\lambda^\alpha(x)\psi^\lambda = 0, \quad (\alpha = 1, \dots, d), \tag{10}$$

is called a system of infinitesimal equations of invariantness of the homogeneous space (F, K, τ).

6. Example: *Let us go back to formulas (8). As a consequence of theorem 3 the equations*

$$dX_a^i + (X^\wedge \chi^{(r)})_a^i = 0, \quad dX_{a_1a_2}^i + (X^\wedge \chi^{(r)})_{a_1a_2}^i = 0,$$

$$\dots, dX_{a_1\dots a_r}^i + (X^\wedge \chi^{(r)})_{a_1\dots a_r}^i = 0$$

are just infinitesimal equations of the homogeneous space $(\mathcal{H}_r^{p,n}, L_r^{p,n}.)$.

We note that a prime integral of (10) yields always some local expression of the action τ, cf. [1], [2], [3].

7. Proposition: *If we take into consideration a right action, σ, such that $\sigma_k\chi := \tau_{k^{-1}}\chi$, then the system (10) turns into the following $dx^\alpha - \sum t_\lambda^\alpha(x)\psi^\lambda = 0$.*

Let $\mathcal{G}$ be a Lie subgroup in K such that $G := K/\mathcal{G}$ is an invariant Lie subgroup. Denote by G a manifold $F/\mathcal{G}$, i.e. a manifold of orbits in F by the action of $\tau/\mathcal{G}$.

8. Theorem: *If K is isomorphic to a semidirect product $\mathcal{G} \geqslant H$ then the mapping τ induces a left action $\vartheta : H \times G \to G$.*

Proof: Denote by $\tau_{\mathcal{G}}a$ an $\mathcal{G}$-orbit of some point $\in F$. Then we put $\vartheta_h(\tau_{\mathcal{G}}a) := \tau_{\mathcal{G}}(\tau_h a)$ for $h \in H$. In order to show that this definition is correct we must prove that if we take another point, $a' = \tau_j a$, which belongs to $\tau_{\mathcal{G}}a$, we have $\vartheta_h(\tau_{\mathcal{G}}a') = \vartheta_h(\tau_{\mathcal{G}}a)$. In fact,

$$\vartheta_h(\tau_{\mathcal{G}}a') = \tau_{\mathcal{G}}\tau_h a' = \tau_{\mathcal{G}}\tau_{hj} a' = \tau_{\mathcal{G}}\tau_{hjh^{-1}}\tau_h a = \tau_{\mathcal{G}}\tau_h a = \vartheta_h(\tau_{\mathcal{G}} a), \text{ q. e. d.}$$

Now we take into consideration the new homogeneous space (G, H, ϑ) and we have to deduce the corresponding equations of invariantness. Assume that $\dim G = \delta < d$. Suppose that a certain surface defined by the equations

$$\chi^1 = c^1, \dots, \chi^{d-\delta} = c^{d-\delta} \tag{11}$$

has exactly one point of coincidence with each $\mathcal{G}$-orbit in the domain $\mathcal{U}$. If we substitute (11) into (10), we obtain the system of equations

$$\sum_\lambda t_\lambda^z(C^1, \dots, C^{d-\delta}, \xi^1, \dots, \xi^\delta)\psi^\lambda = 0, \quad z = 1, \dots, d-\delta. \tag{12}$$

Assume that this system is of rank δ'. Then it is equivalent to the following

$$\psi^q = \sum_{\mu=1}^{k-\delta'} S_\mu^q(\xi^1, \dots, \xi^\delta)\psi^\mu, \quad q = k - \delta' + 1, \dots, k.$$

If we substitute the above expressions into the last $d - \delta$ equations of the system (10) we obtain the system

$$d\xi^\nu + \sum_{\sigma=1}^{k-\delta'} p_\sigma^\nu(\xi^1, \dots, \xi^\delta)\psi^\sigma = 0.$$

These are equation of invariantness of the homogeneous space (G, H, ϑ). In view of proposition (7) the above treatment may be extended onto the case of a homogeneous space of orbits with respect to a right action of a Lie group $\mathcal{G}^*$ such that there exists an injection $\mathcal{G}^* \to K$.

9. Example: *We consider a homogeneous space $(\mathcal{H}_r^{p,n}, L_r^{p,n}\cdot)$ where $\cdot$ denotes the action described in section I. The equations of invariantness of this space are given by example 6*

Then we are going to obtain a projection of $(\mathcal{H}_r^{p,n}, L_r^n\cdot)$ to another Klein space, $(\mathcal{G}_r^{p,n}\, T_r^p L_1^n, \cdot)$ where $\mathcal{G}_r^{p,n}$ is a generalized Grassmannian introduced in sect. 1 and the action is induced by the action of $L_r^{p,n}$ on $\mathcal{H}_r^{p,n}$ and by projections

$$\mathcal{H}_r^{p,n} \to \mathcal{G}_r^{p,n} := \mathcal{H}_r^{p,n}/L_r^p, \quad L_r^{p,n} \to T_r^p L_1^n = L_r^{p,n}/L_r^p,$$

In order to find the invariant equations of the space $(\mathcal{G}_r^{p,n}\, T_r^p L_1^n, \cdot)$, we write equations of $(\mathcal{H}_r^{p,n}, L_r^{p,n}, \cdot)$. These follow immediately by theorem 3 (see formula 8). We have

$$\begin{gathered} dX_a^b + \sum_c \Phi_c^b X_a^c + \sum_u \Phi_u^b X_a^u - \sum_c X_c^b \Omega_a^c = 0, \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ dX_{a_1\dots a_s}^b + \sum_c \Phi_{c\{a_1\dots a_s}^b X_{\cdot\}}^c + \sum_u \Phi_{u\{a_1\dots a_s}^b X_{\cdot\}}^u - \sum_c X_{c\{a_1\dots a_s}^b \Omega_{\cdot\}}^c = 0; \end{gathered} \tag{13}$$

$$\begin{gathered} dX_a^w + \sum_c \Phi_c^w X_a^c + \sum_u \Phi_u^w X_a^u - \sum_c X_c^w \Omega_a^c = 0, \\ \dots\dots\dots\dots\dots\dots \\ \dots\dots\dots\dots\dots\dots \\ dX_{a_1\dots a_s}^w + \sum_c \Phi_{c\{a_1\dots a_s}^w X_{\cdot\}}^c + \sum_u \Phi_{u\{a_1\dots a_s}^w X_{\cdot\}}^u - \sum_c X_{c\{a_1\dots a_s}^w \Omega_{\cdot\}}^c = 0. \end{gathered} \tag{14}$$

We remind our previous convention: indices a, b, c vary from 1 to p; u, w vary from $p+1$ to n. Then we restrict ourselves to a subset in $\mathcal{H}_r^{p,n}$ containing these L_r^p - orbits which pass through a point with coordinates

$$X_c^b = \delta_c^b, \quad X_{c_1c_2}^b = 0, \dots, X_{c_1c_s}^b = 0. \tag{15}$$

If we substitute above quantities into (13), the following relations follow:

$$\Omega_a^c = \Phi_a^c + \Phi_u^c X_a^u, \dots, \Omega_{a_1\dots a_s}^c = \Phi_{a_1\dots a_s}^c + \Phi_{u\{a_1\dots a_s}^b X_{.\}}^u . \tag{16}$$

Formulas (16) allow to express $\mathbf{L}_r^p$ -components by $\mathbf{T}_r^p\, \mathbf{L}_1^n$-components in the action of $L_r^p \,|\!\times\, T_r^p\, L_1^n$ on the orbit space. The numbers $X_a^w, X_{a_1a_2}^w, \dots, X_{a_1\dots a_r}^w$ may be considered as local coordinates of an orbit, i. e. of a point of a Grassmannian of order r. Of course, not every orbit passes through a point with coordinates (15). But every orbit passes through a point with coordinates like (15), such that the indices a, b, c vary in a convenable subset $\{g_1, \dots, g_p\} \subset \{1, \dots, n\}$ where $g_1 < g_2 < \dots < g_p$. As $\{1, \dots, n\}$ contains $C_n^p = n!/[p!(n-p)!]$ of such subsets, we see that the orbit space $\mathcal{G}_r^{p,n}$ may be covered in the whole with C_n^p charts. Coordinates which are determined by these charts are of the form $X_a^w, \dots, X_{a_1\dots a_r}^w$ where w varies in the set $\{1, \dots, n\} \setminus \{g_1, \dots, g_p\}$.

In order to obtain the infinitesimal equations of invariantness of a Grassmann space we have to substitute (16) into (14). After performing some computations we obtain:

$$\begin{gathered}
dX_a^w + \Phi_a^w + \Phi_u^w X_a^u - X_c^w \Phi_a^c - X_c^w X_a^u \Phi_u^c = 0, \\
dX_{a_1a_2}^w + \Phi_{a_1a_2}^w + \Phi_{ua_1}^w X_{a_1}^u + \Phi_{ua_2}^w X_{a_1}^u + \Phi_u^w X_{a_1a_2}^u - X_{ca_1}^w \Phi_{a_2}^c - X_{ca_2}^w \Phi_{a_1}^c - X_c^w \Phi_{a_1a_2}^c - X_c^w X_{a_1}^u \Phi_{ua_2}^c - \\
- X_c^w X_{a_2}^u \Phi_{ua_1}^c - X_{ca_1}^w X_{a_2}^u \Phi_u^c - X_{ca_2}^w X_{a_1}^u \Phi_u^c - X_c^w X_{a_1a_2}^u \Phi_u^c = 0, \\
\dots\dots\dots\dots\dots\dots\dots\dots \\
\dots\dots\dots\dots\dots\dots\dots\dots \\
dX_{a_1\dots a_r}^w + \Phi_{a_1\dots a_r}^w + \Phi_{u\{a_1\dots a_r}^w X_{.\}}^u - X_{c\{a_1\dots a_r}^w \Phi_{.\}}^c - X_{c\{a_1\dots a_r}^w X_{.}^u \Phi_{|u|\}}^c = 0.
\end{gathered} \tag{17}$$

Then we are able to formulate the following result:

10. Theorem: *The generalized Grassmannian is a basic manifold of a Klein space with the structure group $T_r^p\, L_1^n$. Equations* (17) *are the corresponding invariance equations.*

Let us turn to a general situation. Consider a homogeneous space (F, K, τ). Let (P, M, K) be a principal fibre bundle over some manifold M. P is just a bundle manifold and K is a structure group. We fix a point $a \in P$ and a local cross-section σ by a. Let $\mathbf{x}$ be a vector tangent to σ at a. Then we decompose $\mathbf{x}$ in two components $\mathbf{x}^v$ and $\mathbf{x}^h$ so that $\mathbf{x}^v$ is tangent to the fibre and $\mathbf{x}^h$ is tangent to the cross-section σ. This decomposition depends essentially on the jet $j^1\sigma$. A manifold of 1-jets of all such cross-sections is a bundle over M. This bundle is called a first *regular prolongation* of (P, M, K). We denote by $\Theta(\mathbf{X})$ the vector in $\mathbf{K}$ which generates $\mathbf{x}^v$. If we let a, $j^1\sigma$ and $\mathbf{x}$ vary in the prolongated bundle, (P_1, M, K_1), then we define a field of $\mathbf{K}$-valued linear forms in P_1 [16]. This is called a field of canonical form. Its components satisy the identities

$$d\theta^\lambda + \sum C_{\mu\nu}^\lambda\, \theta^\mu \wedge \theta^\lambda + \sum \theta_\alpha^u \wedge \omega^\alpha = 0, \tag{18}$$

where λ, μ, ν vary from 1 to $\dim K$, α varies from 1 to $\dim M$ and θ_α^μ are components of the canonical form on a second prolongation of P, ω^α are components of the co-frame at the projection of a, (cf. [6], [16], [25]).

Let us take into consideration a product $P \times F$. Thus the group K acts on it by the rule

$$(k, (p, f)) \to (pk^{-1}, \tau_k f) \in P \times F,$$

where pk^{-1} denotes a result of the canonical action of $k^{-1} \in K$ on $p \in P$. Let $\mathcal{A}$ denote the orbit space by this action. Thus $\mathcal{A}$ is a bundle manifold of a fibre bundle which is associated with (P, M, K). We denote this bundle by $(\mathcal{A}, M, P, (F, K, \tau))$. The following proposition is due to R. Crittenden:

There exists a $1-1$ correspondence between cross-sections $M \to \mathcal{A}$ and the mappings $\mathfrak{X}: P \to F$ which satisfy the identity $\tau_k \mathfrak{X}(p) = \mathfrak{X}(pk^{-1})$.

A proof follows almost directly by the definition of the associated bundle. Thus a mapping $\rho: M \to \mathcal{A}$ defining a cross-section which corrresponds to $\mathfrak{X}$ is given by ρ(projection of p) = orbit of $(p, \mathfrak{X}(p))$.

In what follows we fix some mapping $\mathfrak{X}$ and we shall derive certain infinitesimal relations. Let $t \to \gamma(t)$ be a parametrization of a curve in P. Thus $j_0^1\gamma =: \mathbf{x}$ is a vector tangent to P at some point a. We have

$$j_0^1 \mathfrak{X} \circ \gamma = (d\mathfrak{X}(a))\, \mathbf{x} = (d\mathfrak{X}(a))\mathbf{x}^v + (d\mathfrak{X}(a))\, \mathbf{x}^h.$$

Then we check a parametrization γ^v which corresponds to the vertical component $\mathbf{x}^v$, that is $\mathbf{x}^v = j_0^1 \gamma^v$. We have

$$(d\mathfrak{X})\mathbf{x}^v = j_0^1 \mathfrak{X} \circ \gamma^v = j_{t;0}^1\, \mathfrak{X}(\gamma_0 k_t)$$

where $t \to k_t$ describes a convenable curve in K such that $j_{t;0}^1 k_t = \theta(\mathbf{x}^v)$. In view of (19) we obtain

$$(d\mathfrak{X})\mathbf{x}^v = -\overset{+}{\tau}_{\theta(\mathbf{x}^v)}\, \mathfrak{X}(a).$$

By summing we have

$$(d\mathfrak{X})_{|a}\, \mathbf{x} + \overset{+}{\tau}_{\theta(\mathbf{x}^v)}\, \mathfrak{X}(a) = (d\mathfrak{X})_{|a}\, \mathbf{x}^h . \tag{20}$$

We have an equivalent formula in local coordinates which has been introduced above, viz.

$$d\xi^A + t_\lambda^A(\xi)\, \theta^\lambda = \xi_\alpha^A\, \omega^\alpha . \tag{21}$$

Thus (ξ^A, ξ_α^A), $A = 1, \dots, \dim F$, $\alpha = 1, \dots, \dim M$ are local components of the jet $j^1\mathfrak{X} \circ \sigma$. The equalities (21) are sometimes called infinitesimal equations of a field of a geometric object, [18], [19], [23]. Notice that the left hand member in (21) may be obtained directly from (10) by formaly substituting θ^λ instead of ψ^λ and by adjoining the right hand member $\xi_\alpha^A\, \omega^\alpha$.

11. Example: *Consider the manifold* $H_r^d := \mathrm{Reg}^r(\mathbf{R}_{|0}^d, M)$ *where* $d = \dim M$. There exists a natural bundle structure over M, with L_r^d structural group, [15]. In particular, if $r = 1$, then H_1^d is a common bundle of linear frames. In order to construct some associated bundle, we take $\mathbf{R}^d$ to be a standard fibre and we assume a canonical left action of L_1^d on $\mathbf{R}^d$. Then the corresponding space of orbits in $(H_1^d, \mathbf{R}^d)$ is a vector bundle TM over M. A vector field on M may be viewed as a mapping $\tilde{\omega}: H_1^d \to \mathbf{R}^d$ such that

$$\tilde{\omega}((j_0^1 s)(j_0^1 \sigma)^{-1}) = (j_0 \sigma)\, \tilde{\omega}(j_0^1 s),$$

where $j_0^1 s$ is a linear frame and $j_0^1 \sigma$ is an element of L_1^d. Then we consider the r-th prolongation of $\tilde{\omega}$ and of TM. We put $\mathbf{R}_r^d := \mathcal{J}^n(\mathbf{R}_{|0}^d\, \mathbf{R}^d)$ and we consider a group action

$$\overset{r}{\tau}: L_r^d \times \mathbf{R}_r^d \to \mathbf{R}_r^d, \quad (j_0^r \sigma, j_0^r w) \to j_0^{r-1}((j_0^1 \sigma). w).$$

Since $j_0^1 \sigma \in L_1^n$ and $w \in \mathbf{R}^d$, the result $(j_0^1 \sigma)w$ of a canonical action is in $\mathbf{R}^d$. The orbits of the action

$$L_{r+1}^d \times (H_{r+1}^d \times \mathbf{R}_r^d) \to H_{r+1}^d \times \mathbf{R}_r^d, \; (j_0^{r+1} \sigma, (j_0^{r+1} s, j_0^r w)) \to (j_0^{r+1} s \circ \sigma, \overset{r}{\tau}_{j_0^r \sigma}\, j_0^r w)$$

form a new bundle, $T_r M$, which is associated with the principal one $(H_{r+1}^d, M, L_{r+1}^d)$. If we denote the local components of some cross-section $M \to T_r M$ by $v^\alpha, v_\beta^\alpha, \dots, v_{\beta_1, \dots, \beta_r}^\alpha$, then the infinitesimal equations of the corresponding field are

$$dv^\alpha + v^\gamma \omega_\gamma^\alpha = v_\gamma^\alpha \omega^\gamma, \qquad dv_\beta^\alpha + v_\beta^\gamma \omega_\gamma^\alpha + v^\gamma \omega_{\gamma\beta}^\alpha - v_\gamma^\alpha \omega_\beta^\gamma = v_{\beta\gamma}^\alpha \omega^\gamma,$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$dv_{\beta_1 \dots \beta_r}^\alpha + v^\gamma_{\{\beta_1 \dots \beta_r} \omega^\alpha_{|\gamma|\}} - v^\alpha_{\gamma\{\beta_1 \dots \beta_r} \omega^\gamma_{\}} = v^\alpha_{\beta_1 \dots \beta_\gamma} \omega^\gamma$$

where $\omega^\alpha, \omega_\beta^\alpha, \dots, \omega^\alpha_{\beta_1 \dots \beta_\gamma}$ are components of the canonical form in the bundle H_{r+1}^d (cf. [1], [5], [15], [21], [24]). These satisfy the well known differential identities

$$(22) \qquad \begin{gathered} d\omega^\alpha + \omega_\gamma^\alpha \wedge \omega^\gamma = 0, \qquad d\omega_\beta^\alpha + \omega_\gamma^\alpha \wedge \omega_\beta^\gamma + \omega_{\beta_\gamma}^\alpha \wedge \omega^\gamma = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ d\omega^\alpha_{\beta_1 \dots \beta_\gamma} + \omega^\alpha_{\gamma\{\beta_1 \dots \beta_\gamma} \wedge \omega^\gamma_{\}} = 0, \end{gathered}$$

which are called *structures equations*.

III. Infinitesimal equations of moving frame. Let $\mathcal{W}$ be an n-dimensional manifold and $\mathcal{V}$ a p-dimensional submanifold. Let us put

$$Q = \mathrm{Reg}^1 (\mathbf{R}^n /_0, \mathcal{W}) \mid \{j_0^1 s / s(0) \in \mathcal{V}\}.$$

Obviously Q is a fibered manifold. More adequately Q is a manifold of principal bundle over $\mathcal{V}$ and L_1^n is here a structure group. Simultaneously we consider another bundle of r-frames, namely

$$P_r := \mathrm{Reg}^1 (\mathbf{R}^p /_0, \mathcal{V}).$$

Obviously (P_r, V, L_r^p) is a principal bundle. Then we define the following prolongation Q_r of Q:

$Q_r := \{j_m^{r-1} c \circ p \mid c$ is a local cross-section of $\mathcal{V}$ to Q, $j_m^r\; p \in P_r$, m varies on $\mathcal{V}\}$.

We see that Q_r is a manifold of a principal bundle with a structure group $T_r^P L_1^n$.

Then we define a right action of the group $L_r^{p,n}$ on the product $P_r \times Q_r$ as follows:

$$(23) \qquad \begin{gathered} < j_0^r f, j_{y|0}^{r-1} j_{x|0}^1 F(x, f(y)) > < j_0^r \gamma, j_{y|0}^{r-1} j_{x|0}^1 \Gamma(x, y) > \\ := < j_0^r f \circ \gamma, j_{y|0}^{r-1} [(j_{x|0}^1 F(x, f \circ \gamma(y))) \cdot (j_{x|0}^1 \Gamma(x, y))] \end{gathered}$$

where $j^r f \in \mathrm{Reg}^r (\mathbf{R}^p_{|0}, \mathcal{V}_{|f(0)})$, $j_{x|0}^1 F(x, q) \in \mathrm{Reg}(\mathbf{R}^n_{|0}, \mathcal{W}_q)$ and the second factor in the left hand member is an element of $L_r^{p,n}$. It is easy to see that $P_r \times Q_r$ is a manifold of a certain principal bundle over $\mathcal{V}$ and the structure group is just $L_r^{p,n}$.

Now we specialize a construction of the canonical form to our bundle $(P_r \times Q_r, \mathcal{V}, L_r^{p,n})$. We fix a vector $\mathbf{X}$ which is tangent to $P_r \times Q_r$ at the point u. Let $\mathbf{X}_0$ (resp. u_0) be the image of $\mathbf{X}$ (resp. of u) by the canonical projection $P_r \times Q_r \to P_{r-1} \times Q_{r-1}$. There exists a local cross-section of $P_r \times Q_r$ by u_0 such that a first order jet of this section is just equal to u. Let h denote a subspace in $T_{u_0} P_{r-1} \times Q_{r-1}$ which is tangent to the above mentionned

section at the point u_0. Then we decompose $\mathbf{X}_0$ into its vertical component $\mathbf{X}^v$ and its horizontal component $\mathbf{X}^h$ so that $\mathbf{X}^v$ is tangent to the fibre and $\mathbf{X}^h$ is tangent to h. Note that this decomposition depends essentially on u. Then we define a value of the canonical form on $\mathbf{X}$ to be a pair of vectors. The first of them, $\omega_u^{(r)}(\mathbf{X})$, is the value of a canonical form on the bundle $(P_r, \mathcal{V}, L_r^p)$ and the second one is the $T_r^p \mathbf{L}_1^n$-component of this element of $L_r^{p,n}$ which generates the vertical component $\mathbf{X}^v$ of $\mathbf{X}^v$. We denote this second component by $\zeta_u^{(r)}$. Then we have the following decomposition

$$(24) \qquad \begin{gathered} \omega^{(r)} = I_a \otimes \omega^a + I_a^b \otimes \omega_b^a + \dots + I_a^{b_1 \dots b_{r-1}} \otimes \omega^a_{b_1 \dots q_{r-1}}, \\ \zeta^{(r)} = \mathbf{I}_k^i \otimes \zeta_i^k + \mathbf{I}^{\cdot\cdot} \otimes \zeta_{ib}^k + \dots\dots + \mathbf{I}_k^{ib_1 \dots b_{r-2}} \otimes \zeta^k_{ib_1 \dots b_{r-2}}. \end{gathered}$$

The basic vectors $\mathbf{I}_a$ constitute the canonical basis in $\mathbf{R}^n$, and $\mathbf{I}_a^b$, $\mathbf{I}_a^{b_1 b_2}$, ... have been introduced in section II. (compare with formulas (7)). We remind that the components of $\omega^{(r)}$ satisfy the differential identities (22).

14. Theorem : *The* $\mathbf{T}_r^p \mathbf{L}_1^n$*-components of the canonical form on* $P_r \times Q_r$ *satisfy the following differential identities;*

$$d\zeta_i^k + \zeta_j^k \wedge \zeta_i^j + \zeta_{ia}^k \wedge \omega^a = 0,$$

$$d\zeta_{ia}^k + \zeta_{ja}^k \wedge \zeta_i^j + \zeta_j^k \wedge \zeta_{ia}^j + \zeta_{jb}^k \wedge \omega_a^b + \zeta_{jab}^k \wedge \omega^b = 0,$$

$$d\zeta^k_{ia_1 \dots a_s} + \zeta^k_{j\{a_1 \dots a_s} \wedge \zeta^j_{|i|\}} + \zeta^k_{jb\{a_1 \dots a_s} \omega^b_{\}} = 0, \quad s = 1, \dots, r-2.$$

For a proof, cf. [16]. Other treatments are in [6] and [22].

The principal fibre bundle $(P_r \times Q_r, \mathcal{V}, L_r^{p,n})$ will be denoted by $\mathcal{P}_r$. Observe that $\mathcal{P}_r$ is uniquely determined by an immersion : $\iota : \mathcal{V} \to \mathcal{W}$.

15. Lemma : *Let* ξ_m *denote the r-jet of the immersion of* $\mathcal{V}$ *at the point m. Then for each element* $w \in P_r \mid$ fibre over m, *there exists a mapping* w^0 *which sends* ξ_m *to a point* $w^0 \xi_m \in \mathcal{H}_{r-1}^{p,n}$. *Moreover, if* $z = wg$ *for some* $g \in L_r^{p,n}$ *(see formula (23)) we have* $z \cdot \xi_m = g^{-1}(w \cdot \xi)$ *(see (2) and (6)).*

Proof : If we assume

$$w = < j_0^r f, j_{y|0}^r j_{x|0}^1 F(x, f(y)) >$$

and $\xi_m = j_m^r \iota$ then we set

$$w \cdot \xi_m := j_{y|0}^{r-1} ((j_{x|0}^1 F(x, f(y)))^{-1} (\iota \circ f(y))),$$

where $(j_{x|0}^1 F(x, q))^{-1}$ is a 1-jet reciprocal to $j_{x|0}^1 F(x, q)$ which maps the linear space $T_q W$ into $\mathbf{R}^n$. q varies in some neighbourhood of m in $\mathcal{V}$. The right hand member of (25) is an element of the Stiefel manifold $\mathcal{H}_{r-1}^{p,n}$. The proof of the second statement is a matter of brief calculation by means of formulas (23) and (25). In fact, we have

$$< j_0^r f \circ \gamma, j_{y|0}^{r-1} (j_{x|0}^1 F(x, f \circ \gamma(y))) (j_{x|0}^1 \Gamma(x, y)) > \cdot j_m^r \iota =$$

$$= j_{y|0}^{r-1} ((j_{x|0}^1 \Gamma(x, y))^{-1} (j_{x|0}^1 F(x, f \circ \gamma(y))^{-1} (\iota \circ f \circ \gamma)) =$$

$$= < j_{y|0}^r \gamma^{-1}(y), j_{y|0}^{r-1} (j_{x|0}^1 \Gamma(x, \gamma^{-1}(y)))^{-1} > \times j_{x|0}^r ((j_{x|0}^1 F(x, f(y)))^{-1} (\iota \circ f(y))).$$

The above equalities prove the second statement of the lemma.

The following proposition is a consequence of considerations in section II and lemma 15.

16. Proposition : *The field* $m \to \xi_m$ *of* $(r-1)$ *jets of the immersion* $\mathcal{V} \to \mathcal{W}$ *is a cross-section in a bundle* $(\mathcal{A}_r, \mathcal{V}, P_r, (\mathcal{H}_{r-1}^{p,n}, L_{r-1}^{p,n}))$ *which is associated with* P_r.

Let us consider a class A_m of local automorphisms of $\mathcal{V}$ which send a point m to itself. Let w and ξ have the same meaning as above and ρ be an r-jet of an element $a \in A_m$. As a consequence of (14), we have $w(\xi\rho) = \tilde{\rho}^{-1}(w \cdot \xi)$, where $\tilde{\rho} = j_0^r(f^{-1} \circ a \circ f) \in L_r^p$. Definition 4 and lemma 15, allow to state the following proposition.

17. Proposition : *To each frame w at m there corresponds a mapping $\tilde{w}$ which sends the orbit*

$$\{\xi_m \rho \mid \rho = j_m^r a, \ a \in A_m\} \tag{26}$$

to the element $\{w^e(\xi_m \rho g) \mid g \in L_r^p\}$ of the Grassmann manifold $\mathcal{G}_r^{p,n}$.

The orbit (26) is called a contact element of order r, cf. [17]. It characterizes the inclusion of $\mathcal{V}$ in $\mathcal{W}$ independently of a choice of frames and of the immersion. Thus the coordinates of this point in $\mathcal{G}_{r-1}^{p,n}$ which is an image by $\tilde{\omega}$ of a contact element relative to coordinates of Ξ. Then proposition 17 implies the following :

18. Theorem : *Let $\mathcal{V}$ be a submanifold of $\mathcal{W}$. Then for any $r \geqslant 1$ there exists a bundle $\mathcal{B}_r$ of contact elements of order r over $\mathcal{V}$. The structural group of this bundle is $T_r^p L_1^n$ and the standard fibre is the generalized Grassmann manifold $\mathcal{G}_r^{p,n}$.*

19. Theorem : *Any cross-section in the bundle $\mathcal{B}_r$ of contact elements satisfies the following infinitesimal equations*

$$\begin{aligned} & d\Lambda_a^v + \Lambda_a^u \zeta_u^v + \zeta_a^v - \Lambda_b^v \zeta_a^b - \Lambda_a^u \Lambda_b^v \zeta_u^b = \Lambda_{ab}^v \omega^b, \\ & d\Lambda_{a_1a_2}^v + \Lambda_{a_1a_2}^u \zeta_u^v + \Lambda_{a_1}^u \zeta_{ua_2}^v + \Lambda_{a_2}^u \zeta_{ua_1}^v + \zeta_{a_1a_2}^v - \\ & - \Lambda_b^v \zeta_{a_1a_2}^b - \Lambda_{ba_1}^v \zeta_{a_2}^b - \Lambda_{ba_2}^v \zeta_{a_2}^b - \Lambda_{a_1a_2}^u \Lambda_b^v \zeta_u^b - \Lambda_{a_1}^u \Lambda_{ba_2}^v \zeta_u^b \\ & - \Lambda_{a_2}^u \Lambda_{ba_1}^v \zeta_u^b - \Lambda_{a_2}^u \Lambda_b^v \zeta_{ua_1}^b - \Lambda_{a_1}^u \Lambda_b^v \zeta_{ua_2}^b = \Lambda_{a_1a_2b}^v \omega^b, \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & \cdots\cdots\cdots\cdots\cdots\cdots \\ & d\Lambda_{a_1\ldots a_s}^v + \Lambda_{\{a_1\ldots a_s}^u \zeta_{|u|\}}^v + \zeta_{a_1\ldots a_s}^v - \Lambda_{b\{a_1\ldots a_s}^v \zeta_{\cdot\}}^b - \Lambda_{b\{a_1\ldots a_s}^v \Lambda_{\cdot}^u \zeta_{|u|\}}^b = \Lambda_{a_1\ldots a_s b}^v \omega^b, \end{aligned} \tag{27}$$

where $s \leqslant r - 1$, $\Lambda_{a,\ldots,}^v \Lambda_{a_1,\ldots,a_s}^v$ are local components on the standard fibre $\mathcal{G}_r^{p,n}$ of $\mathcal{B}_r$ and the forms $\zeta_{a\ldots,}^b$ have the same meaning as in theorem 14.

Proof : The theorem follows from theorem 18 and the general formula (21) for cross-sections. In order to obtain equations (27), we have to substitute in the left hand members of (21), the left hand mambers of the invariance equations of $\mathcal{G}_{r-1}^{p,n}$ and zeros in the right hand of (21) are to be changed into horizontal forms $\Lambda_{ab}^v \omega^b, \ldots, \Lambda_{a_1\ldots ar-1b}^v \omega^b$ respectively.

Equations (27) are also called *equations of moving frame* of E. Cartan. They play an essential role in a process of canonization of frame fields on submanifolds. The theory is exposed in papers [8]—[12], [14] [19]. In concrete situations there are used rather direct methods of finding invariants. Advanced examples are to be found in [7]. Paper [27] contains a clear and exhausting exposition of this method.

IV. Final remarks. In this section we compare the above deducing of equations of moving frame to a method of direct obtaining these equations. This is founded on the following lemma.

Lemma of Cartan and Laptev : Let be given two matrices of exterior forms $[\alpha_\nu]$ and $[A_\nu]$, $\deg \alpha_\nu = 1$, $\deg A_\nu = q$, $\nu = 1, \ldots, p$. If $\sum_{\nu=1}^{p} A_\nu \wedge \alpha_\nu = 0$ holds then there exist $(q-1)$ forms $\gamma_{\mu\nu}$ $\mu, \nu = 1, \ldots, p$ such that

$$1^\circ \ \gamma_{\mu\nu} = \gamma_{\nu\mu}, \quad 2^\circ \ A_\nu = \sum_{\mu=1}^{p} \gamma_{\nu\mu} \wedge \alpha_\mu,$$

3° if $[\gamma_{\nu\mu}]$ and $[\tilde{\gamma}_{\nu\mu}]$ satisfy 2° then there exists a matrix of $(p-2)$-forms, $[\beta_{\mu\nu\lambda}]$ such that $\tilde{\gamma}_{\nu\mu} - \gamma_{\nu\mu} = \sum_\lambda \beta_{\nu\mu\lambda} \wedge \alpha^\lambda$ and $[\beta_{\nu\mu\lambda}]$ is symmetric with respect to lower indices.

Cf. [3]. An extension and an algebraic treatment of this lemma is given in [20].

Let the manifolds $\mathcal{V}$ and $\mathcal{W}$ be like in the above section. Let ω^a (and, respectively, $\overline{\omega}^i$) be components of coframes on $\mathcal{V}$ (respectively, on $\mathcal{W}$ in points of $\mathcal{V}$). Then the Pfaff system of the immersion $\mathcal{V} \to \mathcal{W}$ is the following

$$X_a^i \omega^a - \overline{\omega}^i = 0. \tag{28}$$

We perform an exterior differentiation. By using structure equations

$$d\omega^a = -\omega_b^a \wedge \omega^b, \quad d\overline{\omega}^i = -\omega_j^i \wedge \omega^j. \tag{29}$$

we obtain the following equalities

$$(dX_a^i + X_a^j \overline{\omega}_j^i - X_b^i \omega_a^b) \wedge \omega^a = 0.$$

We apply Cartan-Laptev's lemma and we conclude that there exists a matrix field $[X_{a_1a_2}^i]$, such that $X_{a_1a_2}^i = X_{a_2a_1}^i$ and

$$dX_a^i + X_a^j \overline{\omega}_j^i - X_b^i \omega_a^b = X_{ab}^i \omega^b. \tag{30}$$

Then we return to considerations in sections III. We conclude that (30) is a system of infinitesimal equations of a cross-section in a bundle over $\mathcal{V}$ with a Stiefel manifold as a standard fibre. In order to project this bundle onto a bundle with a Grassmannian fibre, we have to restrict ourselves to co-frames $[\overline{\omega}^i]$ such that the first p components $\overline{\omega}^a$ constitute a co-frame on $\mathcal{V}$ and $\overline{\omega}^v (v = p+1, \ldots, n)$ are transversal with respect to $\mathcal{V}$. Thus we have to write ζ^v instead of $\overline{\omega}^v$ and $\overline{\omega}^a = \omega^a$. Then the system (28) takes the form

$$\Lambda_a^v \omega^a = \zeta^v \tag{31}$$

and the structure equations (29) are to be replaced by

$$\begin{aligned} d\zeta^v &= -(\zeta_u^v \Lambda_a^u + \zeta_a^v) \wedge \omega^a, \\ d\omega^a &= -\zeta_b^a \wedge \omega^b - \zeta_u^a \Lambda_b^u \wedge \omega^b, \end{aligned} \tag{32}$$

where ζ^v, ζ_a^v are components of the canonical form on a principal bundle over $\mathcal{V}$ with $T_2^p L_n^1$ as its group. By an exterior differentiation of (31) and using (32) and Cartan-Laptev's lemma, we obtain infinitesimal equations of the form

$$d\Lambda_a^v + \Lambda_a^u \zeta_u^v + \zeta_a^v - \Lambda_b^v \zeta_a^b - \Lambda_a^u \Lambda_b^v \zeta_u^b = \Lambda_{ab}^v \omega^b.$$

These are equations of a cross-section of a bundle with a Grassmannian $\zeta_1^{p,n}$ as a standard fibre. In order to obtain infinitesimal equations of higher order, like (17), we have to differentiate equations (33) and to use the structure equations of higher order (see theorem 14). We shall not deal with details of computation which are to be found in many papers, e. g. [3] and, recently, a most part of papers published in Trudy Geometricheskogo Seminara (see References). A purpose of our above considerations was to derive the equations of moving frame by using suitable bundles of higher order. Grassmann spaces of higher order appear there naturally as standard fibres. We considered only the case when full linear group is the original structure group. A treatment of the moving frame in G-strucures is to be found in quoted papers of Gh. Gheorghiev.

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ON ALMOST TANGENT STRUCTURES OF ORDER k

BY

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1. Introduction. Almost tangent structures of order k were introduced by Eliopoulos. [13] (**), and they are a generalization of the almost tangent ones (Clark and Bruckheimer [5]). Catz [7], [8] obtained some results, and Clark and Goel [10] deal with the case $k = 2$. In this article we get some results on such structures. Concretely, we give a definition equivalent to Catz's, and some examples, for instance the existence of an integrable almost tangent structure of order $k-1$ on the torus $S^1 \times \cdots \times S^1$ (k factors). §3 studies the general problem of existence, and we get a theorem on the existence of associated structures in Bernard's way [1].

In §4 we approach the obstruction to the existence of the almost tangent structures of order k on a certain kind of manifolds by using structure connections and Chern, Pontrjagin and Stiefel-Whitney classes; showing how a chain of foliations $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{k-1}$, and a theorem analogous to Bott's relative to the foliations, are given in this situation. § 5 studies the almost tangent structures of order k, integrable case; and at last, some results on prolongations ot the k-jets fiber bundles are given in § 6.

2. Definition and examples. Definition 2.1. *Let M be a $(k+1)$ n-dimensional differentiable manifold ($k \in \mathbb{N}$). A G-structure with group G of matrices of the form*

$$\left\| \begin{matrix} A_1 & 0 & \dots & 0 \\ A_2 & A_1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_{k+1} & A_k & \dots & A_1 \end{matrix} \right\|, \quad A_1 \in Gl(n, \mathbb{R}) \tag{2.1}$$

will be said an almost tangent structure of order k.

We will say, M is an *almost tangent manifold of order k* if such a structure is defined on M.

Let $P(M, G)$ be the adapted frame fiber bundle. We define a (1, 1) tensor field, such that its components are

$$J = \left\| \begin{matrix} 0 & 0 & \dots & 0 & 0 \\ I_n & 0 & \dots & 0 & 0 \\ 0 & I_n & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I_n & 0 \end{matrix} \right\|$$

relative to any adapted frame of M. Any local cross-section is an adapted moving frame on M. It can be identified with a set of independent vector fields $X_1, \dots, X_{(k+1)n}$, such that

$$JX_{a+kn} = 0, \quad JX_{a+in} = X_{a+(i+1)n} \quad i = 0, \dots, k-1; \quad a = 1, \dots, n.$$

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**) Numbers in brackets refer to the references at the end of the paper.