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ON ALMOST TANGENT STRUCTURES OF ORDER k

BY

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1. Introduction. Almost tangent structures of order k were introduced by Eliopoulos. [13] (**), and they are a generalization of the almost tangent ones (Clark and Bruckheimer [6]). Catz [7], [8] obtained some results, and Clark and Goel [10] deal with the case $k = 2$. In this article we get some results on such structures. Concretely, we give a definition equivalent to Catz's, and some examples, for instance the existence of an integrable almost tangent structure of order $k - 1$ on the torus $S^1\times \cdots \times S^1$ (k factors). §3 studies the general problem of existence, and we get a theorem on the existence of associated structures in Bernard's way [1].

In §4 we approach the obstruction to the existence of the almost tangent structures of order k on a certain kind of manifolds by using structure connections and Chern, Pontrjagin and Stiefel-Whitney classes; showing how a chain of foliations $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_{k-1}$, and a theorem analogous to Bott's relative to the foliations, are given in this situation. § 5 studies the almost tangent structures of order k, integrable case; and at last, some results on prolongations ot the k-jets fiber bundles are given in § 6.

2. Definition and examples. Definition 2.1. *Let M be a $(k + 1)$ n-dimensional differentiable manifold ($k \in \mathbb{N}$). A G-structure with group G of matrices of the form*

$$\left\| \begin{matrix} A_1 & 0 & \dots & 0 \\ A_2 & A_1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_{k+1} & A_k & \dots & A_1 \end{matrix} \right\|, \quad A_1 \in Gl(n, \mathbb{R}) \tag{2.1}$$

will be said an almost tangent structure of order k.

We will say, M is an *almost tangent manifold of order k* if such a structure is defined on M.

Let $P(M, G)$ be the adapted frame fiber bundle. We define a (1, 1) tensor field, such that its components are

$$J = \left\| \begin{matrix} 0 & 0 & \dots & 0 & 0 \\ I_n & 0 & \dots & 0 & 0 \\ 0 & I_n & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I_n & 0 \end{matrix} \right\|$$

relative to any adapted frame of M. Any local cross-section is an adapted moving frame on M. It can be identified with a set of independent vector fields $X_1, \dots, X_{(k+1)n}$, such that

$$JX_{a+kn} = 0, \quad JX_{a+in} = X_{a+(i+1)n} \quad i = 0, \dots, k - 1; \quad a = 1, \dots, n.$$

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**) Numbers in brackets refer to the references at the end of the paper.

The (1, 1) tensor field J has constant rank kn and it satisfies the equation $J^{k+1} = 0$. In this sense, we see that a field, so defined, determines an almost tangent structure of order k on M.

It has been proved (see Eliopoulos [13], Catz [7]) that this condition is also sufficient, in the following sense

Theorem 2.2. *A necessary and sufficient condition for a manifold M to admit an almost tangent structure of order k, is the existence of a (1, 1) tensor field with constant rank kn, where $\dim M = (k+1)n$, and such that $J^{k+1} = 0$.*

Example 2.3. The natural almost tangent structure on the bundle of k-jets ($T^k M$). The tangent vectors of order k of any differentiable manifold M form the manifold $T^k M$ A local chart $x = (x^h)_{h=1,\dots,n}$ in M defines through the natural projection $\pi_k : T^k M \to M$ another chart $(x, y_1, \dots, y_k)$ in $T^k M$.

$$y_i = \left(y_i^{h_i}\right), \quad (h_i = 1, \dots, n), \qquad y_i^h = \frac{1}{i!}\,\frac{d^i x^h}{dt^i}.$$

The change of charts in domains with common intersection is given by $\partial y_\nu^{h'}/\partial y_\mu^h$ Yano proved that

$$\frac{\partial y_\nu^{h'}}{\partial y_\mu^h} = \left(\frac{\partial x^{h'}}{\partial x^h}\right)^{(\nu-\mu)},$$

where $(\ )^{(\nu-\mu)}$ represents the prolongation of order $(\nu-\mu)$. Then, the change of charts is given by a matrix of the type considered in (2 — 1). Therefore the fiber bundle of k-jets (on an arbitrary differentiable manifold M) admits, in a natural way, an almost tangent structure of order k.

Example 2.4. Existence of an integrable almost tangent structure of order k—1 on the torus $S^1 \times \cdots \times S^1$ (k factors).

This question is proved in §5.

§3. Existence of almost tangent structures of order k. Definition 3.1. *A G'-structure $P'(M, G')$ is said equivalent to a G-structure $P(M, G)$ is there exists $l \in Gl(n, \mathbf{R})$ such that $P' = P \circ l$.*

A G'-structure P', is a G-structure if and only if $l \in N(G)$ (the normalizer of G in $Gl(n,\mathbf{R})$; $\dim M = n$). We then say that P' is a G-structure associated to P.

In fact; if $\tau_{j.x} = \tau_j/\pi^{-1}(x)$ represents the linear map $(\pi^{-1}(x) \to G)$ and if $l^{-1}\tau_{i.x}$ is the induced one in the G'-structure by $\tau_{i.x}$, we can obtain :

$$\tau'_{j.x} \circ \tau'^{-1}_{i.x} \in G'\,; \quad \tau'_{j.x} = l^{-1}\,\tau_{j.x} : \pi'^{-1}(x) \to G'$$

$$\tau'_{j.x} \cdot \tau'^{-1}_{i.x} = l^{-1}\,\tau_{j.x} \circ \tau^{-1}_{i.x}\, l \in G \Leftrightarrow l \in N(G).$$

Consequently, in a given G-structure, the cardinal of the set of associated structures corresponds to the index $(N(G) : G)$.

We consider the following well known

Proposition 3.2. *The closed principal fiber bundle subspaces of a differentiable principal fiber space ξ, with structural group G', agree bijectively, with differentiable sections in the spaces ξ/G where G is any closed subgroup of G'. The principal fiber subspaces with structural group G correspond biunivocally to the differentiable sections of the space ξ/G.*

The problem about the determination of all these possible G-structures on a manifold M is thus solved, in the case $N(G) \neq G$, because these G-structures correspond biunivocally to the differentiable sections in the quotient of the linear frame bundle on M FM/G, G being closed in $Gl(n, \mathbf{R})$.

In the particular case of the almost tangent structure of order $k = 1$ we have :

Proposition 3.3. $N(G) \neq G$.

Proof: Let G' be the subgroup of $Gl(kn,\mathbf{R})$ of matrices with the following form:

$$\left\|\begin{matrix} A_1 & 0 & \dots & 0 & 0 \\ A_2 & A_1 & \dots & 0 & 0 \\ A_3 & A_2 & \dots & 0 & 0 \\ \vdots & \vdots & & & \vdots \\ A_k & A_{k-1} & \dots & A_2 & A_1 \end{matrix}\right\|$$

where $Z = \lambda \,.\, I$, $\lambda \in \mathbf{R} - \{0\}$.

We obtain $G \neq N(G)$, because, for a matrix $B \in G$:

$$\left\|\begin{matrix} A_1 & 0 & \dots & 0 \\ A_2 & A_1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_k & A_{k-1} & \dots & A_1\ Z \end{matrix}\right\| \cdot \left\|\begin{matrix} B_1 & 0 & 0 & \dots & 0 \\ B_2 & B_1 & 0 & \dots & 0 \\ \vdots & \vdots & & & \vdots \\ B_k & B_{k-1} & 0 & \dots & B_1 \end{matrix}\right\| =$$

$$= \left\|\begin{matrix} C_1 & 0 & \dots & 0 \\ C_2 & C_1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ C_k & C_{k-1} & \dots & C_1 \end{matrix}\right\| \left|\begin{matrix} A_1 & 0 & \dots & 0 \\ A_2 & A_1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_k & A_{k-1} & \dots & A_1\ Z \end{matrix}\right|$$

where $C_n = (A_n B_1 + A_{n-1} B_2 + \cdots + A_1 B_n - C_{n-1} A_2 - \cdots - C_1 A_n) A_1^{-1}$; $n = 1, \dots, k$

We thus show $N(G) > G' \neq G$.

Corollary 3.4. *If there is an almost tangent structure on a manifold M, all associated structures correspond to the sections in the fiber space (that is, if $N(G) \neq G$).*

Corollary 3.5. *If there is an almost tangent structure on a manifold M, then all associated structures, given by the last corollary, correspond biunivocally to the differentiable sections in the fiber space FM/G.*

In fact, the group G is closed in the space $\mathfrak{M}_{kn}(\mathbf{R})$ of matrices of order $kn \times kn$, and therefore it is closed in $Gl(kn, \mathbf{R})$ because $G \cap Gl(kn,\mathbf{R}) = G$.

4. Obstructions to the existence of the structures. 4.1. Structure connections and Pontrjagin classes. We can define, in a way similar to that of Eliopoulos [13], an almost tangent structure of order k on a differentiable manifold M, of dimension $(k+1)n$, as a (1, 1) tensor field $J : T_x^{\mathbf{C}} \to T_x^{\mathbf{C}}$ (where $T_x^{\mathbf{C}}$ denotes the complexified tangent on each point $x \in M$), with constant rank kn, and such that $J^{k+1} = 0$. This is equivalent to a reduction of the structural group of the complexified tangent bundle to the subgroup

$$G = \left\{ \left|\begin{matrix} A_1 & 0 & \dots & 0 \\ A_2 & A_1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ A_{k+1} & A_k & \dots & A_1 \end{matrix}\right\| \,\Big/\, A_i \in Gl(n,\mathbf{C}) \right\}$$

of $Gl((k+1)n, \mathbf{C})$.

Let M be an almost tangent manifold of order k. A connection on the frame fiber bundle $P(M, Gl((k+1)n, \mathbf{C}))$ is called an *almost tangent connection* if the horizontal distribution in each point, induces an horizontal distribution in the principal subbundle of adapted G-frames, G being the structure group.

Conversely, a connection in $P(M, G)$ induces a single almost tangent connection on $P(M, Gl((k+1)n, \mathbf{C}))$.

Let ∇ be a connection on M, and $X_1, \dots, X_{(k+1)n}$ a local basis in an open set U. Then $\nabla_{X_i} X_j = \Gamma_{ij}^k X_k$. If the connection is an almost tangent one, its 1-form on each open set U, restricted to $P(M, G)$, is valued in the Lie algebra $\mathcal{G}$ of G.

Proposition 4.1.1. *A connection in an almost tangent structure of order k is a structure connection, if and only if the coefficients of the connection in $P(M, G)$, satisfy:*

$$\Gamma^{\alpha+\beta n}_{b.c+\gamma n} = \Gamma^{\alpha+(\beta+1)n}_{b.c+(\gamma+1)n}\,; \quad \text{if} \quad \beta > \gamma, \quad \Gamma^{\alpha+\beta n}_{b.c+\gamma n} = 0.$$

Proposition 4.1.2. (See Eliopoulos [13]). *A necessary and sufficient condition for a connection on an almost tangent manifold of order k, to be a structure connection is that the tensor field J, associated to the structure, have null absolute differential, that is, $\nabla J = 0$.*

Proof: If θ^i_j is the 1-form of the connection ∇, we have $\nabla J(X, Y) = \nabla_Y(JX) - J\nabla_Y X$, that is: $\nabla J = \theta J - J\theta$.

In a matricial form we have

$$\begin{Vmatrix} \nabla J^{\alpha}_{\beta} & \nabla J^{\alpha+n}_{\beta} & \dots & \nabla J^{\alpha+kn}_{\beta} \\ \nabla J^{\alpha}_{\beta+n} & \nabla J^{\alpha+n}_{\beta+n} & \dots & \nabla J^{\alpha+kn}_{\beta+n} \\ \vdots & \vdots & & \vdots \\ \nabla J^{\alpha}_{\beta+kn} & \nabla J^{\alpha+n}_{\beta+kn} & \dots & \nabla J^{\alpha+kn}_{\beta+kn} \end{Vmatrix} = \begin{Vmatrix} \theta^{\alpha}_{\beta} & \theta^{\alpha+n}_{\beta} & \dots & \theta^{\alpha+kn}_{\beta} \\ \theta^{\alpha}_{\beta+n} & \theta^{\alpha+n}_{\beta+n} & \dots & \theta^{\alpha+kn}_{\beta+n} \\ \vdots & \vdots & & \vdots \\ \theta^{\alpha}_{\beta+kn} & \theta^{\alpha+n}_{\beta+kn} & \dots & \theta^{\alpha+kn}_{\beta+kn} \end{Vmatrix} \times$$

$$\begin{Vmatrix} 0 & 0 & \dots & 0 \\ I^{\alpha}_{\beta+n} & 0 & \dots & 0 \\ 0 & I^{\alpha+n}_{\beta+2n} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & \dots & I^{\alpha+(k-2)n}_{\beta+(k-1)n} & 0 \end{Vmatrix} - \begin{Vmatrix} 0 & 0 & \dots & 0 & 0 \\ I & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & I & 0 \end{Vmatrix} \times \begin{Vmatrix} \theta^{\alpha}_{\beta} & \dots & \theta^{\alpha+kn}_{\beta} \\ \vdots & & \vdots \\ \theta^{\alpha}_{\beta+kn} & \dots & \theta^{\alpha+kn}_{\beta+kn} \end{Vmatrix}$$

$$= \begin{Vmatrix} \theta^{\alpha+n}_{\beta} I^{\alpha}_{\beta+n} & 0 & \dots & 0 \\ \theta^{\alpha}_{\beta+n} - I^{\alpha}_{\beta+n} I^{\alpha}_{\beta+n}\theta^{\alpha}_{\beta} & \theta^{\alpha+n}_{\beta+n} I^{\alpha+n}_{\beta+2n} - I^{\alpha}_{\beta+n}\theta^{\alpha+n}_{\beta} & \dots & 0 \\ \vdots & \vdots & & \\ \theta^{\alpha+n}_{\beta+kn} I^{\alpha}_{\beta+n} - I^{\alpha+n(k-2)}_{\beta+n(k-1)}\theta^{\alpha}_{\beta+(k-1)n} & \cdot & & I^{\alpha+(k-2)n}_{\beta+(k-1)n}\theta^{\alpha+kn}_{\beta+(k-1)n} \end{Vmatrix}$$

If $\nabla J = 0$, we obtain the equalities of the last proposition. Conversely, if the matrix of θ has the following form:

$$\begin{Vmatrix} \theta^{\alpha}_{\beta} & 0 & \dots & 0 \\ \theta^{\alpha}_{\beta+n} & \theta^{\alpha}_{\beta} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \theta^{\alpha}_{\beta+kn} & \theta^{\alpha}_{\beta+(k-1)n} & \dots & \theta^{\alpha}_{\beta} \end{Vmatrix}, \quad \text{we have } \nabla J = 0.$$

Suppose a certain structure-connection has a 1-form of connection (θ^i_j). Its 2-form of curvature is $\Omega^i_j = d\theta^i_j - \theta^i_j \wedge \theta^i_j \quad i, j = 1, \dots, (k+1)n$. Matricially it must be written

$$\begin{Vmatrix} \Omega^{\alpha_1}_{\beta_1} & 0 & \dots & 0 \\ \Omega^{\alpha_1}_{\beta_2} & \Omega^{\alpha_1}_{\beta_1} & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \Omega^{\alpha_1}_{\beta_k} & \Omega^{\alpha_1}_{\beta_{k-1}} & \dots & \Omega^{\alpha_1}_{\beta_1} \end{Vmatrix}$$

The total Pontrjagin class of the tangent bundle verifies:

$$p(TM) = \left[\det\left(I_{kn} - \frac{1}{2\pi}\Omega^i_j\right)\right] = \left[\left(\det\left(I_n - \frac{1}{2\pi}\Omega^{\alpha_1}_{\beta_1}\right)\right)^k\right] = \left[\det\left(I_n - \frac{1}{2\pi}\Omega^{\alpha_1}_{\beta_1}\right)\right]^k$$

and it does not depend, on account of Chern-Weil's theorem, on the elected particular connection. We can state the following

Theorem 4.1.3. *The complete Pontrjagin class of a differentiable manifold X, of dimension kn, and endowed with an almost tangent structure of order $k-1$ can be expressed as a power of order k.*

4.2. Chern classes. By means of an analogous process, we can get.

Theorem 4.2.1. *The total Chern class of a differentiable complex manifold, with complex dimension $(k+1)n$, and endowed with an almost tangent structure of order k, can be expressed by a power of order k.*

4.3. Obstruction to the existence on certain types of manifolds. Corollary 4.3.1. *Complex projective spaces $\mathbf{CP}^{2s}$ do not admit almost tangent structures of order $2k-1$.*

In fact, for $k > s$, there are no such structures by reasons of dimension. It is sufficient to consider $k \leqslant s$. The complete Chern class (see Borel-Hirzebruch [4]) of $\mathbf{CP}^{2s}$ is: $c(\mathbf{CP}^{2s}) = (1+\alpha)^{2s+1}$, α being a generator of $H^2(\mathbf{CP}^{2s}; \mathbf{Z})$.

Therefore, keeping in mind that $\alpha^{2s+1} = 0$, we will get:

$$c(\mathbf{CP}^{2s}) = (1+\alpha)^{2s+1} = \sum_{i=1}^{2s+1}\binom{2s+1}{i}\alpha^{2s+1-i} = 1 + (2s+1)\alpha + \dots + \alpha^{2s+1}.$$

If $\mathbf{CP}^{2s}$ admits an almost tangent structure of order $2k-1$, by 4.3.1, we will write for a convenient ε,

$$c(\mathbf{CP}^{2s}) = \left(\sum_{j=0}^{\varepsilon}\lambda_j\alpha^j\right)^{2k} = \lambda_0^{2k} + 2k\lambda_0^{2k-1}\lambda_1\alpha + \dots$$

Identifying coefficients between both expressions, we get:

$$1 = \lambda_0^{2k}, \quad \lambda_0 = \pm 1, \; 2s+1 = 2k\lambda_0^{2k-1}\lambda_1 = \pm 2k\lambda_1,$$

inconsistency, because the coefficients are integers.

Proposition 4.3.2. *Let M be a manifold with an almost tangent structure of an odd order. Then odd Pontrjagin classes can be expressed as twice a polynomial in Chern's classes.*

It is well known that, for a complex vector fiber bundle ω on M, the Pontrjagin classes $p_k(\omega_{\mathbf{R}})$ of the real vector fiber bundle $\omega_{\mathbf{R}}$, are determined by the Chern classes $c_i(\omega)$, such that

$$(*) \qquad p_k(\omega_{\mathbf{R}}) = c_k(\omega)^2 - 2c_{k-1}(\omega)\,c_{k+1}(\omega) \pm \dots \pm 2c_1(\omega)\,c_{2k-1}(\omega) \pm 2c_{2k}(\omega).$$

By the last theorem the odd Chern classes can be expressed as twice a polynomial in these Chern classes. Therefore, from $(*)$, we deduce an analogous result for the odd Pontrjagin ones.

Corollary. (New demonstration). *The complex projective spaces $\mathbf{CP}^{2s}$ do not admit almost tangent structures of odd order.*

In fact, the complete Pontrjagin class of $\mathbf{CP}^{2s}$ is $p(\mathbf{CP}^{2s}) = (1+\alpha^2)^{2s+1}$, α generator of $H^2(\mathbf{CP}^{2s}; \mathbf{Z})$; that is,

$$p_k(\mathbf{CP}^{2s}) = \binom{2s+1}{k}\alpha^{2k}, \quad 1 \leqslant k \leqslant \frac{n}{2}.$$

Therefore, $p_1(\mathbf{CP}^{2s}) = (2s+1)\alpha^2$; and by the last proposition, $\mathbf{CP}^{2s}$ does not admit almost tangent structures of an odd order.

Corollary 4.3.3. *Let M be a manifold, such that $(\omega_{2j}(M))^2 \neq 0$, for a certain odd j. In this case, M does not admit almost tangent structures of odd order.*

Keeping in mind that for a fibre bundle ξ of the manifold M, it verifies $p_j(\xi) = (\omega_{2j}(\xi))^2$, (mod 2) and by the last proposition, in these conditions, for an odd j, it is

$$(\omega_{2j}(\xi))^2 = p_j(\xi) = 0 \mod 2.$$

Corollary 4.3.4. *The real projective spaces $\mathbf{RP}^{4s+2}$ do not admit almost tangent structures of order $2k-1$.*

It is well known that $\omega(\mathbf{RP}^{2s}) = (1+\alpha)^{2s+1}$, α generator of $H^1(\mathbf{RP}^{2s}; \mathbf{Z})$, and similarly the statement is confirmed. On the other hand (see Borel [2]) there are the following six classes of symmetric, hermitian, irreducible, compact, spaces:
I) Complex Grassmannians $W(m,n) = U(m+n)/U(m)\times U(n)$. II) Spaces $F_n = SO(2n)/U(n)$, III) Spaces $G_n = Sp(n)/U(n)$. IV) Complex quadrics $SO(n+2)/SO(2)\times SO(n)$
and two exceptional spaces.

The first Chern class of Grassmannian spaces is $c_1(W(m, n)) = -(m+n)\sigma_1$, σ_1 being a generator of the infinite cyclic group $H^2(W(m, n); \mathbf{Z})$. Therefore in accordance with 4.3.1. we deduce:

Corollary 4.3.5. *$W(m, n)$ does not admit almost tangent structures of order $2k-1$, if m is even and n is odd, or m odd and n even.* In a particular case we find again that *the complex projective spaces $\mathbf{CP}^{2s}$ do not admit such structures.*

It is sufficient to consider $W(1, n) = \mathbf{CP}^n$. In the case of the spaces $G_n = Sp(n)/U(n)$, we have $c_1(G_n) = (n+1)\sigma_1$, where σ_1 is a generator of the group $H^2(G_n; \mathbf{Z})$; and so:

Corollary 4.3.6. *The spaces G_{2n} do not admit almost tangent structures of order $2k-1$.*

In the case of complex quadrics, $Q_n = SO(n+2)/SO(2)\times SO(n)$ it verifies

$$c(Q_n) = (1+\alpha)^{n+2}(1+2\alpha)^{-1} = (1+\alpha)^{n+2}(1-2\alpha+4\alpha^2-8\alpha^3+\ldots) = 1+n\alpha+\ldots$$

that is, $c_1(Q_n) = n\alpha$, and therefore:

Corollary 4.3.7. *The quadrics Q_n do not admit almost tangent structures of odd order.*

5. Integrable almost tangent structurés of order k. A manifold M, modelled in $\mathbf{R}^n \oplus \mathbf{R}^n \oplus \ldots \oplus \mathbf{R}^n$ (k-times) admits a foliation (of dimension n) if it has a pseudogroup structure, whose coordinate transformations are local diffeomorphisms of the type:

$$(z, w) = (z, w_1, \ldots, w_{k-1}) \to (f(z), g_1(z, w), \ldots, g_{k-1}(z, w)).$$

Similarly M admits a foliation $\mathcal{F}_1$ of dimension n, and another one $\mathcal{F}_2$ (with dimension $2n$) such that $\mathcal{F}_1 \to \mathcal{F}_2$, if these coordinate transformations are of the type:

$$(z, w_1, \ldots, w_{k-1}) \to (f(z), g_1(z, w_1), g_2(z, w_1, \ldots, w_{k-1}), \ldots, g_{k-1}(z, w_1, \ldots, w_{k-1})).$$

In a general way, M admits a *chain* of foliations $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_j$, with respective dimensions $n, 2n, \ldots, jn$ ($j = 1, \ldots, k-1$, a fixed number), if these transformations are of the type

$$(z, w_1, \ldots, w_{k-1}) \to (f(z), g_1(z, w_1), g_2(z, w_1, w_2), \ldots, g_{j-1}(z, w_1, \ldots, w_{j-1}),$$
$$, g_j(z, w_1, \ldots, w_{k-1}), \ldots, g_{k-1}(z, w_1, \ldots, w_{k-1})),$$

that is, if the first j transformation functions (after a convenient removing), depend on the first jn coordinates. It is an easy thing to see in each case, which charts determine the leaves $(\mathcal{F}_1)_m, (\mathcal{F}_2)_m, \ldots, (\mathcal{F}_j)_m$ of foliations, from an adapted chart in a point $m \in M$.

An integrable almost tangent structure of order $k-1$ on M is a pseudogroup structure with coordinate transformations of the type:

$$(z, w_1, \ldots, w_{k-1}) \to (f(z), (Df)_z w_1 + b_1(z), (D_h f)_z w_2^h + (D_{ij}^2 f)_z w_1^i w_1^j + b_2(z), \ldots,$$
$$\ldots, (D_h f)_z w_{k-1}^h + (D_{ij}^2 f)_z w_{k-2}^i w_{k-2}^j + \ldots + b_{k-1}(z)) \quad (5-1)$$

where f is a local diffeomorphism in $\mathbf{R}^n$, and the local functions $b_1, \ldots, b_{k-1}$: $\mathbf{R}^n \to \mathbf{R}^n$ are differentiable. An integrable almost tangent *manifold admits* $k-1$ foliations $\mathcal{F}_1, \mathcal{F}_2 \ldots \mathcal{F}_{k-1}$ of respective dimensions $n, 2n, \ldots, (k-1)n$, the form of the charts adapted to the integrable almost tangent structure is easily found.

A pseudogroup structure with coordinate transformations of the type

$$(z, w_1, \ldots, w_{k-1}) \to (f(z), (Df)_z w_1, \ldots, (D_h f)_z w_{k-1}^h + (D_{ij}^2 f)_z w_{k-2}^i w_{k-2}^i + \ldots +$$
$$(D_{i_1 \ldots i_j}^{k-1} f) w_1^{i_1} \ldots w_1^{i_j}),$$

that is, of $(5-1)$ type, where $b_1 = b_2 = \ldots = b_{k-1} = 0$ is a *„nearly" tangent* structure of order k. A manifold with an integrable almost tangent structure of order k does not necessarily admit an order k subordinate „nearly" tangent structure. Let us see an example.

Let us consider the circle S^1 and an atlas $\mathcal{A}$, the charts of which are restrictions of the global function $(\cos\alpha, \sin\alpha) \to \alpha$. Two of these charts differ in a coordinate transformation, locally given by $z \to z + c$, and so, $\mathcal{A}$ determines a locally affine complete structure (see Brickell-Clark [6]) on S^1. Now, we consider the torus $T^k = S^1 \times \ldots \times S^1$ (k factors). The atlas with charts $f_1 \times \ldots \times f_h$, $(f_1, \ldots, f_h \in \mathcal{A})$, defines an integrable almost tangent structure of order $k-1$, on T^k

$$(Df_j) = 1 \quad (D^n f_j) = 0; \quad j = 1, \ldots, k; \quad n \geq 2.$$

The change is then, $(z_1, z_2, \ldots, z_k) \to (z_1 + c_1, z_2 + c_2, \ldots, z_k + c_k)$ (Its type is like $(5-1)$). The leaves of foliation $\mathcal{F}_1$ are the circles $(\mathcal{F}_1)_p = S^1 \times \{p\}$, the ones of $\mathcal{F}_2$ are $(\mathcal{F}_2)_p = S^1 \times S^1 \times \{p\}$, and so on. T^k is compact, then it cannot be a manifold of k-jets. In this sense, it follows a theorem (the proof of which is in [14]), generalization of the Brickell-Clark's [6], and according to which, the almost tangent structure Σ, cannot admit any subordinate „nearly" tangent structure. By means of the considerations made at the beginning of the paragraph, any kn-dimensional manifold M, endowed with an integrable almost tangent structure, admits a chain of foliations $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_{k-1}$ with respective codimensions $(k-1)n, (k-2)n, \ldots, n$. A theorem similar to Bott's [5] can then be stated

Theorem 5.1. *Let M be a kn-manifold endowed with an integrable almost tangent structure of order k; and let be $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \ldots \subset \mathcal{F}_{k-1}$ the carried chain of foliations, and $E_i = TM/T\mathcal{F}_i$ the fiber bundle, orthogonal to $\mathcal{F}_i$. Then $Pont^s(E_i) = 0$, if $s > 2(k-i)n$.*

In the particular case of the almost tangent structure of order k on the k-jets fiber bundles, it is possible to obtain these conditions, following the techniques of Cordero [11].

6. Prolongations of the structures to the k-jets fiber bundle. Yano and Ishihara define (see [18]) the prolongations $F^{0)}, F^{I)}, F^{II)}, \ldots, F^{k)}$ from a given $(1, 1)$ tensor-field F, which are $(1, 1)$ tensor fields on the bundle T^kM of k-jets. According to definitions and properties of the almost tangent structure of order k and the prolongations, we can obtain:

Proposition 6.1. *Let M be a manifold; and $J \in \mathcal{T}_1^1(M)$, a $(1, 1)$ tensor field, such that $J^{k+1} = 0$, rank $J = [k/(k+1)]$. $\dim M = [k/(k+1)]n$. In this case, the prolongation $J^{k)} \in \mathcal{T}_1^1(\mathcal{T}^k, M)$ is an almost tangent structure of order k on $T^k M$.*

Observe $J^{k+1} = 0$, rank $J^{k)} = [k/k+1] \dim T^kM = kn$.

Proposition 6.2. *Let $F \in T_1^1(H)$ be the identity tensor field on the manifold M. The prolongation $F^{r)} \in \mathcal{T}_1^1(T^r M)$ coincides with the natural almost tangent structure of order r defined in T^rM.*

Proof: In fact, if $F = I$

$$F^{r)} = \begin{Vmatrix} 0 & 0 & \ldots & 0 & 0 \\ I & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & I & 0 \end{Vmatrix} \qquad \begin{array}{l} (F^{r)})^{r+1} = 0 \\ \text{rank } F^{r)} = r \,.\, n. \end{array}$$

Corollary 6.3. *The natural almost tangent structure of order $r+s$ defined in $T^{r+s} M$ is the one given by the prolongation of order r, of the identity tensor field on T^sM.*

Proposition 6.4. *The prolongation of order s of the r-order almost tangent structure coincides with the prolongation of order r, of the s-order structure defined in T^sM, both prolongations defined in $T^{r+s}M$.*

Proof.: Owing to 6.3 it is sufficient to prove that the prolongation of order r, of the natural structure defined in T^sM, coincides with the prolongation of order $r+s$, of identity tensor field on M.

In fact,

$$\begin{array}{c} (s+1) \\ F = \begin{Vmatrix} 0 & 0 & \ldots & 0 & 0 \\ I & 0 & \ldots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \ldots & I & 0 \end{Vmatrix} \end{array}$$

As it is well known, the prolongation of order r, (see Y a n o - I s h i h a r a [18]) is

$$F^{r)} = \begin{Vmatrix} F & 0 & \dots & 0 \\ 0 & F & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & F \end{Vmatrix}, \text{ that is } \quad F^{r)} = \begin{array}{c} (r+s+1 \\ \begin{Vmatrix} 0 & 0 & \dots & 0 & 0 \\ I & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{Vmatrix} \end{array}.$$

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BI-RECURRENT LORENTZIAN MANIFOLDS

BY

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Introduction. Let **L** be an even dimensional Lorentzian C^∞-manifold and let $B(\mathbf{L}) = \bigcup \mathcal{F}_S$ be the quasi-orthonormal frame bundle over **L**. We say that **L** is *bi-recurrent* (denoted by $\mathbf{L}_r$) if both *isotropic* (real) vectors of $\mathcal{F}_\xi$ are recurrent.

The tangent space $T_p(\mathbf{L}_r)$ at $p \in L_r$ may split as $T_p(\mathbf{L}_r) = H_p(\mathbf{L}_r) + S_p(\mathbf{L}_r)$ where $H_p(\mathbf{L}_r)$ is a hyperbolic 2-plane and $S_p(\mathbf{L}_r)$ is a spatial $2n$-plane.

The distribution defined by H_p and S_p is involutive, and any $\mathbf{L}_r$ manifold may be considered as a product manifold $\mathbf{L}_r = V_H \times V_S$ of 2 minimal manifolds, which are hyperbolic (of dimension 2) and spatial (of dimension $2n$) respectively.

The case when the connection on V_S is *almost flat* (in the sens of R. R o ş c a [1]) is especially investigated. Such a connection (denoted by $\nabla_{r.f.}$) introduces in a natural manner a closed 1-form α and its associated vector field $\mathbf{X}_a$, which has the property to be concurrent (in the sense of K. Y a n o [2]) over V_S in $\mathbf{L}_r$. According to a definition given by R. R o ş c a [1] the connection $\nabla_{r.f}$ defines on $\mathbf{L}_r$ a *conformal bi-cosymplectic structure* defined by a spatial 1-form ds, by α and the isotropic co-vectors of R^*_ξ. Some properties of this structure are studied. Then, after having proved that the immersion $x : V_S \to \mathbf{L}_r$ is *totally geodesic*, we study the immersion of an even dimensional spatial manifold V in V_S.

It is proved that the immersion $x : V \to V_S$ possesses the following properties:

(i) x is umbilical;

(ii) the mean curvature vector field **H** associated with x is the normal component of the restricted vector field $\mathbf{X}_a$ on V. Furthermore the necessary and sufficient condition that **H** be concurrent over V is that it be parallel in the normal bundle;

(iii) V is Einsteinian;

(iv) the *shape operator* of the restriction $\mathbf{X}_a$ and $\mathbf{X}_\alpha$ on V ($\mathbf{X}_\alpha$ is the dual field with respect in Ω_S of α) is conformal to **H**.

Finally if $\mathbf{L}_r$ is a general space-time ($n = 1$) the manifold is symplectic and belongs to *type N* in Petrov's classification.

§ 1. Let $\mathbf{L}(\equiv \mathbf{L}^{2n+2})$ be a Lorentzian C^∞-manifold (or hyperbolic manifold) of dimension $2n + 2$ and $\mathcal{B}(\mathbf{L})$ be the *quasi-orthonormal* [6] frame bundle over **L**. Denote by $\{ \xi_A ; A = 1, 2n + 2 \}$ the basis of a quasi-orthonormal frame $\mathcal{F}_\xi \in \mathcal{B}(\mathbf{L})$. Such a basis is defined by 2 *isotropic* vectors (ξ_a ; $a = 1, 2n + 2$) and $2n$ *space-like* vectors (ξ_r ; $r = 2, \dots, 2n + 1$) which satisfy:

$$< \xi_1, \xi_{2n+2} > = 1, \; < \xi_r, \xi_A > = 0 \text{ for } r \neq A. \tag{1.1}$$

If $T_p(\mathbf{L})$ is the tangent space to **L** at $\mathbf{p} \in \mathbf{L}$ we may split it as

$$T_p(\mathbf{L}) = H_p(\mathbf{L}) \oplus S_p(\mathbf{L}). \tag{1.2}$$

In (1.2) $H_p(\mathbf{L}) = \{ \xi_1, \xi_{2n+2} \}$ is a Lorentzian (or hyperbolic) 2-plane and $S_p(\mathbf{L}) = \{ \xi_r \}$ is a spatial $2n$-plane. If $\{ \alpha^A \}$ is the dual basis of $\{ \xi_A \}$, the line element $d\mathbf{p}$ ($d\mathbf{p}$ is a cannonical differential 1-form on the Lie group G acting on $\mathcal{B}(\mathbf{L})$) is written as

$$d\mathbf{p} = \alpha^A \oplus \xi_A. \tag{1.3}$$