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LINEARIZED CONSTITUTIVE EQUATIONS FOR SIMPLE FLUIDS WITH MEMORY

BY

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1. Introduction. The aim of this paper is to construct a linear theory of simple fluids with memory, fully compatible with the thermodynamics principles [1], 96. It is found that $\mathbf{T}$, ψ, and q are independent of the histories of (θ, g), g and $(\mathbf{F}, \theta)$, respectively. Finally, two classes of steady flows are considered.

2. Linearized constitutive relations. accorind to the principle of equipresence, we consider the following constitutive relations for a simple fluid with memory

$$(\psi, \mathbf{q}) = (\underset{s=0}{\overset{\infty}{\psi, \mathbf{q}}})(\mathbf{F}^t(s),\ \theta^t(s),\ \mathbf{g}^t(s)\ ;\ \rho), \tag{2.1}$$

where ψ is the free energy, $\mathbf{q}$ is the heat flux, ρ is the density and $\mathbf{F}_t^t$, θ^t and $\mathbf{g}^t$ are, respectively, the relative deformation gradient, the temperature and the temperature gradient hystories. Using the objectivity and symmetry principles we get from (2.1):

$$(\psi \cdot \mathbf{q}) = (\underset{s=0}{\overset{\infty}{\psi, \mathbf{q}}})(\mathbf{G}^t(s),\ \theta^t(s),\ \tilde{\mathbf{g}}^t(s)\ ;\ \rho) = (\psi, \mathbf{q})(\Gamma^t\ ;\ \rho), \tag{2.2}$$

where $\mathbf{G}^t = \mathbf{F}_t^{t\mathrm{T}}\mathbf{F}_t^t - 1$, $\tilde{\mathbf{g}}^t = \mathbf{F}_t^{t\mathrm{T}}\mathbf{g}^t$ while ψ and $\mathbf{q}$ must satisfy

$$\underset{s=0}{\overset{\infty}{\psi}}(\mathbf{G}^t(s),\ \theta^t(s),\ \tilde{\mathbf{g}}^t(s)\ ;\ \rho) = \underset{s=0}{\overset{\infty}{\psi}}(Q\mathbf{G}^t(s)\,Q\mathrm{T},\ \theta^t(s),\ Q\tilde{\mathbf{g}}^t(s)\ ;\ \rho), \tag{2.3}$$

$$Q\,\underset{s=0}{\overset{\infty}{\mathbf{q}}}(\mathbf{G}^t(s), \theta^t(s),\ \tilde{\mathbf{g}}^t(s)\ ;\ \rho) = \underset{s=0}{\overset{\infty}{\mathbf{q}}}(Q\mathbf{G}^t(s)\,Q\mathrm{T}, \theta^t(s),\ Q\,\tilde{\mathbf{g}}^t(s)\ ;\ \rho),$$

for any orthogonal tensor $\mathbf{Q}$.

The idea that the material has a fading memory will be precised with the aid of an influence function $h(\,.\,)$ [1, 2]. It is clear that the collection of all measurable functions Γ^t for which [2])

$$\|\Gamma^t\| = \|\Gamma_r^t\|_h + |\Gamma^t(0)| = \left\{\int_0^\infty h(s)^2\,\Gamma_r^t(s)\,.\,\Gamma_r^t(s)\,ds\right\}^{1/2} + |\Gamma^t(0)|, \tag{2.4}$$

is finite, can be regarded as a Hilbert space $H = H_G \otimes H_\Theta \otimes H_g$.

We shall assume that there exists an influence function h, such that for each value of ρ, our constitutive functionals, considered as functions of Γ^t, are defined and continuously Fréchet differentiable with respect to the norm (2.4) in a neighbourhood $\mathcal{D}$ of the constant history $\Gamma_0^t = (0^t, 0_0^t, 0^t)$. We further assume that, for each fixed $\Gamma^t \in \mathcal{D}$ they are continuous and differentiable with respect to ρ.

1) $\mathbf{T}$, η and ε will be determined with the aid of ψ.

2) Here Γ_r^t is the past history of Γ^t, and the scalar product of two triples is $\Gamma_1 \cdot \Gamma_2 = \mathrm{tr}\,(\mathbf{G}_1\mathbf{G}_2) + \theta_1\theta_2 + \mathbf{g}_1 \cdot \mathbf{g}_2$.

These smoothness assumptions enable us to write the following Taylor expansions of ψ and $\mathbf{q}$ when $(\Gamma^t ; \rho) \in D \times v(\rho_0)$

$$\psi = \psi(\Lambda_0) + \partial_\rho \psi(\Lambda_0)(\rho - \rho_0) + \delta\psi(\Lambda_0 / \Gamma^t - \Gamma_0^t) + r_\psi(\Lambda_0, \Gamma^t ; \rho)$$

$$\mathbf{q} = \mathbf{q}(\Lambda_0) + \partial_\rho \mathbf{q}(\Lambda_0)(\rho - \rho_0) + \delta_q(\Lambda_0 / \Gamma^t - \Gamma_0^t) + r_q(\Lambda_0, \Gamma^t ; \rho) \tag{2.5}$$

where $\Lambda_0 = (\Gamma_0^t ; \rho_0)$, $\delta\psi = \delta\Gamma + \delta_\Gamma \psi$ [2] and

$$\lim_{\|\Gamma^t - \Gamma_0^t\| + |\rho - \rho_0| \to 0} \frac{r_\psi(\Lambda_0, \Gamma^t ; \rho) \text{ or } r_q(\Lambda_0, \Gamma^t ; \rho)}{\|\Gamma^t - \Gamma_0^t\| + |\rho - \rho_0|} = 0$$

From the conditions (2.3) we can easily obtain (taking $Q = -1$)

$$\partial_g \psi(\Lambda_0) = \delta_g \psi(\Lambda_0) = \mathbf{q}(\Lambda_0) = \partial_\rho \mathbf{q}(\Lambda_0) = \partial_\theta \mathbf{q}(\Lambda_0) = \delta_\theta \mathbf{q}(\Lambda_0) = \delta_G \mathbf{q}(\Lambda_0) = 0. \tag{2.6}$$

Now, let us use the fact $\delta_\Gamma \psi$ and $\delta_g \mathbf{q}$ are linear, i. e.

$$\delta_\Gamma \psi(\Lambda_0 / \Phi) = \int_0^\infty P(s) \cdot \Phi(s)\, h(s)^2\, ds, \qquad \Phi \in D,$$

$$\delta_g \mathbf{q}(\Lambda_0 / \Gamma) = \int_0^\infty \mathbf{R}(s)\, v(s)\, h(s)^2 ds, \quad v \in H_g \subset D, \tag{2.7}$$

where $P(\cdot) \in H$ and $\mathbf{R}(\cdot)$ is a tensor function on $(0, \infty)$ with components in H_Θ (see [3]). Using again (2.4) we can prove that

$$P(s) = (V(s)\,\mathbf{1},\ \lambda(s),\ 0),\ \mathbf{R}(s) = r(s)\,\mathbf{1},\ \partial_g \mathbf{q}(\Lambda_0) = k\,\mathbf{1}. \tag{2.8}$$

Let $\zeta(s)$, $k(s)$ and $\alpha(s)$ $(s \in [0, \infty))$ be unique solutions of the following initial value problemes

$$\frac{d\zeta(s)}{ds} = v(s)\, h(s)^2, \quad \frac{dk(s)}{ds} = -r(s)\, h(s)^2, \quad \frac{d\alpha(s)}{ds} = \lambda(s)\, h^2(s) \tag{2.9}$$

$$\zeta(0) = 0 \qquad k(0) = -k \qquad \alpha(0) = \partial_\theta \psi(\Lambda_0).$$

Neglecting r_ψ and r_q in (2.5) and using (2.6)—(2.9) we get

$$\psi = \psi_0 + \tilde{\psi}_0 \rho + \alpha(0)\theta + \int_0^\infty \alpha'(s)\, \theta^t(s)\, ds + \int_0^\infty \zeta'(s)\, tr G^t(s) ds,$$

$$\mathbf{q} = -k(0)\,\mathbf{g} - \int_0^\infty k'(s)\, \tilde{\mathbf{g}}^t(s)\, ds, \tag{2.10}$$

where $\psi_0 = \psi(\Lambda_0) - \tilde{\psi}_0 \rho_0 - \alpha(\infty)\,\theta_0$ and $\tilde{\psi}_0 = \partial_\rho \psi(\Lambda_0)$.

In the following, assuming that the body force and the heat supply can be determined such that the momentum and energy laws be satisfied, we shall find the restrictions imposed by the entropy inequality [1, 2]. So, in view of the equation of continuity and of the relation

$$\mathbf{G}^t(s) = -\frac{d}{ds}\mathbf{G}^t(s) - D\mathbf{G}^t(s) - \mathbf{G}^t(s)D + W\mathbf{G}^t(s) - \mathbf{G}^t(s)W - 2D,$$

where $\mathbf{W}$ is the spin tensor (see [1] § 24), we shall have

$$\mathbf{T} = \rho(\rho\psi_0 - 2\zeta(\infty))\mathbf{1} - 2\rho \int_0^\infty \zeta'(s)\, \mathbf{G}^t(s)\, ds,$$

$$\eta = -\alpha(0), \quad \sigma - \frac{1}{\rho\Theta}\, \mathbf{q} \cdot \mathbf{g} \geqslant 0, \quad \sigma \geqslant 0. \tag{2.11}$$

Here $\sigma = \int_0^\infty \alpha'(s)\, d\theta^t(s) + \int_0^\infty \zeta'(s)\, \mathrm{tr} d\mathbf{G}^t(s)$ is the internal dissipation per unit mass.

R e m a r k 1. If $(\Gamma^t ; \rho) = (0, \theta_0^t, \mathbf{g}_0^t ; \rho_0)$ then the generalized dissipation inequality $(2.11)_3$ reduces to the usual heat conduction inequality, which implies $k(\infty) = k(0) - \delta \mathbf{g}\mathbf{q}(\Lambda_0 / \mathbf{1}^t) > 0$.

R e m a r k 2. From (2.10) and (2.11) we see that: ψ is not influenced by the history of g, q by the histories of θ and $\mathbf{G}$, T by the histories of θ and $\mathbf{g}$ and η is a constant.

R e m a r k 3. In the linear theory of a simple fluid with memory, the problem of the determination of a motion, reduces to a system of differentiable and functional equations in $\mathbf{G}^t$ and ρ and to a coupled equation in θ^t, $\mathbf{G}^t$ and ρ.

R e m a r k 4. In the case of an incompressible fluid, ρ is constant and

$$\mathbf{T} = -p\,\mathbf{1} - 2\rho \int_0^\infty \zeta'(s)\, \mathbf{G}^t(s)\, ds, \tag{2.12}$$

where p is an indeterminate pressure.

3. Steady flows of incompressible fluids. In this section we shall discuss two classes of steady flows whose velocity fields in Cartesian and cylindrical coordinates are

$$\dot{x} = 0, \quad \dot{y} = v(x, z), \quad \dot{z} = 0, \tag{3.1}$$

$$\dot{r} = 0, \quad \dot{\theta} = w(r), \quad \dot{z} = v(r). \tag{3.2}$$

The first flow can be obtained between three walls, which are at rest and the second one, is a helical flow between two infinite coaxial cylinders which rotate around their axis. Assuming that the fluid adheres to the walls, the boundary conditions can be easy fixed.

Following [4], we can reduce the problem of the determination of the flow (3.1) to two equations of Laplace and Poisson types. The problem of the determination of the flow (3.2) can be completely solved, obtaining exact solutions. In each case the material constants can be determined by experiments.

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