Elle est continue et la démonstration du théorème se ramène à trouver, pour tout voisinage ouvert U de $j^r(f' \circ f)$, des voisinages ouverts V de $j^r f$ et W de $j^r f'$ tels que $c(V x_Z W) \subset U$. En remarquant que $\pi | j^r f = f$ est propre et que $\pi' | j^r f' = \mathrm{Id}_Z$ l'est aussi, le lemme suivant nous donne la réponse voulue :

Lemme. *Soient A, B et P trois espaces métriques complets, π et π' deux applications continues respectivement de A dans P et de B dans P. Soient K une partie de A et L une partie de B telles que $\pi | K$ et $\pi' | L$ soient propres et donnons-nous un voisinage U de $K x_P L$ dans $D = A x_P B$. Alors il existe un ouvert V, voisinage de K dans A, et un ouvert W, voisinage de L dans B, tels que : $V x_P W \subset U$.*

4.3 Théorème. *Soient Y et Z deux variétés banachiques de classe C^r et X une sous-variété (fermée) de classe C^r de Z. Alors l'application restriction χ de $\mathrm{Hom}^r(Z, Y)$ dans $\mathrm{Hom}^r(X, Y)$ qui à g associe g/X est continue si on munit les deux espaces d'applications de la topologie $\mathcal{C}^r$.*

Il suffit de voir que χ se décompose de la manière suivante :

$$\begin{array}{ccccc} \mathrm{Hom}^r(Z, Y) & \to & \mathrm{Hom}^r_p(X, Z) \times \mathrm{Hom}^r(Z, Y) & \xrightarrow{1} & \mathrm{Hom}^r(X, Y) \\ g & \to & (i, g) & \longrightarrow & g \circ i \end{array}$$

où i est l'injection de X dans et on applique le théorème précédent.

5. Conclusion. Nous avons ainsi construit une topologie qui nous permet d'ignorer l'hypothèse de locale compacité et d'avoir néanmoins des théorèmes d'approximation très utiles dans l'étude de certains systèmes hydrodynamiques.

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Reçu le 24. III. 1975

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Analele ştiinţifice ale Universităţii „AL. I. CUZA" din Iaşi
Tomul XXII, s. I a, 1976, f. 2

ON SOME NONLINEAR VOLTERRA INTEGRODIFFERENTIAL EQUATIONS

BY

B. G. PACHPATTE

1. Introduction. In recent years much attention has been given to the study of nonlinear Volterra integral and integrodifferential equations. Excellent accounts on this subject may be found in two recent monographs by R. K. Miller [6] and C. Corduneanu [1]. The first one is a pure mathematics product while the second emphasizes the significance of these equations in the theory of feedback systems. The purpose of this paper is to study the behavior of solutions of a Volterra integrodifferential system of the form

$$x'(t) = Ax(t) + \int_0^t B(t-s)\,x(s)ds + x(t)\,H(t, x(t), \sigma(t)), \quad x(0) = x_0, \tag{1}$$

with

$$\sigma(t) = f(t) + \int_0^t k(t, s, x(s))ds, \tag{2}$$

where $0 \leqslant t < \infty$, A and $B(t)$ are $n \times n$ matrices. The system will be studied as a perturbation of the linear system

$$y'(t) = Ay(t) + \int_0^t B(t-s)\,y(s)ds, \quad y(0) = x_0. \tag{3}$$

Here x, y, H, σ, f, k are vector functions with n components. The allowable perturbations $xH(t, x, \sigma)$ are to be thought of as being small and motivated by recent studies of Corduneanu [1], [2], Grossman and Miller [3], [4], Keller [5], Miller [6], [7], Nohel [8], and Pachpatte [9]—[12].

The mathematical literature provides a good deal of information concerning the existence, uniqueness and continuous dependence of solutions of various classes of Volterra integral and integrodifferential equations, see, [1], [3]—[6], [8]. We therefore turn our attention to other problems. One of the problems of interest that arise in this connection is concerned with the question of whether the stability, boundedness and other problems of the system (1) are shared with the system (3). The present paper takes up this question when the perturbation satisfy certain conditions. Although such problems have been considered by various investigators (see, [1]—[8]), the results in the present paper are of interest because of the weak assumptions on the functions involved and the approach to the problem is different from those of the cited results.

2. Preliminaries. Let R^n denote real n-dimensional Euclidean space of column vectors with Euclidean norm $| \cdot |$. R^+ denotes the set of all t such that $0 \leqslant t < \infty$. For p in the interval $1 \leqslant p < \infty$, L^p is the usual Lebesgue space of measurable functions g such that

$$\| g \|_p = \left(\int_0^\infty | g(t) |^p \, dt \right)^{\frac{1}{p}} < \infty .$$

LL^p is the set of all functions which are locally L^p on R^+. Let $C[X, Y]$ denote the space of continuous functions from X to Y, where X and Y are convenient spaces. We shall assume that $H \in C[R^+ \times R^n \times R^n, R^n]$, $f(t) \in C[R^+, R^n]$, and $k \in C[R^+ \times R^+ \times R^n, R^n]$.

The differential resolvent $R(t)$ associated with the linear system (3) is the unique solution of the matrix equation

$$R'(t) = AR(t) + \int_0^t B(t-s)\, R(s) ds, \quad R(0) = I,$$

where I is the identity matrix.

If $B(t)$ is locally L^1 on R^+, then (1) may be written in the equivalent variation of constants form

$$x(t) = R(t)x_0 + \int_0^t R(t-s)\, x(s) H(s, x(s), \sigma(s))ds, \; t \geqslant 0.$$

See, Grossman and Miller [3] for details.

Let us remark that the equation (1)–(2) when $xH(t, x, \sigma) = \Phi(\sigma)$, $\sigma = \langle c, x \rangle$ has been investigated by Corduneanu [2] by using frequency technique. Very interesting contributions concerning the frequency technique and stability theory for integral equations have been made in a recent book by C. Corduneanu [1, Chapter 3].

In order to establish our main results in this paper we require the following integral inequality which is new to the literature.

Theorem 1. *Let $x(t)$, $p(t)$, $q(t)$, and $h(t)$ be real-valued nonnegative continuous functions defined on R^+, for which the inequality*

$$x(t) \leqslant x_0 + \int_0^t p(s)x(s)\left(x(s) + h(s) + \int_0^s q(\tau)x(\tau)d\tau\right)ds, \; t \in R^+, \tag{4}$$

holds, where x_0 is a positive constant. Then

$$x(t) \leqslant x_0 \exp\left(\int_0^t p(s)\left\{h(s) + \left[x_0 \exp\left(\int_0^s (p(\tau)h(\tau) + q(\tau))d\tau\right)\right]\Big/ \left[1 - x_0\int_0^s p(\tau)\exp\left(\int_0^\tau (p(k)h(k) + q(k))\,dk\right)d\tau\right]\right\}ds\right), \tag{5}$$

for all $t \in R^+$.

Proof. Define a function $m(t)$ by the right member of (4). Then

$$m'(t) = p(t)x(t)\left(x(t) + h(t) + \int_0^t q(\tau)\, x(\tau)\, d\tau\right), \; m(0) = x_0,$$

which in view of (4) implies

$$m'(t) \leqslant p(t)\, m(t)\left(m(t) + h(t) + \int_0^t q(\tau)\, m(\tau)\, d\tau\right). \tag{6}$$

If we put

$$v(t) = m(t) + \int_0^t q(\tau)m(\tau)d\tau, \quad v(0) = m(0) = x_0, \tag{7}$$

it follows from (6), (7) and the fact that $m(t) \leqslant v(t)$, the inequality

$$v'(t) \leqslant (p(t)\, h(t) + q(t))\, v(t) + p(t)v^2(t), \tag{8}$$

is satisfied. The inequality (8) can be written as

$$v^{-2}(t)\, v'(t) - (p(t)\, h(t) + q(t)^{-1}v(t) \leqslant p(t). \tag{9}$$

Put $v^{-1}(t) = n(t)$, so that $v^{-2}(t)\, v'(t) = -n'(t)$, and $n(0) = x_0^{-1}$, then we obtain

$$n'(t) + (p(t)\, h(t) + q(t))\, n(t) \geqslant -p(t), \tag{10}$$

which implies the estimation for $n(t)$ such that

$$n(t) \exp\left(\int_0^t (p(s)h(s) + q(s))ds\right) \geqslant n(0) - \int_0^t p(s)\exp\left(\int_0^t (p(\tau)\, h(\tau) + q(\tau))\, d\tau\right) ds.$$

Now, substituting $n(t) = v^{-1}(t)$ in the above inequality, we have

$$v(t) \leqslant \left[x_0 \exp\left(\int_0^t (p(s)\, h(s) + q(s))\, ds\right)\right]\Big/\left\{1 - x_0\int_0^t p(s)\exp\left(\int_0^s (p(\tau)\, h(\tau) + q(\tau))\, d\tau\right) ds\right\}$$

since $n(0) = 1/x_0$. Then, from (6), we have

$$m'(t) \leqslant p(t)\, m(t)\left\{h(t) + \left[x_0 \exp\left(\int_0^t (p(s)h(s) + q(s))ds\right)\right]\Big/\left[1 - x_0\int_0^t p(s)\exp\left(\int_0^s (p(\tau)\, h(\tau) + q(\tau))d\tau\right) ds\right]\right\},$$

which implies the estimate for $m(t)$ such that

$$m(t) \leqslant x_0 \exp\left\{\int_0^t p(s)\left[h(s) + \left[x_0 \exp\left(\int_0^\tau (p(\tau)\, h(\tau) + q(\tau))d\tau\right)\right]\Big/\left[1 - x_0\int_0^t p(\tau)\exp\left(\int_0^\tau (p(k)\, h(k) + q(k))\, dk\right) d\tau\right]\right] ds\right\}.$$

Now, substituting the value of $m(t)$ in (4) we obtain the desired bound in (5).

We point out that inequalities similar to that proved have been successfully applied by this author to study the behavior of solutions of some integral equations in [9]–[12]. The success of our results in this paper totally depend upon the development of a nonlinear variation of constants formula given by Grossman and Miller [3] and the integral inequality established in the above theorem.

3. Main Results. In this section we state and prove our main results on the behavior of solutions of (1)–(2). Our first theorem deals with the boundedness of the solutions of (1)–(2) under some suitable conditions on the perturbation term and on the solutions of (3).

Theorem 2. *Let $B \in LL^1(R^+)$ and suppose that the functions H and K in (1)–(2) satisfy*

$$|R(t-s)\,H(s,\,x(s),\,\sigma(s))| \leqslant p(s)\,(\,|x(s)| + |\sigma(s)|\,), \tag{11}$$

$$|k(t,\,s,\,x(s))| \leqslant q(s)\,|x(s)|\,, \tag{12}$$

for $0 \leqslant s \leqslant t < \infty$, where $R(t)$ is the resolvent associated with the linear system (3), $p,\,q \in C[R^+,\,R^+]$ and

$$\int_0^\infty p(s)\left[\,|f(s)| + \left[c\exp\left(\int_0^s (p(\tau)\,|f(\tau)| + q(\tau))\,d\tau\right)\right]\right/ \tag{13}$$
$$\left/\left[1 - c\int_0^s p(\tau)\exp\left(\int_0^\tau (p(k)\,|f(k)| + q(k))\,dk\right)d\tau\,\right]\right]ds < \infty,$$

where $c > 0$ is a constant. Then for every bounded solution $y(t)$ of (3) on R^+, the corresponding solution $x(t)$ of (1)–(2) is bounded on R^+.

Proof. By using the variation of constants formula developed by Grossman and Miller [3] any solution $x(t)$ of (1)–(2) is represented by the integral equation

$$x(t) = y(t) + \int_0^t R(t-s)\,x(s)\,H(s,x(s),\sigma(s))ds, \tag{14}$$

where $y(t) = R(t)x_0$ is the solution of the linear system (3). From (14) and (11) we obtain

$$|x(t)| \leqslant c + \int_0^t p(s)\,|x(s)|\,[\,|x(s)| + |\sigma(s)|\,]\,ds, \tag{15}$$

where $c > 0$ is the upper bound for $|y(t)|$. Further using (15), (2), and (12) we obtain

$$|x(t)| \leqslant c + \int_0^t p(s)\,|x(s)|\left[\,|x(s)| + |f(s)| + \int_0^s q(\tau)\,|x(\tau)|\,d\tau\right]ds.$$

Now an application of Theorem 1 yields

$$|x(t)| \leqslant c\exp\left(\int_0^t p(s)\left[\,|f(s)| + \left[c\exp\left(\int_0^s (p(\tau)\,|f(\tau)| + q(\tau))\,d\tau\right)\right]\right/\right.$$
$$\left.\left/\left[1 - c\int_0^s p(\tau)\exp\left(\int_0^\tau (p(k)\,|f(k)| + q(k))\,dk\right)d\tau\,\right]\right]ds\right).$$

The above estimation in view of the assumption (13) implies the boundedness of $x(t)$ on R^+ and the theorem is proved.

We note that Theorem 2 implies not only the boundedness, but the stability of the solution $x(t)$ of (1)–(2), if c is small enough.

Our next theorem shows that under some suitable conditions on the functions H and k and on the resolvent $R(t)$ associated with the linear system (3), all the solutions of (1)–(2) approach zero as $t \to \infty$.

Theorem 3. *Let $B \in LL^1(R^+)$ and the resolvent $R(t)$ associated with the linear system (3) verifies*

$$|R(t)| \leqslant Me^{-\alpha t}, \quad t \in R^+, \tag{16}$$

where M and α are positive constants. Suppose that the functions H and k in (1)–(2) satisfy

$$|H(t,\,x(t),\,\sigma(t))| \leqslant p(t)\,(\,|x(t)| + |\sigma(t)|\,), \quad t \in R^+, \tag{17}$$

$$|k(t,\,s,\,x(s))| \leqslant e^{-\alpha t}\,q(s)\,|x(s)|\,, \quad 0 \leqslant s \leqslant t < \infty, \tag{18}$$

where $\alpha > 0$ is a constant, and $p,\,q \in C[R^+,\,R^+]$ such that

$$\int_0^\infty Mp(s)\,e^{-\alpha s}\left[\,|f(s)|\,e^{\alpha s} + \left[M\,|x_0|\exp\left(\int_0^s (Mp(\tau)\,|f(\tau)| + q(\tau)e^{-\alpha\tau})d\tau\right)\right]\right/ \tag{19}$$
$$\left/\left[1 - M\,|x_0|\int_0^s Mp(\tau)\,e^{-\alpha\tau}\exp\left(\int_0^\tau (M\,p(k)\,|f(k)| + q(k)\,e^{-\alpha k})\,dk\right)d\tau\,\right]\right]ds < \infty,$$

where $x_0 \neq 0$ is a constant. Then all solutions of (1)–(2) approach zero as $t \to \infty$.

Proof. Using the variation of constants formula, any solution $x(t)$ of (1)–(2) is represented by the integral equation

$$x(t) = R(t)x_0 + \int_0^t R(t-s)\,x(s)\,H(s,\,x(s),\,\sigma(s))ds. \tag{20}$$

where $x_0 \neq 0$ is a constant. Using (20), (16), and (17) we obtain

$$|x(t)| \leqslant M\,|x_0|\,e^{-\alpha t} + \int_0^t Me^{-\alpha(t-s)}\,p(s)\,|x(s)|\,[\,|x(s)| + |\sigma(s)|\,]\,ds. \tag{21}$$

Further using (21), (2), and (18) we obtain

$$|x(t)| \leqslant M\,|x_0|\,e^{-\alpha t} + \int_0^t Mp(s)\,e^{-\alpha(t-s)}\,p(s)\,|x(s)|\,[\,|x(s)| + |f(s)| + e^{-\alpha s}\int_0^s q(\tau)\,|x(\tau)|\,d\tau]\,ds.$$

The above inequality can be rewritten as

$$|x(t)|\,e^{\alpha t} \leqslant M|x_0| + \int_0^t M\,p(s)\,e^{-\alpha s}\,|x(s)|\,e^{\alpha s}\left[\,|x(s)|e^{\alpha s} + |f(s)|e^{\alpha s} + \int_0^s q(\tau)e^{-\alpha\tau}x(\tau)|e^{\alpha\tau}d\tau\,\right]ds.$$

Now applying Theorem 1 with $x(t) = |x(t)|\,e^{\alpha t}$, then multiplying by $e^{-\alpha t}$, we obtain

$$|x(t)| \leqslant M|x_0|e^{-\alpha t}\exp\left(\int_0^t Mp(s)e^{-\alpha s}\left[\,|f(s)|\,e^{\alpha s} + \left[M\,|x_0|\exp\left(\int_0^s (Mp(\tau)\,|f(\tau)| + \right.\right.\right.\right.$$
$$\left.\left.\left.\left. + q(\tau)\,e^{-\alpha\tau})\,d\tau\right)\right]\right/\left[1 - M\,|x_0|\int_0^s Mp(\tau)\,e^{-\alpha\tau}\exp\left(\int_0^\tau (Mp(k)\,|f(k)| + q(k)e^{-\alpha k})dk\right)d\tau\right]\right]ds\right).$$

The above estimation in view of the assumption (19) yields the desired result if we choose M and $|x_0|$ small enough, and the proof of the theorem is complete.

We note from [12] that a continuous function $z(t)$ is slowly growing if and only if for every $\alpha > 0$ there exists a constant M which may depend on α such that $|z(t)| \leq Me^{\alpha t}$, $t \geq 0$. Theorem 4 below demonstrates that, the solution of (1)–(2) grow more slowly than any positive exponential.

Theorem 4. *Let $B \in LL^1(R^+)$ and the resolvent $R(t)$ associated with the linear system (3) verifies $|R(t)| \leq Me^{\alpha t}$, $t \in R^+$, where M and α are positive constants. Suppose that the functions H and k in (1)–(2) satisfy*

$$|H(t, x(s), \sigma(s))| \leq p(t)(|x(s)| + |\sigma(s)|),\ t \in R^+,$$

$$|k(t, s, x(s))| \leq e^{\alpha t} q(s)|x(s)|, \quad 0 \leq s \leq t < \infty,$$

where $\alpha > 0$ is a constant, and $p, q \in C[R^+, R^+]$ such that

$$\int_0^\infty Mp(s)e^{\alpha s}\Big[|f(s)|e^{-\alpha s} + \Big[M|x_0|\exp\Big(\int_0^s (Mp(\tau)|f(\tau)| + q(\tau)e^{\alpha\tau})d\tau\Big)\Big]\Big/$$

$$\Big/\Big[1 - M|x_0|\int_0^s Mp(\tau)e^{\alpha\tau}\exp\Big(\int_0^\tau (Mp(k)|f(k)| + q(k)e^{\alpha k}\,dk\Big)d\tau\Big]\Big]ds < \infty,$$

where $x_0 \neq 0$ is a constant. Then all solutions of (1)–(2) are slowly growing.

The proof of this theorem follows by the similar argument as in the proof of Theorem 3 with suitable modifications, and hence we omit the details.

4. Applications. In this section we wish to apply the technique used in section 3 to study the integrodifferential system of the form

$$x'(t) = Ax(t) + \int_0^t B(t-s)x(s)\,ds + \int_0^t b(t-s)x(s)H(s, x(s), \sigma(s))ds,\ x(0) = x_0 \tag{22}$$

with

$$\sigma(t) = f(t) + \int_0^t k(t, s, x(s))ds, \tag{23}$$

as a perturbation of the integrodifferential system

$$y'(t) = Ay(t) + \int_0^t B(t-s)x(s)ds, \quad y(0) = x_0, \tag{24}$$

where the meaning of the quantities occurring above is the same as in the preceding sections, excepting $b(t)$ which now denotes a measurable vector function from the positive half-axis into R^n.

Let us remark that the equation (22)–(23) when $xH(t, x, \sigma) = \Phi(\sigma)$, $\sigma = \langle c, x\rangle$ has been investigated by C o r d u n e a n u [2] by using Popov's frequency method. The interested reader will find there an extensive study of the most salient and effective criteria of stability and asymptotic behavior for various classes of control systems.

The equation (22) leads easily to the equivalent variation of constants form

$$x(t) = R(t)x_0 + \int_0^t R(t-s)ds\int_0^s b(s-u)x(u)H(u, x(u), \sigma(u))du,$$

which can be also written as

$$x(t) = R(t)x_0 + \int_0^t\Big\{\int_0^{t-u} R(t-u-v)b(v)dv\Big\}x(u)H(u, x(u), \sigma(u))du. \tag{25}$$

Of course, we assume that the function $b(t)$ is sufficiently smooth in order to be able to change the order of integration.

Corresponding to the Theorem 2 given in section 3, we can state the following result regarding the boundedness of the solutions of (22)–(23).

Theorem 5. *Let B, $b \in LL^1(R^+)$ and*

$$\int_0^{t-s} |R(t-s-v)b(v)|\,dv \leq M < \infty,\ 0 \leq s \leq t < \infty,$$

where $M > 0$ is a constant and $R(t)$ is the resolvent associated with the linear system (24) Suppose that the functions H and k in (22)–(23) satisfy

$$|H(t, x(t), \sigma(t))| \leq p(t)(|x(t)| + |\sigma(t)|),\ t \in R^+,$$

$$|k(t, s, x(s))| \leq q(s)|x(s)|, \quad 0 \leq s \leq t < \infty,$$

where $p, q \in C[R^+, R^+]$ and

$$\int_0^\infty Mp(s)\Big[|f(s)| + \Big[c\exp\Big(\int_0^s (Mp(\tau)|f(\tau)| + q(\tau))d\tau\Big)\Big]\Big/$$

$$\Big/\Big[1 - c\int_0^s Mp(\tau)\exp\Big(\int_0^\tau (Mp(k)|f(k)| + q(k))dk\Big)d\tau\Big]\Big]ds < \infty,$$

where $c > 0$ is a constant. Then for every bounded solution $y(t)$ of (24) on R^+, the corresponding solution $x(t)$ of (22)–(23) is bounded on R^+.

The proof follows by using the variation of constants formula (25) and following the similar argument as in the proof of Theorem 2.

To this end we note that one can very easily state two more theorems on the behavior of solutions of (22)–(23) corresponding to Theorem 3 and 4 given in section 3 with suitable modifications. These theorems will not be given again since there are no new essential ideas to explain.

Finally, we shall present a simple example to illustrate our Theorem 3.

Example. Consider the integrodifferential equation

$$x'(t) = -2x(t) + \int_0^t \frac{2e^{-(t+s)}}{(1-e^{-2t})}x(s)ds + x(t)H(t, x(t), \sigma(t)),\ x(0) = x_0, \tag{26}$$

where $\sigma(t)$ is as defined in (2), a perturbation of the integrodifferential equation

$$y'(t) = -2y(t) + \int_0^t \frac{2e^{-(t+s)}}{(1-e^{-2t})}y(s)ds, \quad y(0) = x_0, \quad x_0 \neq 0. \tag{27}$$

Suppose that the functions H and k in (26)–(2) satisfy the hypotheses (17) and (18) of Theorem 3 with $\alpha = 1$. The resolvent kernel $R(t)$ associated with (27) is given by $R(t) = e^{-t}$, which clearly satisfies the condition (16) of Theorem 3.

The variation of constants formula can be employed in order to rewrite (26) in the form

$$x(t) = x_0 e^{-t} + \int_0^t e^{-(t-s)} x(s)\, H(s, x(s), \sigma(s))ds. \tag{28}$$

Using (28), (17), (2), (18) and following the similar argument as in the proof of Theorem 3 we obtain $|x(t)| \to 0$ as $t \to \infty$, and the conclusion of Theorem 3 is true.

In concluding this paper, we note that the method used above applies equally well to study the various classes of integral and integrodifferential equations recently studied by this author in [9]—[12]. As pointed by the authors in [3], [5] integrodifferential systems of the form considered in this paper occur in nuclear reactor dynamics with the perturbation term $xH(t, x, \sigma)$ having the form

$$x(t) \int_0^t c(t, s)\, x(s)ds \text{ or } \int_0^t c(t, s) x^2(s)ds.$$

A detailed discussion of the applications of such systems in nuclear reactor dynamics can be found in the papers by Grossman and Miller [3] and Keller [5].

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Received 24 IV. 1976

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Analele ştiinţifice ale Universităţii „AL. I. CUZA" din Iaşi
Tomul XXII, s. I a, 1976, f. 2

ON A DIFFERENCE EQUATION OF TYPE $1/n$

BY

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Introduction. It was shown earlier [4], that a purely analytic solution of the high order difference equation

$$\Delta z_n = \lambda \beta_n z_{n+r}, \quad r \geqslant 0 \tag{1.1}$$

with λ a parameter, can be obtained by a numerical method. Such a solution is found to be analytic up to the first zero of a fundamental integral function.

We now consider a particular case of (1.1) viz.,

$$\Delta z_n = \frac{\lambda}{n+s} z_{n+r}, \quad r,\ s \geqslant 0, \tag{1.2}$$

here referred to as an r-th order difference equation of type $1/n$. The analytic solution of (1.2) is obtained by a method already discussed [4]. This leads to the computation of the first zero of the fundamental integral which determines the boundary of the domain of analyticity.

2. The analytic solution. Let be:

$$z_n(\lambda) = \int_C (1 - e^{-t})^{n-1} \tau_\lambda(t)dt, = \tag{2.1}$$

$$= \int_C e^t \tau_\lambda(t) \frac{d}{dt} [(1 - e^{-t})^n]. \tag{2.2}$$

Then

$$n z_n(\lambda) = e^t(1 - e^t)^n \tau_\lambda(t) \Big|_C - \tag{2.3}$$

$$- \int_C (1 - e^{-t})^n (e^t \tau_\lambda(t))' dt.$$

Substituting (2.2) and (2.3) into (1.2) we have,

$$e^t(1 - e^{-t})^{n-1} [1 - e^{-t} - 1]\tau_\lambda(t) \Big|_C - \int_C \{[(1 - e^{-t})^n - (1 - e^{-t})^{n-1})](e^t \tau_\lambda(t))'\}dt +$$

$$+ \int_C [3(1 - e^{-t})^{n-1} - (s + 1)(1 - e^{-t})^{n-2} \lambda (1 - e^{-t})^{n+r+1}]\tau_\lambda(t)dt = 0.$$

For the choice of the contour C, we have