The variation of constants formula can be employed in order to rewrite (26) in the form

$$x(t) = x_0 e^{-t} + \int_0^t e^{-(t-s)} x(s)\, H(s, x(s), \sigma(s))ds. \tag{28}$$

Using (28), (17), (2), (18) and following the similar argument as in the proof of Theorem 3 we obtain $|x(t)| \to 0$ as $t \to \infty$, and the conclusion of Theorem 3 is true.

In concluding this paper, we note that the method used above applies equally well to study the various classes of integral and integrodifferential equations recently studied by this author in [9]–[12]. As pointed by the authors in [3], [5] integrodifferential systems of the form considered in this paper occur in nuclear reactor dynamics with the perturbation term $xH(t, x, \sigma)$ having the form

$$x(t)\int_0^t c(t, s)\, x(s)ds \quad \text{or} \quad \int_0^t c(t, s) x^2(s)ds.$$

A detailed discussion of the applications of such systems in nuclear reactor dynamics can be found in the papers by Grossman and Miller [3] and Keller [5].

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ON A DIFFERENCE EQUATION OF TYPE $1/n$

BY

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Introduction. It was shown earlier [4], that a purely analytic solution of the high order difference equation

$$\Delta z_n = \lambda \beta_n z_{n+r}, \quad r \geqslant 0 \tag{1.1}$$

with λ a parameter, can be obtained by a numerical method. Such a solution is found to be analytic up to the first zero of a fundamental integral function.

We now consider a particular case of (1.1) viz.,

$$\Delta z_n = \frac{\lambda}{n+s} z_{n+r}, \quad r, \; s \geqslant 0, \tag{1.2}$$

here referred to as an r-th order difference equation of type $1/n$. The analytic solution of (1.2) is obtained by a method already discussed [4]. This leads to the computation of the first zero of the fundamental integral which determines the boundary of the domain of analyticity.

2. The analytic solution. Let be:

$$z_n(\lambda) = \int_C (1 - e^{-t})^{n-1} \tau_\lambda(t)dt, \tag{2.1}$$

$$= \int_C e^t \tau_\lambda(t) \frac{d}{dt}[(1 - e^{-t})^n]. \tag{2.2}$$

Then

$$n z_n(\lambda) = e^t (1 - e^t)^n \tau_\lambda(t)\Big|_C - \int_C (1 - e^{-t})^n (e^t \tau_\lambda(t))' dt. \tag{2.3}$$

Substituting (2.2) and (2.3) into (1.2) we have,

$$e^t(1 - e^{-t})^{n-1}[1 - e^{-t} - 1]\tau_\lambda(t)\Big|_C - \int_C \{[(1 - e^{-t})^n - (1 - e^{-t})^{n-1})](e^t\tau_\lambda(t))'\}dt + \int_C [3(1 - e^{-t})^{n-1} - (s+1)(1 - e^{-t})^{n-2}\lambda(1 - e^{-t})^{n+r+1}]\tau_\lambda(t)dt = 0.$$

For the choice of the contour C, we have

$$(2.5)\qquad (1 - e^{-t})^{n-1}\, \tau_\lambda(t) \Big|_C = 0$$

while $\tau_\lambda(t)$ satisfies the differential equation

$$-\tau'_\lambda - \{(s+1)[1 - (1 - e^{-t})^{-1}] - \lambda(1 - e^{-t})^r\}\tau_\lambda = 0,$$

from which

$$(2.6)\qquad \tau_\lambda(t) = C_\lambda e^{\lambda t}(1 - e^{-t})^{s+1} e^{-\lambda\Psi(t)},$$

where

$$(2.7)\qquad \Psi(t) = \sum_{i=1}^{r} \frac{(1 - e^{-t})^i}{i}.$$

(2.5) becomes

$$C_\lambda e^{+\lambda t}(1 - e^{-t})^{n+s} e^{-\lambda\Psi(t)} \Big|_C = 0.$$

The last equality holds if $\lambda < i$, $\forall i \geqslant 0$, in particular for $\lambda < 0$, while $t \to \infty$. It is of course true if $t = 0$. Thus the desired path of integration is the open contour from 0 to $+\infty$ Hence the solution sought for is

$$(2.8)\qquad Z_n(\lambda) = C_\lambda \int_0^\infty e^{\lambda t}(1 - e^{-t})^{n+s} e^{-\lambda\Psi(t)}dt.$$

Putting $T = 1 - e^{-t}$ we have that

$$Z_n(\lambda) = C_\lambda \int_0^1 (1 - T)^{-(\lambda+1)}\, T^{n+s}\, e^{-\lambda\overline{\Psi}(t)}\, dT,$$

where

$$(2.9)\qquad \overline{\Psi}(T) = \sum_{i=1}^{r} T^i/i,$$

which gives an analytic solution, $z_n(\lambda)$ for $\lambda \leqslant -1$. Similarly, if we had, in (1.2), $\lambda = -\bar{\lambda}$, we would still obtain a solution similar to (2.8) but with λ replaced by $-\bar{\lambda}$ and subject to the analyticity condition $\bar{\lambda} > 0$. Equivalently we obtain (2.9) with $\lambda = -\bar{\lambda}$, subject to $\bar{\lambda} > 1$. In both of these latter cases the solution is obtained either in the right half plane or the left half plane.

3. Domain of analyticity. The above regions may be extended provided the first singularities, on either side of the origin in the λ-plane, are simple poles. The extent to which we can achieve this will depend partly on the value of s. Integrating (2.9), denoting the i^{th} derivative with respect to t by D^i, we have after s repeated integrations:

$$(3.1)\qquad Z_n(\lambda) = \frac{(-1)^s C_\lambda}{\lambda(\lambda-1)\dots(\lambda-s+1)} \int_0^1 (1 - T)^{-\lambda+s-1} D^s[T^{n+s} e^{-\lambda\overline{\Psi}}]dT,$$

from which we obtain

$$(3.2)\qquad Z_0(\lambda) = \frac{(-1)^s C_\lambda}{\lambda(\lambda-1)\dots(\lambda-s+1)} \int_0^1 (1 - T)^{-\lambda+s-1} D^s[T^s e^{-\lambda\overline{\Psi}}]\, dT$$

and by theorem 1 of [4] the solution of the corresponding iterative equation converges to

$$(3.3)\qquad y_n^{(\lambda)} = \frac{Z_n(\lambda)}{Z_0(\lambda)} = \left[\int_0^1 (1 - T)^{-\lambda+s-1} D^s[T^{n+s} e^{-\lambda\overline{\Psi}}]dT\right] \Big/ \int_0^1 (1 - T)^{-\lambda+s-1} D^s[T^s e^{-\lambda\overline{\Psi}(t)}]\, dT.$$

The inital conditions demand that $y_0(\lambda) = 1 = y_n(0)$ for $n \geqslant 0$. That these condition are met can be seen by inspection.

We see here that $Z_n(\lambda)$ can be continued analytically by removing s simple poles at integer points. This enables us, therefore, to extend the original region of existence of the solution. However, we note that this extension is limited by the value of s since we cannot integrate $Z_0(\lambda)$ more than s-times. Thus the existence of a convergent solution of the problem posed in (1.2), will depend on the first zeros of $Z_0(\lambda)$ on either side of the origin. The first zero of $Z_0(\lambda)$ in a particular half plane (depending on the sign of the right side of (1.2)) fundamentally determines the boundary of the half plane within which an analytic solution exists. We here take a particular case which will be followed by a table of computed zeros of the „fundamental integral", $Z_0(\lambda)$.

Case $r = 0$.

In (3.1) if $r = 0$, $\lambda > 0$, $n \geqslant 1$, then $\overline{\Psi}(T) \equiv 0$ and

$$Z_n(\lambda) = \frac{(-1)^s C_\lambda}{\lambda(\lambda-1)\dots(\lambda-s+1)} \int_0^1 (1 - T)^{-\lambda+s-1} D^s[T^{n+s}]dT =$$

$$= \frac{(-1)^s C_\lambda (n+s)!}{n!\,\lambda(\lambda-1)\dots(\lambda-s+1)} \int_0^1 (1 - T)^{-\lambda+s-1}\, T^n\, dT$$

while

$$Z_0(\lambda) = \frac{(-1)^s C_\lambda s!}{\lambda(\lambda-1)\dots(\lambda-s+1)} \int_0^1 (1 - T)^{-\lambda+s-1}dT =$$

$$= \frac{(-1)^s C_\lambda\, s!}{\lambda(\lambda-1)\dots(\lambda-s+1)(s-\lambda)}.$$

Therefore, the analytic solution of the equation

$$\Delta y_{n-1} = \frac{\lambda}{n+s}\, y_n,\quad y_n(0) = 1$$

when it exists, will be given by

$$y_n(\lambda) = \frac{(n+s)!\,(s-\lambda)}{n!\,s!} \int_0^1 (1 - T)^{-\lambda+s-1} T^n\, dT$$

Furthermore it is worthy of note that when $r > 0$, the region of convergence of the solution will be further reduced due to the presence of the exponential term $e^{-\lambda\overline{\Psi}(T)}$ in the integrand. As far as the practicability of this method of obtaining an analytic solution is concerned, we note that the limitation imposed here by s does not necessarily imply that the region, D_{λ}, with radius R, obtained here cannot be further extended. In fact, if we transform the plane into a circle, there will exist some positive number, ε, say, such that the solution still exists and in fact converges inside $|\lambda| < R + \varepsilon$. Thus in the case of $r = 0$, we have that $|\lambda| < R + \varepsilon = s + \varepsilon = 1$. This therefore suggests that a better estimate of the region D_{λ}, may be obtained when the solution is constructed fromally. The table I below shows a few of the first zeros of the integral

$$Z_0(\lambda) \equiv \int_0^1 (1 - T)^{-\lambda+s-1} D^s[T^s e^{-\lambda \overline{T}(T)}]dT$$

for different values of r and s. These indicate the boundary of domain of analyticity for the solution of (1.2) for the values of r and s indicated.

TABLE I

First zeros of $Z_0(\lambda)$ for $(0,0) \leqslant (r, s) \leqslant (8, 5)$

s \ r	1	2	3	4	5	6	7	8
0	0.6017	0.4905	0.4339	0.3983	0.3733	0.3545	0.3397	0.3277
1	1.0000	0.7655	0.6531	0.5847	0.5379	0.5036	0.4769	0.4556
2	1.3507	0.0000	0.8351	0.7358	0.6704	0.6219	0.5848	0.5553
3	1.6809	1.2155	1.0000	0.8738	0.7887	0.7271	0.6801	0.6428
4	2.0000	1.4207	1.1569	1.0000	0.8989	0.8246	0.7680	0.7234
5	2.3093	1.6187	1.3067	1.1247	1.0000	0.9169	0.8512	0.7993

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LYAPUNOV STABILITY OF CONTINUOUS SOLUTIONS OF SOME FUNCTIONAL EQUATIONS

BY

KAROL BARON and ROMAN GER

Here we are looking for a stability of the functional equation

(1) $$\varphi(x) = h(x, \varphi[f_1(x)], \varphi[f_2(x)], \ldots)$$

with an unknown function φ. We start from

Theorem 1. *Suppose that* $h: X \times Y^N \to Y$ (*) *and* $f_n: X \to X$, $n \in N$, *where* X *is a closed and convex subset of a finite dimensional Banach space normed by* $\|\cdot\|$, *whereas* Y *is a metric space with metric* σ, *fulfil*

$$\sigma(h(x, y), h(x, \overline{y})) \leqslant \sum_{n=1}^{\infty} \gamma_n(x)\sigma(y_n, \overline{y}_n), \quad (x, y, \overline{y}) \in X \times Y^N \times Y^N$$

and

$$\sup\{\|f_n(x) - \xi\| : n \in N\} < \|x - \xi\|, \; x \in X \setminus \{\xi\},$$

with functions $\gamma_n: X \to [0, \infty)$ *and a* $\xi \in X$, *respectively. If*

$$\sum_{n=1}^{\infty} \gamma_n(x) \leqslant 1, \quad x \in X$$

and the family $\{f_n: n \in N\}$ *is locally equicontinuous, then for every two solutions* $\varphi_\omega: X \to Y$ $\omega = 1, 2$, *of* (1) *continuous at* ξ, *the inequality*

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant \sigma(\varphi_1(\xi), \varphi_2(\xi)), \quad x \in X$$

holds.

Proof. (cf. the proofs of Lemmas 2 and 3 in [3]). Let $\varphi_\omega: X \to Y$, $\omega = 1, 2$, be two solutions of (1) continous at ξ and let us take an arbitrary positive real number ε. Then

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant \sum_{n=1}^{\infty} \gamma_n(x)\,\sigma(\varphi_1[f_n(x)], \varphi_2[f_n(x)]), \quad x \in X$$

and there exists a neighbourhood U of ξ such that

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant \sigma(\varphi_1(\xi), \varphi_2(\xi)) + \varepsilon, \quad x \in U.$$

Therefore, by Lemma 1 in [3]

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant \sigma(\varphi_1(\xi), \varphi_2(\xi)) + \varepsilon, \; x \in X.$$

(*) By Y^N, where N is the set of all positive integers, we denote the set of all sequences with elements from Y. Moreover, if y is a member of Y^N then y_n will denote the n-th coordinate of y.