As it is well known, the prolongation of order r, (see Yano-Ishihara [18]) is

$$F^{r)} = \begin{Vmatrix} F & 0 & \dots & 0 \\ 0 & F & \dots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & F \end{Vmatrix}, \text{ that is } \quad F^{r)} = \begin{array}{c} \scriptstyle (r+s+1 \\ \begin{Vmatrix} 0 & 0 & \dots & 0 & 0 \\ I & 0 & \dots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 \end{Vmatrix} \end{array}.$$

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BI-RECURRENT LORENTZIAN MANIFOLDS

BY

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Introduction. Let **L** be an even dimensional Lorentzian C^∞-manifold and let $B(\mathbf{L}) = \bigcup \mathcal{F}_S$ be the quasi-orthonormal frame bundle over **L**. We say that **L** is *bi-recurrent* (denoted by $\mathbf{L}_r$) if both *isotropic* (real) vectors of $\mathcal{F}_\xi$ are recurrent.

The tangent space $T_p(\mathbf{L}_r)$ at $p \in L_r$ may split as $T_p(\mathbf{L}_r) = H_p(\mathbf{L}_r) + S_p(\mathbf{L}_r)$ where $H_p(\mathbf{L}_r)$ is a hyperbolic 2-plane and $S_p(\mathbf{L}_r)$ is a spatial $2n$-plane.

The distribution defined by H_p and S_p is involutive, and any $\mathbf{L}_r$ manifold may be considered as a product manifold $\mathbf{L}_r = V_H \times V_S$ of 2 minimal manifolds, which are hyperbolic (of dimension 2) and spatial (of dimension $2n$) respectively.

The case when the connection on V_S is *almost flat* (in the sens of R. Roşca [1]) is especially investigated. Such a connection (denoted by $\nabla_{r.f.}$) introduces in a natural manner a closed 1-form α and its associated vector field $\mathbf{X}_a$, which has the property to be concurrent (in the sense of K. Yano [2]) over V_S in $\mathbf{L}_r$. According to a definition given by R. Roşca [1] the connection $\nabla_{r.f}$ defines on $\mathbf{L}_r$ a *conformal bi-cosymplectic structure* defined by a spatial 1-form ds, by α and the isotropic co-vectors of R^*_ξ. Some properties of this structure are studied. Then, after having proved that the immersion $x : V_S \to \mathbf{L}_r$ is *totally geodesic*, we study the immersion of an even dimensional spatial manifold V in V_S.

It is proved that the immersion $x : V \to V_S$ possesses the following properties:

(i) x is umbilical;

(ii) the mean curvature vector field **H** associated with x is the normal component of the restricted vector field $\mathbf{X}_a$ on V. Furthermore the necessary and sufficient condition that **H** be concurrent over V is that it be parallel in the normal bundle;

(iii) V is Einsteinian;

(iv) the *shape operator* of the restriction $\mathbf{X}_a$ and $\mathbf{X}_\alpha$ on V ($\mathbf{X}_\alpha$ is the dual field with respect in Ω_S of α) is conformal to **H**.

Finally if $\mathbf{L}_r$ is a general space-time ($n = 1$) the manifold is symplectic and belongs to *type N* in Petrov's classification.

§ 1. Let $\mathbf{L}(\equiv \mathbf{L}^{2n+2})$ be a Lorentzian C^∞-manifold (or hyperbolic manifold) of dimension $2n + 2$ and $\mathfrak{B}(\mathbf{L})$ be the *quasi-orthonormal* [6] frame bundle over **L**. Denote by $\{ \xi_A ; A = 1, 2n + 2 \}$ the basis of a quasi-orthonormal frame $\mathcal{F}_\xi \in \mathfrak{B}(\mathbf{L})$. Such a basis is defined by 2 *isotropic* vectors (ξ_a ; $a = 1, 2n + 2$) and $2n$ *space-like* vectors (ξ_r ; $r = 2, \dots, 2n + 1$) which satisfy:

$$< \xi_1, \xi_{2n+2} > = 1, \quad < \xi_r, \xi_A > = 0 \text{ for } r \neq A. \tag{1.1}$$

If $T_p(\mathbf{L})$ is the tangent space to **L** at $\mathbf{p} \in \mathbf{L}$ we may split it as

$$T_p(\mathbf{L}) = H_p(\mathbf{L}) \oplus S_p(\mathbf{L}). \tag{1.2}$$

In (1.2) $H_p(\mathbf{L}) = \{ \xi_1, \xi_{2n+2} \}$ is a Lorentzian (or hyperbolic) 2-plane and $S_p(\mathbf{L}) = \{ \xi_r \}$ is a spatial $2n$-plane. If $\{ \alpha^A \}$ is the dual basis of $\{ \xi_A \}$, the line element dp (dp is a cannonical differential 1-form on the Lie group G acting on $\mathfrak{B}(\mathbf{L})$) is written as

$$dp = \alpha^A \oplus \xi_A. \tag{1.3}$$

We may expand dp as $dp = (dp)_H + (dp)_S$ where $(dp)_H = \alpha^a \otimes \xi_a$ and $(dp)_s = \alpha^r \otimes \xi_r$ are the Lorentzian and spatial components of dp respectively. So as a consequence of (1.1) the metric of L is expressed in terms of the quasi-orthonormal co-vectors α^A, by

(1.4) $$ds^2 = \langle dp, dp \rangle = 2\alpha^1 \alpha^{2n+2} - \sum_r (\alpha^r)^2.$$

The connection ∇ on V_L is given by

(1.5) $$\nabla \xi_A = \alpha_A^B \otimes \xi_B$$

where $\alpha_A^B = \Gamma_{AC}^B \alpha^C$ are the connection 1-forms on $\mathcal{B}(V_L)$. From (1.1) and (1.5) one finds easily [3]

(1.6) $$\begin{gathered} \alpha_r^{2n+2} - \alpha_1^r = 0, \quad \alpha_{2n+2}^r - \alpha_r^1 = 0, \\ \alpha_s^r + \alpha_r^s = 0, \quad \alpha_1^{2n+2} = 0 = \alpha_{2n+2}^1, \quad \alpha_1^1 + \alpha_{2n+2}^{2n+2} = 0. \end{gathered}$$

The connection ∇ being torsion-less the first and second group of structural equations (E. Cartan) are respectively

(1.7) $$d \wedge \alpha^A = \alpha^B \wedge \alpha_B^A,$$

(1.7′) $$d \wedge \alpha_B^A = \alpha_B^C \wedge \alpha_C^A + \Omega_B^A.$$

In (1.7) $\alpha_1^1 (= -\alpha_{2n+2}^{2n+2})$ is the *homothety component* of ∇ and Ω_B^A are the curvature 2-forms.

§ 2. According to the definition given in [13] we say that the manifold L is *bi-recurrent* if both isotropic base vectors ξ_a are *recurrent*. One must write

(2.1) $$\nabla \xi_a = u \otimes \xi_a; \quad u \in T_p^*(\mathrm{L})$$

and taking into account (1.5), conditions (2.1) give

(2.2) $$\alpha_1^r = 0, \quad \alpha_r^1 = 0.$$

So we may say that (2.1) implies that all *mixed connection* forms [3] vanish. Consequently one has

(2.4) $$d \wedge \alpha^1 = \alpha^1 \wedge \alpha_1^1 \quad d \wedge \alpha^{2n+2} = -\alpha^{2n+2} \wedge \alpha_1^1.$$

We shall call a connection ∇ for which (2.2) holds a *bi-recurrent connection*, and denote it by ∇_r. Equations (2.4) show that on a manifold structured by ∇_r, both isotropic co-vectors are *completely integrable* (such a manifold will be denoted by L_r)

Now let us denote by

(2.5) $$\eta_H = \alpha^1 \wedge \alpha^{2n+2}$$

and

(2.6) $$\eta_S = \alpha^2 \wedge \ldots \wedge \alpha^{2n+1}$$

the *unit simple* forms corresponding to H_p and S_p respectively.

By (2.4) and (2.2) one finds by exterior differentiation of (2.5) and (2.6)

(2.7) $$d \wedge \eta_H = 0 \qquad \text{(2.8)} \qquad d \wedge \eta_S = 0.$$

From the above equations we infer that both distributions $H(\mathrm{L}_r)$ and $S(\mathrm{L}_r)$ are *involutive*. If $GL(p; R)$ is the group of linear transformations of R which leave the 2-plane $H_p(V_L)$ invariant, then one may also say on a manifold L_r structured by ∇_r, $GL(2; R) \times G_L(2n; R)$ defines a $(2, 2n)$ *foliation*.

Furthermore since L_r is orientable the volume element (simple unit form corresponding to $T_p(\mathrm{L}_r)$) is given by

(2.9) $$\eta = \eta_H \wedge \eta_S.$$

Hence (2.7) and (2.8) express that both forms η_H and η_S are *harmonic*, that is

(2.10) $$\nabla \eta_H = \nabla \eta_S = 0; \quad (\nabla = d\delta + \delta d).$$

Consider now the integral submanifold $V_S \subseteq \mathrm{L}_r$ of $T_p(\mathrm{L}_r)$.

Such a manifold is defined by $\alpha^1 = 0 = \alpha^{2n+2}$ and denoting by α^A/S, α_B^A/S the restrictions of α^A, α_B^A on V_S, the line element of V_S is

(2.11) $$(dp)/_S = (\alpha^r)/_S \otimes \xi_r.$$

Denoting like usual by $*$ the *star isomorphism*, we get from (2.11):

$$*(dp)/_S = \Sigma(-1)^{r-1} (\alpha^2)/_S \wedge \ldots \wedge \widehat{(\alpha^r)}/_S \wedge \ldots \wedge (\alpha^{2n-1})/_S \otimes \xi_r$$

where the < roof > indicates the missing term. Using now (1.7) and taking into account 2.2) one readily finds

((2.12) $$d \wedge (dp)/_S = 0.$$

Thus according to the well known definition of minimal submanifolds, equation (2.12) proves that V_S is *minimal*. In the same manner one finds that the integral submanifold of $H_p(\mathrm{L}_r)$ is also minimal. These properties are in accordance with (2.10) and following Tachibana [4] one may say that a L_r manifold is covered with 2 families of 2-dimensional and $2n$-dimensional oriented minimal submanifolds which are orthogonal to each other.

Denote now by

(2.13) $$\mathrm{X}_S = \Sigma t_r \xi_r \in S_p(\mathrm{L}_r)$$

a spatial vector field. If $\mathcal{L}_{\mathrm{X}_S}$ is the Lie derivative in the direction of X_S, one has taking into account (2.8)

(2.14) $$\mathcal{L}_{\mathrm{X}_S} \eta_S = d \wedge (i_{\mathrm{X}_S} \eta_S).$$

So denoting by t_{sr} the Pfaffian derivative, one finds by straightforward computation

(2.15) $$\mathcal{L}_{\mathrm{X}_S} \eta_S = (\Sigma_s (\Sigma_r t^s_{s,r} + t_s \operatorname{div} \xi_s)) \eta_S.$$

Thus any field X_S is a *conformal infinitesimal transformation* of η_S. If $(\mathcal{F}_\xi)_s$ the spatial component of $\mathcal{F}_\xi$ is divergence-free and X_S is a constant spatial section, then

$$\mathcal{L}_{\mathrm{X}_S} \eta_S = 0$$

and X_S is an *infinitesimal* automorphism of η_S.

Theorem. *Any bi-recurrent Lorentzian manifold L_r of dimension $2n+2$ is covered by 2 families of minimal submanifolds which are hyperbolic surfaces and $2n$-dimensional spatial submanifolds V_S, respectively. If η_S is the volume element of V_S, then any space-like tangent vector filed X_S to V_S is an infinitesimal conformal transformation of η_S.*

§ 3. We recall that Ω_1^1 is the *homothety curvature* on L_r, and from (2.4) we get by exterior differentiation:

(3.1) $$\Omega_1^1 = h\alpha^1 \wedge \alpha^{2n+2}; \quad h \in \mathcal{D}(\mathrm{L}_r).$$

Let us investigate now if there exist a hyperbolic 2-dimensional vector field $X_H \in H_p(\mathrm{L}_r)$ which is recurrent. One may put

(3.2) $$\mathrm{X}_H = g_1 \xi_1 + g_{2n+2} \xi_{2n+2}; \quad g_1, g_{2n+2} \in \mathcal{D}(\mathrm{L}_r)$$

and if u is a certain 1-form of the co-tangent bundle $T^*(\mathbf{L}_r)$ we must write

$$\nabla \mathbf{X}_H = u \otimes \mathbf{X}_H. \tag{3.3}$$

Making use of (2.3) and (2.4) one finds

$$dg_1/g_1 + \alpha_1^1 = u, \quad dg_{2n+2}/g_{2n+2} - \alpha_1^1 = u. \tag{3.4}$$

and one infers

$$u = \frac{d(g_1 g_{2n+2})}{g_1 g_{2n+2}}. \tag{3.5}$$

Differentiating (3.4) one easily finds using also (3.5) that Ω_1^1 vanishes ($h = 0$).

Proposition. *Being given a bc-recurrent Lorentzian manifold $\mathbf{L}_r$, the necessary and sufficient condition that there exist on $\mathbf{L}_r$ a hyperbolic recurrent vector field, is that the homothety curvature of $\mathbf{L}_r$ vanishes.*

§ 4. We shall suppose that the connection ∇_r on V_S is *almost flat*. Following R. Roșca [1] we must write

$$d \wedge \alpha^r = \alpha \wedge \alpha^r, \tag{4.1}$$

where α is a closed 1-form of the co-tangent spatial bundle

$$S^*(\mathbf{L}_r) = US_p^*(\mathbf{L}_r).$$

Putting

$$\alpha = \Sigma a_r \alpha^r; \quad a_r \in \mathcal{D}(\mathbf{L}_r) \tag{4.2}$$

one gets from $d \wedge \alpha = 0$ and (4.1)

$$\Sigma_r\, da_r \wedge \alpha^r = 0. \tag{4.3}$$

The 1-form α is termed the *principal form* associated with ∇_r.

Making now use of (4.1) one finds

$$\mathcal{L}_{\xi_r} \eta = \mathcal{L}_{\xi_r} \eta_H \wedge \eta_S = \eta_H \wedge \mathcal{L}_{\xi_r} \eta_S = (\operatorname{div} \xi_{\eta_r})\, \eta = (2n-1) a_r\, \eta. \tag{4.4}$$

Hence the functions a_r have an invariant meaning and represent (up to the factor $2n-1$) the *divergences* of the base vectors ξ_r (the indices correspond) respectively.

Next, comparison of (3.1) with (1.7) gives after some calculations

$$\alpha_s^r = a_s \alpha^r - a_r \alpha^s. \tag{4.5}$$

Consider now the spatial vector field $\mathbf{X}_a \in S\mathrm{p}(\mathbf{L}_r)$ defined by

$$\mathbf{X}_a = \Sigma_r\, a_r\, \xi_r \tag{4.6}$$

and put

$$\tau = |\mathbf{X}_a|^2 = -\Sigma_r (a_r)^2. \tag{4.7}$$

Making use of (1.5) and taking into account (4.5) one finds from (4.6)

$$\nabla \mathbf{X}_a = \Sigma_r\, da_r \otimes \xi_r - \tau (d\mathrm{p})_S - \alpha \otimes \mathbf{X}_a \tag{4.8}$$

where $(d\mathrm{p})_S$ is the spatial component of $d\mathrm{p}$. Now, using (3.3) one finds after a short calculation $(d\mathrm{p})_S \wedge \nabla \mathbf{X}_a = 0$. Consequently, one may write $d\mathrm{p}_s + f \nabla \mathbf{X}_a = 0$, $f \in \mathcal{D}(\mathbf{L}_r)$. and this shows that $\mathbf{X}_a$ is concurrent over V_S in $\mathbf{L}_r$ [2].

The field $\mathbf{X}_a$ which is the dual of α with respect to the spatial metric $< (d\mathrm{p})_S, (d\mathrm{p})_S >$, will be called, in the following, the *associated vector field* of α.

Theorem. *Given a bi-concurrent Lorentzian manifold $\mathbf{L}_r$ $(= V_S \times V_H)$, if the connection on $\mathbf{L}_r$ is almost flat, then the associated vector field $\mathbf{X}_a$ of the principal form α is concurrent over V_s in $\mathbf{L}_r$.*

§5. Under the hypothesis of an almost flat spatial connection, consider on $\mathbf{L}_r$ the spatial 2-form $\Omega_S \in \Lambda^2(V_S)$ defined by

$$\Omega_S = \alpha^2 \wedge \alpha^3 + \ldots + \alpha^{2n-2} \wedge \alpha^{2n-1}. \tag{5.1}$$

We readily see that Ω_S is of rank $2n-2$, that is

$$\Omega\wedge^{(n-1)} \neq 0, \quad \Omega\wedge^{(n)} = 0 \tag{5.2}$$

(one may also say that $\dim \ker \Omega_S = 2$) and that

$$\Omega\wedge^{(n-1)} \wedge \alpha^1 \wedge \alpha^{2n+2} \neq 0. \tag{5.3}$$

On the other hand by means of 4.1 one gets

$$d \wedge \Omega_S = 2\alpha \wedge \Omega_S. \tag{5.4}$$

It follows from (5.3), (5.4) and (2.4) that the quadruple $\{\Omega_S, \alpha, \omega_1(\equiv \alpha^1), \omega_2(\equiv \alpha^{2n+2})\}$ defines a *conformal bi-cosymplectic structure* (in the sens of R. Roșca [12]).

Let now $\mu : T(V_S) \to T^*(V_S)$ be *fibred isomorphism* defined by $i_X \Omega_S \to \omega$.

Then if we denote by

$$\mathbf{X}_\alpha = \mu^{-1}(\alpha) = a_3\, \xi_2 - a_2\, \xi_3 + a_1\, \xi_n - \ldots - a_{2n-2}\, \xi_{2n-1} \in S\mathrm{p}(\mathbf{L}_r) \tag{5.5}$$

one finds

$$\mathcal{L}_{\mathbf{X}_\alpha} \Omega_S = 0 \tag{5.6}$$

that is, $\mathbf{X}_\alpha$ is an *infinitesimal automorphism* of $\{\Omega_S, \omega_1, \omega_2\}$. Furthermore since $\eta_H = \alpha_1^1 \wedge \wedge\, \alpha^{2n+2} \equiv (\omega_1 \wedge \omega_2)$ is closed, one has

$$\mathcal{L}_{\mathbf{X}_\alpha} \eta = \frac{1}{(n-1)!} \mathcal{L}_{\mathbf{X}_\alpha} \eta_H \wedge \Omega_S^{\wedge(n-1)} = 0 \Rightarrow \operatorname{div} \mathbf{X}_\alpha = 0. \tag{5.7}$$

Let now

$$\mathbf{X}_v = a_1\, \xi_1 + a_{2n+2}\, \xi_{2n+2}; \quad a_1, a_{2n+2} \in D(\mathbf{L}_r) \tag{5.8}$$

be a *vertical* field [1] with respect to $\{\Omega_S, \alpha, \omega_1, \omega_2\}$. Taking into account (2.4), one gets

$$\mathcal{L}_{\mathbf{X}_v} \eta_H = (da_1 + a_1 \alpha_1^1) \wedge \alpha^{2n+2} - (da_{2n+2} - a_{2n+2} \alpha_1^1) \wedge \alpha^1$$

and therefore

$$da_1 + a_1 \alpha_1^1 = 0, \quad da_{2n+2} - a_{2n+2}\, \alpha_1^1 = 0 \Rightarrow \mathcal{L}_{\mathbf{X}_v} \eta_H = 0. \tag{5.9}$$

Now with the help of (2.1) and (2.3) one finds also that condition (5.9) implies

$$\nabla \mathbf{X}_v = 0 \tag{5.10}$$

that is $\mathbf{X}_v$ is *parallel*. From (5.9) it follows: $d(a_1, a_{2n+2}) = 0$, and therefore $|\mathbf{X}_v| = \text{const}$

$$(< \mathbf{X}_v, \mathbf{X}_v > = 2a_1\, a_{2n+2}).$$

In addition we see at once from (5.9) that

(i) α_1^1 is *exact* and this means that the connection ∇_r is *equi-affine* [5].

(ii) the curvature homothety Ω_1^1 vanishes.

Consider next the associated vector field $\mathbf{X}_a$ (of α) defined by (4.6). One has

$$i_{\mathbf{X}_a} \Omega = \beta = a_2\, \alpha^3 - a_3\, \alpha^2 + \ldots + a_{2n-2}\, \alpha^{2n-1} \tag{5.11}$$

and introducing the operator L of degree 2 defined by $L : \alpha \to \alpha \wedge \Omega_S$ one finds

$$(5.12)\qquad {}_*\beta = -(L^{n-2}\alpha)\wedge\eta_H\,;$$

where $L^{n-2}\alpha = \alpha\wedge\Omega^{\wedge(n-2)}$.

By virtue of (2.7) and (5.4) one finds after some calculations

$$(5.13)\qquad \alpha\wedge{}_*\beta = 0 \Rightarrow <\alpha,\ {}_*\beta> = 0 \qquad (5.14)\qquad d\wedge{}_*\beta = 0 \Rightarrow \delta\beta = 0.$$

The above relations show that α at β are *orthogonal* and that β is *co-closed*.

Theorem. *Let $\mathbf{L}_r$ be a bi-recurrent Lorentzian manifold structured by an almost flat spatial connection defined by α, and let $\mathbf{X}_a$ be associated vector field of α. One may endow $\mathbf{L}_r$ with a conformal bi-cosymplectic structure $\{\Omega_S,\ \alpha,\ \alpha^1,\ \alpha^{2n+2}\}$ defined by a spatial 2-form Ω_S, the 1-form α, and the isotropic co-vectors $\{\alpha^a\,;\ a = 1,\ 2n+2\}$ on $\mathbf{L}_r$. If μ is the isomorphism $X\to i_X\Omega_S$, then*

(i) *the vector field $\mu^{-1}(\alpha)$ is divergence free on $\mathbf{L}_r$.*

(ii) *the 1-form $\beta = \mu(\mathbf{X}_a)$ is co-closed and orthogonal to α.*

§ 6. Equations (2.4) show that isotropic co-vectors are completely integrable. This proves that V_S may be considered as the intersection of 2 degenerated associated hypersurfaces $V_a^{2n+1}(1)$ $(a = 1,\ 2n+2)$ of defect 1 [6].

On the other hand the normal to $V_1^{2n+1}(1)$ (which is defined by $\alpha^{2n+2} = 0$) being ξ_1 [6], we readily see by (2.3) that $V^{2n+1}(1)$ is *totally geodesic* (the second fundamental quadratic form vanishes).

In a similar manner we see that $V_{2n+2}^{2n+1}(1)$ is also totally geodesic. The vertical vector field $\mathbf{X}_v$ of the conformal bi-cosymplectic structure of $\mathbf{L}_r$ is now a normal vector field over V_S. Since according to (5.10) there exist parallel normal fields, one infers that the immersion $x: V_S\to\mathbf{L}_r$ in *not substantial*. In order to simplify we shall denote with the same letters the induced forms by x. Consequently the line element of V_S is

$$(6.1)\qquad dx(\mathbf{p}) = \alpha^r\otimes\xi_r\,;\ r = 2,\dots,2n+1$$

and one readily finds by (2.3) that both fundamental quadratic forms associated with x vanish. This also proves by computation that V_S is totally geodesic (this includes as is well known the case of minimality).

Next it is an easy matter to see that on V_S the induced value of Ω_S becomes *conformal symplectic* and α is the corresponding *co-vector of Lee*.

Proposition. *The immersion $x: V_S\to\mathbf{L}_r$ is totally geodesic and the restriction to x of the conformal bi-cosymplectic structure of $\mathbf{L}_r$ is a conformal symplectic structure having α as co-vector of Lee.*

§ 7. Consider now the immersion $x: V\to V_S$ of a $2p$-dimensional spatial manifold V in V_S. Further we will consider the following ranges of indices

$$i,\ j,\ k = 2,\dots,2p+1\,;\quad l,\ m = 2p+2,\dots,2n+1.$$

Thus the co-dimension of V is $2(n-p)$ and the immersion x is defined by

$$(7.1)\qquad \alpha^a = 0.$$

If $T(V)$ and $T^{\perp}(V)$ are the tangent and the normal bundle of V then one knows that the tangent bundle of ∇_S restricted to V is the direct sum

$$(7.2)\qquad T(V_S)/_V = T(V)\oplus T^{\perp}(V).$$

By (7.1) and (4.5) one finds that all the normal connection forms ∇_m^l vanish. It follows that the normal connection $\nabla_r^{\perp}$ of V in V_S is *flat* [7] ($\nabla^{\perp}$ denotes the induced connection in the normal bundle $T(V)$) and since it is flat one may check that there exist $2(n-p)$ mutually orthogonal vector fields ξ_a which are each *parallel in the normal bundle* and consequently according to E. C a r t a n [8] all the second fundamental forms $\varphi_l = -<dx(\mathbf{p}),\ \nabla\xi_l>$ are *diagonal*.

Since the line element of V is

$$(7.3)\qquad dx(\mathbf{p}) = \alpha^i\otimes\xi_i\,,$$

one finds by (2.1) and (4.5)

$$(7.4)\qquad \varphi_l = -a_l\sum(\alpha^i)^2 = a_l<dx(\mathbf{p}),\ dx(\mathbf{p})>.$$

Hence each ξ_l defines on *umbilical section* and therefore the immersion x is *umbilical*. Mŏreover since

$$(7.5)\qquad K(x(\mathbf{p}),\ \xi_l) = \det(\varphi_l),$$

represent the *Lipschitz-Killing curvature* associated with x, one has from (7.4)

$$(7.6)\qquad K(x(\mathbf{p}),\ \xi_l) = (a_l)^{2p}.$$

Thus one may say that all the Lipschitz-Killing curvatures are positiv. Next, since

$${}_*dx(\mathbf{p}) = \Sigma(-1)^{i-1}\alpha^2\wedge\dots\wedge\hat{\alpha}^i\wedge\dots\wedge\alpha^{2p+1}\otimes\xi_i,$$

represents the *mean curvature vector form* associated with X, one finds after a direct calculation

$$(7.7)\qquad d\wedge{}_*x(\mathbf{p}) = 2p(\Sigma_l a_l\xi_l)\eta.$$

In (7.7)

$$(7.8)\qquad H = \Sigma\, a_l\xi_l\in T^{\perp}(V)$$

represents the *mean curvature* vector (or Bompiani vector) associated with x.

Referring to (7.6) we see that H is the (normal) component on $\nabla^{\perp}(V)$ of the restricted vector field $\mathbf{X}_a$ on V. From (7.8), using also (4.5), we deduce:

$$(7.9)\qquad \nabla\mathbf{H} = \Sigma_l\, da_l\,\xi_l - \sigma_H dx(\mathbf{p}),$$

where

$$\sigma_H = <H, H>.$$

So following the general definition [2] if H is *parallel* in the *normal bundle* $T^{\perp}(V)$ (i. e. the normal component of $\nabla\mathbf{H}$ in $T(V_S)/_V$ vanishes) then the mean curvature vector H is concurrent over V. We remind that if dp is the line element of a certain C^∞–manifold M, then the following invariants

$$(7.10)\qquad \psi_2 = <d\mathbf{p},\ d\mathbf{p}> = ds^2,\quad \psi_4 = <d^2\mathbf{p},\ d^2\mathbf{p}>\ \dots\ \psi_{2k} = <d^k\mathbf{p},\ d^k\mathbf{p}>$$

have been introduced by E. B o m p i a n i [9]. These invariants define the Riemannian or pseudo-Riemannian geometry of K-species associated with M. Coming back to the manifold V under discussion, we derive from (7.3), by neglecting the algebraic differentials of α_B^A

$$(7.11)\qquad d^2x(\mathbf{p}) = \alpha\, dx(\mathbf{p}) - (\mathbf{X}_a)_T\, ds^2 - \mathbf{H}ds^2.$$

In (7.11) $(\mathbf{X}_a)_T$ represents the tangential component of the restriction $\mathbf{X}_a$ on V. Putting

$$(7.12)\qquad \sigma_T = -<(\mathbf{X}_a)_T,\ (\mathbf{X}_a)_T>\,,$$

we get after a short calculation

$$(7.13)\qquad \psi_4 = -<d^2x(\mathbf{p}),\ d^2x(\mathbf{p})> = (\sigma_T+\sigma_H)\,ds^4 - \alpha^2 ds^2.$$

Proceeding in similar manner, we finally reach the general remarkable formula:

$$(7.14)\qquad \psi_{2k} = (\sigma_T+\sigma_H)\,ds^{2k} - \alpha^2 ds^{2k-2}.$$

We shall call a manifold whose Bompiani's invariant of order k has the above form, a metric *recurrent manifold*. The manifold V_S being totally geodesic one may consider the Gauss map associated with $x: V\to V_S$ [10]. The third fundamental form III corresponding to this Gauss map is, according to the general theory [10]:

$$(7.15)\qquad \mathrm{III} = \Sigma_{l,i}(\alpha_i^l)^2.$$

Referring to (4.5) and (7.1) one finds

$$\text{III} = -\sigma_{II} ds^2 \tag{7.16}$$

that is III is *conformal* to ds^2.

But the immersion X being umbilical, it follows from (7.16) and from a result of M. Obata [10] that the manifold V *is Einsteinian*.

We recall now that being given two tangent vectors X and Y over a manifold M, the *„shape operator"* $S_X(Y)$ is the liniar map $S_X : T(M) \to T^{\perp}(M)$ defined by [11]:

$$S_X(Y) = l_{ij}^m X^i Y^j \xi_m, \tag{7.17}$$

(i, j and m are tangential and normal indices respectively). Let us consider the restrictions to V of the spatial vector fields $\mathbf{X}_a$ and $\mathbf{X}_\alpha$ (see (4.6) and (5.5)).

With the help of (4.5) one finds $SX_a(\mathbf{X})_a = \gamma \mathbf{H}$ where $\gamma = < \mathbf{X}_a . \mathbf{X}_\alpha >$. We formulate the

Theorem. *Let $x : V \to V_S$ be an immersion of a certain spatial manifold V (of even dimension) in the spatial manifold V_S which defines a bi-recurrent Lorentzian manifold $\mathbf{L}_r$. Suppose that V_S is structured by an almost flat connection and that α, $\mathbf{X}_a$ and $\mathbf{X}_\alpha$ are the principal 1-form over V_S, the associated vector field and the dual vector field with respect to the conformal symplectic form of V_S.*

Then

(i) *the immersion x is umbilical and all Lipschitz-Killing curvatures are positive;*

(ii) *the mean curvature vector field* $\mathbf{H}$ *associated with x is the normal component of the restricted vector field $\mathbf{X}_a$ on V;*

(iii) *the necessary and sufficient condition that* $\mathbf{H}$ *be concurrent over V is that it be parallel in the normal bundle;*

(iv) *V is a metric recurrent manifold;*

(v) *V is Einsteinian;*

(vi) *the shape operator of the induced vector fields X_a and X_α on V is conformal to* $\mathbf{H}$.

§8. We shall make now some considerations when the bi-concurrent Lorentzian manifold is a general space-time ($n = 1$). To this purpose we shall use the *complex vectorial formalism* constructed in [12]. This formalism is based on the local isomorphism

$$g : L(4) \to SO^3(C),$$

where $L(4)$ is the four dimensional Lorentz group and $SO^3(C)$ is the three dimensional complex group. By means of the transformations

$$\xi_1 = \mathbf{h}_1,\ \xi_3 = \frac{\mathbf{h}_2 + \mathbf{h}_3}{\sqrt{2}},\quad i\xi_2 = \frac{\mathbf{h}_2 - \mathbf{h}_3}{\sqrt{2}},\ \xi_4 = \mathbf{h}_4,\ i = \sqrt{-1}. \tag{8.1}$$

one obtains a Sachs frame $\{\mathbf{p}, \mathbf{h}_A;\ A = 1, \ldots, n\}$ or a null tetrad [12]. Let $\{\theta^A\}$ be the dual basis of $\{\mathbf{h}_A\}$. There exists an isomorphism between the 2-forms $\theta^A \wedge \theta^B$ of the six dimensional space $L^* \wedge^{(2)}$ and the self-dual forms $Z^\alpha (\alpha = 1, 2, 3)$ of the complex space C^3. One takes as basis of C^3

$$Z^1 = \theta^3 \wedge \theta^4,\quad Z^2 = \theta^1 \wedge \theta^3,\quad Z^3 = -\frac{1}{2}(\theta^1 \wedge \theta^4 - \theta^2 \wedge \theta^3) \tag{8.2}$$

and their complex conjugates

$$\overline{Z}^1 = \theta^2 \wedge \theta^4,\quad \overline{Z}^2 = \theta^1 \wedge \theta^3,\quad \overline{Z}^3 = -\frac{1}{2}(\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^3). \tag{8.3}$$

The Riemannian connection defined on $\mathbf{L}^4$ by the metric

$$< d\mathbf{p}, d\mathbf{p} > = 2(\theta^1\theta^4 - \theta^2\theta^3) \tag{8.4}$$

is determinated by a 1-form σ with values in $SO^3(C)$, such that

$$d \wedge Z^1 = \frac{1}{2}\sigma_3 \wedge Z^1 - \sigma_2 \wedge Z^3,\quad d \wedge Z^2 = \frac{1}{2}\sigma_3 \wedge Z^3 + \sigma_1 \wedge Z^3, \tag{8.5}$$

$$d \wedge Z^3 = \frac{1}{2}\sigma_1 \wedge Z^1 - \frac{1}{2}\sigma_2 \wedge Z^2,$$

where

$$\sigma_\alpha = \sigma_{\alpha_A}\theta^A,\quad \overline{\sigma}_\alpha = \overline{\sigma}_{\alpha_A}\theta^A \tag{8.6}$$

(the coefficients σ_{α_A} and $\overline{\sigma}_{\alpha_A}$ correspond to the spinorial coefficients of Newmann and Penrose). With respect to σ, the manifold $\mathbf{L}^4$ is structured by the following equations;

$$\begin{aligned}
\nabla \mathbf{h}_1 &= \frac{1}{4}(\sigma_3 + \overline{\sigma}_3) \otimes \mathbf{h}_1 + \frac{\overline{\sigma}_2}{2} \otimes \frac{\mathbf{h}_2}{2} + \frac{\sigma_2}{2} \otimes \frac{\mathbf{h}_3}{2}, \\
\nabla \mathbf{h}_2 &= -\frac{\overline{\sigma}_1 \otimes \mathbf{h}_1}{2} + \frac{\overline{\sigma}_3 - \sigma_3}{4} \otimes \mathbf{h}_2 + \frac{\sigma_2}{2} \otimes \frac{\mathbf{h}_4}{2}, \\
\nabla \mathbf{h}_3 &= -\frac{\sigma_1}{2} \otimes \mathbf{h}_1 - \frac{\overline{\sigma}_3 - \sigma_3}{4} \otimes \mathbf{h}_3 + \frac{\overline{\sigma}_2}{2} \otimes \frac{\mathbf{h}_4}{2}, \\
\nabla \mathbf{h}_4 &= -\frac{\sigma_1}{2} \otimes \mathbf{h}_2 - \frac{\overline{\sigma}_1}{2} \otimes \mathbf{h}_3 + \frac{\sigma_3 - \overline{\sigma}_3}{4} \otimes \mathbf{h}_4.
\end{aligned} \tag{8.7}$$

The curvature 2-forms Σ_3 of the complex space C^3 are expressed by

$$\Sigma_1 = d \wedge \sigma_1 + \frac{1}{2}\sigma_1 \wedge \sigma_2,\ \Sigma_2 = d \wedge \sigma_2 - \frac{1}{2}\sigma_2 \wedge \sigma_3, \tag{8.8}$$

$$\Sigma_3 = d \wedge \sigma_3 + \sigma_1 \wedge \sigma_2.$$

With respect to the basis Z^α, $\overline{Z}^\alpha$, Σ_α are expanded as

$$\Sigma_\alpha = C_{\alpha\beta} Z^\beta + E_{\alpha\beta}\overline{Z}^\beta. \tag{8.9}$$

In (8.9) $C_{\alpha\beta}$ are the components of a Weyl symmetric tensor [12] and $E_{\alpha\beta}$ the components of Einstein's tensor.

Referring now to (8.1), we see at once that the isotropic vectors $\mathbf{h}_1$ and $\mathbf{h}_4$ are real. The vector $\mathbf{h}_4$ is *time advanced* [12] and is also called the *distinguished isotropic vector* of the Sachs basis. The case when $\mathbf{h}_4$ is recurrent has been investigated by R. Debever and J. Spelkens [13].

We shall now consider when *both* $\mathbf{h}_1$ and $\mathbf{h}_4$ are recurrent, that is by virtue of (8.1), L^4 is *bi-recurrent*.

It readily follows from (8.7) that

$$\sigma_1 = \sigma_2 = 0 \Rightarrow \overline{\sigma}_1 = \overline{\sigma}_2 = 0, \tag{8.10}$$

and therefore by the third equation (8.5), one has

$$d \wedge Z^3 = 0. \tag{8.11}$$

But, the 2-form Z^3 being of rank 4, equation (8.11) proves that Z^3 is *symplectic* [14]. Furthermore comparing equations (8.8) with (8.9) we see that all the components $C_{\alpha\beta}$ vanish. One concludes following [12] that the space-time L_4^r under consideration belongs to *type N* in Petrov's classification. So we have

Theorem. *Any bi-recurrent general space-time* $\mathbf{L}_r^4$, *is symplectic and belongs to type N in Petrov's classification.*

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ON SOME BOUNDARY VALUE PROBLEMS OF NONLOCAL ELASTICITY

BY

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1. Introduction. The incorporation of nonlocal interatomic forces into continuum mechanics has only a recent origin. However, the literature on this aspect of the field theory is extensive [1], [6], [7], [8], [10]. A review of various models of elastic bodies is given in [9], [13]. The nonlocal theories of continua take into account the distant effects of atomic, molecular and granular interactions. The papers [1], [3] prove that the nonlocal theory has many advantages for the study of molecular and atomic-scale effects.

The main purpose of the present paper is to establish the existence, uniqueness and general properties of regularity of solution for some boundary value problems in linear theory of nonlocal elastostatics. In this context, the existence theory of Fichera [4], [5] is extended to the linear integro-differential operator of nonlocal elasticity. Then, these results are used to prove the existence and uniqueness of solutions of the boundary value problems from the linear theory of nonlocal elastiticity.

A boundary value problem for the onedimensional nonlocal theory of elasticity was considered in [14]. A formulation of various boundary value problems in nonlocal elasticity is given in [12].

2. General existence theorems. Throughout this section we shall use basic concepts and results from [4]. For some additional concepts which we shall use in the paper, we shall give specific definitions.

Let A be a bounded domain of the real r-dimensional cartesian space X^r. Without any loss of generality we may assume that $\bar{A}$ is contained in the open square $Q: |x_k| < \pi$ ($k = 1, \ldots, r$).

Let m and n be non-negative integers. If $\varphi(x, z)$ is a function on $A \times A$, we say that $\varphi(x, z)$ is of class $C^{m,n}$ on $A \times A$, if $\varphi(x, z)$ is continuous on $A \times A$ and the functions

$$D_x^p D_z^q \varphi, \quad 0 \leqslant |p| \leqslant m, \quad 0 \leqslant |q| \leqslant n, \quad |p| + |q| \leqslant \max(m, n),$$

exist and are continuous on $A \times A$. The symbol D_x denote the differentiation vector with respect to the variable x. Finally, we write C^n for $C^{n,n}$ and C^∞ for $n \to \infty$.

Suppose that the $n \times n$ complex matrices $a_s(x)$ $(0 \leqslant |s| \leqslant 2m)$ be defined in A and that $b_s(x, z)$ $(0 \leqslant |s| \leqslant m)$ be defined in $A \times A$. Consider the linear matrix integro-differential operator

$$Lu \equiv \sum_{|s|=0}^{2m} a_s(x) D^s u + \sum_{|s|=0}^{m} \int_A b_s(x, z)\, D_z^s\, u(z) dz,$$

where D_z is the differentiation vector with respect to the variable z. This operator is an elliptic operator in A if, for any real non-zero r-vector ξ the following condition is satisfied

$$\det \sum_{|s|=2m} a_s(x)\, \xi^s \neq 0, \quad \forall x \in A.$$

If $a_s(x) \equiv \| a_s^{hk}(x) \|_{n \times n}$, $b_s(x, z) \equiv \| b_s^{hk}(x, z) \|_{n \times n}$, we denote by $\bar{a}_s(x) \equiv \| \bar{a}_s^{kh}(x) \|_{n \times n}$, $\bar{b}_s(x, z) \equiv \| \bar{b}_s^{kh}(z, x) \|_{n \times n}$ the adjoint matrices, respectively.