the region, D_λ, with radius R, obtained here cannot be further extended. In fact, if we transform the plane into a circle, there will exist some positive number, ε, say, such that the solution still exists and in fact converges inside $|\lambda| < R + \varepsilon$. Thus in the case of $r = 0$, we have that $|\lambda| < R + \varepsilon = s + \varepsilon - 1$. This therefore suggests that a better estimate of the region D_λ, may be obtained when the solution is constructed fromally. The table I below shows a few of the first zeros of the integral

$$Z_0(\lambda) \equiv \int_0^1 (1 - T)^{-\lambda+s-1} D^s[T^s e^{-\lambda \overline{T}(T)}]dT$$

for different values of r and s. These indicate the boundary of domain of analyticity for the solution of (1.2) for the values of r and s indicated.

TABLE I

First zeros of $Z_0(\lambda)$ for $(0,0) \leqslant (r, s) \leqslant (8, 5)$

s \ r	1	2	3	4	5	6	7	8
0	0.6017	0.4905	0.4339	0.3983	0.3733	0.3545	0.3397	0.3277
1	1.0000	0.7655	0.6531	0.5847	0.5379	0.5036	0.4769	0.4556
2	1.3507	0.0000	0.8351	0.7358	0.6704	0.6219	0.5848	0.5553
3	1.6809	1.2155	1.0000	0.8738	0.7887	0.7271	0.6801	0.6428
4	2.0000	1.4207	1.1569	1.0000	0.8989	0.8246	0.7680	0.7234
5	2.3093	1.6187	1.3067	1.1247	1.0000	0.9169	0.8512	0.7993

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BIBLIOGRAPHY

1. Cartwright, M. L. — *The zeros of certain integral functions*, Quarterly J. Math. (Oxford), 1 (1930), 38—59.
2. Harris, W. A., Jr. — *Linear systems of difference equations with a regular singularity*, Proc. Amer. Math. Soc., 19 (1968), 1228—1236.
3. Israpilov, R. B. — *The summation of power series at isolated points outside the circle of convergence*, Sakharth, SSR Meca. Akad. Moambe, 54 (1969), 15—16.
4. Jaiyesimi, S. B. — *On the analytic solution of certain linear difference equations*, An. şt. Univ. Iaşi, (to appear).
5. Makinouchi, S. — *Notes on recurrence techniques for calculating the Bessel functions*, Information processing in Japan, 6 (1966), 47—58.
6. Miller, K. S. — *On linear difference equations*, Amer. Math. Monthly, 75 (1968), 630—632.
7. Milne-Thomson, L. M. — *The calculus of finite Differences*, Macmillan Co., London, 1960.

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LYAPUNOV STABILITY OF CONTINUOUS SOLUTIONS OF SOME FUNCTIONAL EQUATIONS

BY

KAROL BARON and ROMAN GER

Here we are looking for a stability of the functional equation

$$\varphi(x) = h(x, \varphi[f_1(x)], \varphi[f_2(x)], \dots) \tag{1}$$

with an unknown function φ. We start from

Theorem 1. *Suppose that* $h: X \times Y^{\mathrm{N}} \to Y$ (*) *and* $f_n: X \to X$, $n \in \mathrm{N}$, *where* X *is a closed and convex subset of a finite dimensional Banach space normed by* $\|\cdot\|$, *whereas* Y *is a metric space with metric* σ, *fulfil*

$$\sigma(h(x, y), h(x, \bar{y})) \leqslant \sum_{n=1}^{\infty} \gamma_n(x)\sigma(y_n, \bar{y}_n), \quad (x, y, \bar{y}) \in X \times Y^{\mathrm{N}} \times Y^{\mathrm{N}}$$

and

$$\sup\{\|f_n(x) - \xi\| : n \in \mathrm{N}\} < \|x - \xi\|, \; x \in X \setminus \{\xi\},$$

with functions $\gamma_n: X \to [0, \infty)$ *and a* $\xi \in X$, *respectively. If*

$$\sum_{n=1}^{\infty} \gamma_n(x) \leqslant 1, \quad x \in X$$

and the family $\{f_n : n \in \mathrm{N}\}$ *is locally equicontinuous, then for every two solutions* $\varphi_\omega : X \to Y$ $\omega = 1, 2$, *of* (1) *continuous at* ξ, *the inequality*

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant \sigma(\varphi_1(\xi), \varphi_2(\xi)), \quad x \in X$$

holds.

Proof. (cf. the proofs of Lemmas 2 and 3 in [3]). Let $\varphi_\omega : X \to Y$, $\omega = 1, 2$, be two solutions of (1) continous at ξ and let us take an arbitrary positive real number ε. Then

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant \sum_{n=1}^{\infty} \gamma_n(x)\,\sigma(\varphi_1[f_n(x)], \varphi_2[f_n(x)]), \quad x \in X$$

and there exists a neighbourhood U of ξ such that

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant \sigma(\varphi_1(\xi), \varphi_2(\xi)) + \varepsilon, \quad x \in U.$$

Therefore, by Lemma 1 in [3]

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant \sigma(\varphi_1(\xi), \varphi_2(\xi)) + \varepsilon, \; x \in X.$$

(*) By y^{N}, where N is the set of all positive integers, we denote the set of all sequences with elements from Y. Moreover, if y is a member of Y^{N} then y_n will denote the n-th coordinate of y.

This completes the proof taking into account the unrestricted choice of ε.

Concerning the functional equation of the first order

$$\varphi(x) = h(s, \varphi[f(x)]) \tag{2}$$

we have

Theorem 2. *Suppose that $h: X \times Y \to Y$ and $f: X \to X$, where X is a topological space and Y is a metric space with metric σ fulfil*

$$\sigma(h(x, y), h(x, \bar{y})) \leqslant \gamma(x)\sigma(y, \bar{y}), \quad (x, y, \bar{y}) \in X \times Y \times Y$$

and

$$\lim_{n\to\infty} f^n(x) = \xi, \quad x \in X,$$

with a function $\gamma: X \to [0, \infty)$ and a $\xi \in X$, respectively. If

$$\prod_{i=0}^{n} \gamma[f^i(x)] \leqslant M, \quad x \in X, \quad n = 0, 1, 2, \ldots,$$

then for every two solutions $\varphi_\omega: X \to Y$, $\omega = 1, 2$, of (2) continuous at ξ the conditions

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant M\sigma(\varphi_1(\xi), \varphi_2(\xi)), \quad x \in X$$

is satisfied.

The proof of this theorem is similar to that given in [4], Theorem 1.

We return to equation (1) through

Theorem 3. *Suppose that $h: X \times Y^{\mathrm{N}} \to Y$ and $f_n: X \to Y$, $n \in \mathrm{N}$, where (X, ρ) is a bounded metric space whereas (Y, σ) is a metric space, fulfil*

$$\sigma(h(x,y), h(x,\bar{y})) \leqslant \sum_{n=1}^{\infty} l_n \sigma(y_n, \bar{y}_n), \qquad (x,y,\bar{y}) \in X \times Y^{\mathrm{N}} \times Y^{\mathrm{N}},$$

$$\sigma(h(x, H), \eta) \leqslant l\,\rho(x, \xi)^\alpha, \quad x \in X$$

whenever $H_n = \eta$ for every $n \in \mathrm{N}$ and $\eta \in Y_0$, where Y_0 is a subset of the set

$$\{\eta \in Y: \eta = h(\xi, H) \text{ whenever } H_n = \eta \text{ for every } n \in \mathrm{N}\}, \tag{3}$$

$$\rho(f_n(x), \xi) \leqslant s_n \rho(x, \xi), \quad x \in X, \quad n \in \mathrm{N} \tag{4}$$

with l, l_n, s_n, $n \in \mathrm{N}$, being nonnegative real constants, positive real α and a $\xi \in X$. If

$$\sum_{n=1}^{\infty} l_n < \infty, \quad \sum_{n=1}^{\infty} l_n s_n^\alpha < 1,$$

then for every positive real number ε there exists a positive real number δ such that for every two solutions $\varphi_\omega: X \to Y$ of (1) for which $\varphi_\omega(\xi) \in Y_0$ and

$$\sigma(\varphi_\omega(x), \varphi_\omega(\xi)) \leqslant \mu_\omega \rho(x, \xi)^\alpha, \quad x \in X, \ \omega = 1, 2 \tag{5}$$

holds with reals μ_ω, $\omega = 1, 2$, the condition

$$\sigma(\varphi_1(\xi), \varphi_2(\xi)) \leqslant \delta \tag{6}$$

implies

$$\sigma(\varphi_1(x), \varphi_2(x)) < \varepsilon, \quad x \in X.$$

Proof (Cf. the proofs of Theorems 1 and 2 in [6], 1 in [2] and 1 in [9]). For every $\varphi: X \to Y$ such that

$$\sigma(\varphi(x), \varphi(\xi)) \leqslant \mu\rho(x, \xi)^\alpha, \ x \in X, \tag{7}$$

where μ is a real, let us define the constant $\mu(\varphi)$ by

$$\mu(\varphi) = \sup\left\{\frac{\sigma(\varphi(x), \varphi(\xi))}{\rho(x, \xi)^\alpha} : x \in X\right\}$$

and put

$$M = 1 \Big/ \left(1 - \sum_{n=1}^{\infty} l_n s_n^\alpha\right)$$

Therefore, if $\varphi: X \to Y$ fulfils (1), (7) and $\varphi(\xi) \in Y_0$ then

$$\sigma(\varphi(x), \varphi(\xi)) = \sigma(h(x, \varphi[f_1(x)], \varphi[f_2(x)], \ldots), h(\xi, \varphi(\xi), \varphi(\xi), \ldots)) \leqslant$$
$$\leqslant \sigma(h(x, \varphi[f_1(x)], \varphi[f_2(x)], \ldots), h(x, \varphi(\xi), \varphi(\xi), \ldots)) +$$
$$\sigma(h(x, \varphi(\xi), \varphi(\xi), \ldots), h(\xi, \varphi(\xi), \varphi(\xi), \ldots)) \leqslant \sum_{n=1}^{\infty} l_n \sigma(\varphi[f_n(x)], \varphi(\xi)) + l\rho(x, \xi)^\alpha$$
$$\leqslant \left(l + \mu(\varphi)\sum_{n=1}^{\infty} l_n s_n^\alpha\right)\rho(x, \xi)^\alpha,$$

for every $x \in X$, which shows that $\mu(\varphi) \leqslant M$, i.e.

$$\sigma(\varphi(x), \varphi(\xi)) \leqslant M\rho(x, \xi)^\alpha, \ x \in X$$

whenever $\varphi: X \to Y$ is a solution of (1) such that $\varphi(\xi) \in Y_0$ and the condition (7) is satisfied with a real μ.

With the aid of this fact and the induction principle it is easy to prove that

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant \left(\sum_{n=1}^{\infty} l_n\right)^\nu \sigma(\varphi_1(\xi), \varphi_2(\xi)) + 2M\left(\sum_{n=1}^{\infty} l_n s_n^\alpha\right)^\nu \rho(x, \xi)^\alpha, \ x \in X, \ \nu = 0, 1, 2, \ldots$$

for every two solutions $\varphi_\omega: X \to Y$ of (1) which fulfil (5) with reals μ_ω and such that $\varphi(\xi) \in Y_0$, $\omega = 1, 2$.

Now, let us take a positive real number ε and a positive integer ν such that

$$2Md^\alpha\left(\sum_{n=1}^{\infty} l_n s_n^\alpha\right) < \varepsilon,$$

where d is the diameter of X. Consequently, if $\varphi_\omega: X \to Y$ are two solutions of (1) such that $\varphi_\omega(\xi) \in Y_0$, $\omega = 1, 2$, (5) is satisfied and (6) hold with

$$\delta = \left[\varepsilon - 2Md^\alpha\left(\sum_{n=1}^{\infty} l_n s_n^\alpha\right)^\nu\right] \Big/ \left[1 + \sum_{n=1}^{\infty} l_n s_n^\alpha,\right.$$

then

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant \delta\left(\sum_{n=1}^{\infty} l_n\right)^\nu + 2Md^\alpha\left(\sum_{n=1}^{\infty} l_n s_n^\alpha\right)^\nu < \varepsilon$$

for every $x \in X$ and our theorem is proved.

Remark 1. Observe that if in the above theorem

$$\sum_{n=1}^{\infty} l_n \leqslant 1,$$

then, recalling its proof we have

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant \sigma(\varphi_1(\xi), \varphi_2(\xi)) + 2M\left(\sum_{n=1}^{\infty} l_n s_n^{\alpha}\right)^{\nu} \rho(x,\xi)^{\alpha}, \ x \in X, \ \nu = 0, 1, 2, \dots$$

and letting ν tend to infinity we get

$$\sigma(\varphi_1(x), \varphi_2(x)) \leqslant \sigma(\varphi_1(\xi), \varphi_2(\xi)), \quad x \in X,$$

for every two solutions $\varphi_\omega : X \to Y$ of (1) for which $\varphi_\omega(\xi) \in Y_0$, $\omega = 1, 2$, and (5) holds, without the assumption of the boundedness of X (cf. Theorem 1).

In the case where $\sum_{n=1}^{\infty} l_n \leqslant 1$ is not assumed, we are able to omit the assumption of the boundedness of X in a very special case only. Namely, we shall prove

Theorem 4. *Suppose that $h : X \times Y^{\mathrm{N}} \to Y$, $f_n : X \to X$, $n \in \mathrm{N}$, and $P : X \to Y$, where (X, ρ) and $(Y, \|\cdot\|)$ are metric and normed linear spaces, respectively, fulfil*

$$\| h(x, y) - h(x, \overline{y}) \| \leqslant \sum_{n=1}^{\infty} l_n \| y_n - \overline{y}_n \|, \ (x, y, \overline{y}) \in X \times Y^{\mathrm{N}} \times Y^{\mathrm{N}},$$

$$h(x, H + y) = h(x, y) + \eta, \quad (x, y) \in X \times Y^{\mathrm{N}},$$

whenever $H_n = \eta$ for every $n \in \mathrm{N}$ and $\eta \in Y_0$, where Y_0 is a subset of (3), (4) and $P(\xi) = 0$, respectively, with l_n, s_n, $n \in \mathrm{N}$, being nonnegative real constants and a $\xi \in X$. If

$$\sum_{n=1}^{\infty} l_n s_n^{\alpha} < 1$$

for a positive real number α, then for every two solutions $\varphi_\omega : X \to Y$ of (1) which admit a representotion

(8) $$\varphi_\omega(x) = \eta_\omega + P(x) + \rho(x,\xi)^{\alpha} \Phi_\omega(x), \quad x \in X$$

where $\eta_\omega \in Y_0$ and $\Phi_\omega : X \to Y$ are bounded, $\omega = 1, 2$, the relation

$$\varphi_1(x) - \varphi_2(x) = \varphi_1(\xi) - \varphi_2(\xi), \quad x \in X$$

occurs.

Proof. (cf. the proof of Theorem 1 in [5]). It is easily seen that the function $\mathcal{H}$ $(X \setminus \{\xi\}) \times Y^{\mathrm{N}} \to Y$ defined by the formula

$$\mathcal{H}(x, y) = \frac{h(x, P[f_1(x)] + \rho(f_1(x), \xi)^{\alpha} y_1, P[f_2(x)] + \rho(f_2(x), \xi)^{\alpha} y_2, \dots) - P(x)}{\rho(x, \xi)^{\alpha}},$$

$$(x, y) \in (X \setminus \{\xi\}) \times Y^{\mathrm{N}}$$

fulfils

$$\| \mathcal{H}H(x, y) - \mathcal{H}(x, \overline{y}) \| \leqslant \sum_{n=1}^{\infty} l_n s_n^{\alpha} \| y_n - \overline{y}_n \|, \ (x, y, \overline{y}) \in (X \setminus \{\xi\}) \times Y^{\mathrm{N}} \times Y^{\mathrm{N}}$$

and that the function $\Phi : X \setminus \{\xi\} \to Y$ given by

$$\Phi(x) = \frac{\varphi(x) - \varphi(\xi) - P(x)}{\rho(x,\xi)^{\alpha}}, \ x \in X \setminus \{\xi\},$$

where $\varphi : X \to Y$ is a solution of (1) such $\varphi(\xi) \in Y_0$, is a solution of the functional equation

$$\Phi(x) = \mathcal{H}(x, \Phi[f_1(x)], \Phi[f_2(x)], \dots).$$

This shows, that if $\varphi_\omega : X \to Y$ fulfil (1) and (8) holds with $\eta_\omega \in Y_0$ and bounded $\Phi_\omega : X \to Y$, $\omega = 1, 2$, then

$$\frac{\varphi_1(x) - \varphi_1(\xi) - P(x)}{\rho(x, \xi)^{\alpha}} = \Phi_1(x) - \Phi_2(x) = \frac{\varphi_2(x) - \varphi_2(\xi) - P(x)}{\rho(x,\xi)^{\alpha}}$$

for every $x \in X \setminus \{\xi\}$ which ends the proof.

Finally, we consider the system of functional equations (9)

(9) $$\varphi_i(x) = h_i(x ; \varphi_1[f_1(x)], \dots, \varphi_1[f_n(x)] ; \dots ; \varphi_m[f_1(x)], \dots, \varphi_m[f_n(x)]), \quad i = 1, \dots, m$$

and its solutions fulfilling

(10) $$\sigma_i(\varphi_i(x), \varphi_i(\xi)) \leqslant \mu\rho(x,\xi)^{\alpha}, \ x \in X, \ i = 1, \dots, m.$$

Theorem 5. *Suppose that $h_i : X \times X_1^n \times \dots \times Y_m^n \to Y_i$, $i = 1, \dots, m$, and $f_k : X \to X$, $k = 1, \dots, n$, where (X, ρ) is a bounded metric space whereas (Y_i, σ_i), $i = 1, \dots, m$, are metric spaces, fulfil*

$$\sigma_i(h_i(x ; y_{1,1}, \dots, y_{1,n} ; \dots ; y_{m,1}, \dots, y_{m,n}), h_i(x ; \overline{y}_{1,1}, \dots, \overline{y}_{1,n} ; \dots ; \overline{y}_{m,1}, \dots, \overline{y}_{m,n})) \leqslant$$
$$\leqslant \sum_{j=1}^{m} \sum_{k=1}^{n} a_{i,j,k} \sigma_{j,k}(y_{j,k}\, \overline{y}_{j,k}), \ x \in X, \ y_{j,k}, \overline{y}_{j,k} \in Y_j, \ i, j = 1, \dots, m, \ k = 1, \dots, n,$$
$$\sigma_i(h_i(x ; \eta_1, \dots, \eta_1 ; \dots ; \eta_m, \dots, \eta_m), h_i(\xi ; \eta_1, \dots, \eta_1 ; \dots ; \eta_m, \dots, \eta_m)) \leqslant$$
$$\leqslant a_i \rho(x, \xi)^{\alpha}, \ x \in X, \ i = 1, \dots, m$$

whenever

(11) $$\eta_i = h_i(\xi ; \eta_1, \dots, \eta_1 ; \dots ; \eta_m, \dots, \eta_m), \ \eta_i \in Y_{i,0}, \ i = 1, \dots, m,$$

where $Y_{i,0}$ are subsets of Y_i, $i = 1, \dots, m$,

(12) $$\rho(f_k(x), \xi) \leqslant s_k \rho(x, \xi), \quad x \in X, \quad k = 1, \dots, n,$$

with positive real constants α, a_i, $a_{i,j,k}$, s_k, $i, j = 1, \dots, m$, $k = 1, \dots, n$ and a $\xi \in X$. If all the characteristic roots of the matrix $[a_{i,j}]_{i,j=1,\dots,m}$, where

(13) $$a_{i,j} = \sum_{k=1}^{n} a_{i,j,k} s_k^{\alpha}, \ i, j = 1, \dots, m,$$

are less than one in absolute value then for every positive real number ε there exists a positive real number δ such that for every two solutions $\varphi_{i,\omega} : X \to Y_i$ of (9) such that $\varphi_{i,\omega}(\xi) \in Y_{i,0}$ and

(14) $$\sigma_i(\varphi_{i,\omega}'(x), \varphi_{i,\omega}(\xi)) \leqslant \mu_\omega \ \rho(x,\xi), \quad x \in X, \quad i = 1, \dots, m, \quad \omega = 1, 2$$

holds with reals μ_ω, $\omega = 1, 2$, the condition

(15) $$\sigma_i(\varphi_{i,1}(\xi), \varphi_{i,2}(\xi)) \leqslant \delta, \quad i = 1, \dots, m$$

implies

$$\sigma_i(\varphi_{i,1}(x), \varphi_{i,2}(x)) < \varepsilon, \ x \in X, \quad i = 1, \dots, m.$$

Remark 2. The hypothesis that all the characteristic roots of the matrix $[a_{i,j}]_{i,j=1,\dots,m}$ given by (13) are less than one in absolute value is equivalent to the statement that

$$0 < a_{\lambda,\lambda}^{\varkappa}, \quad \varkappa = 1, \dots, m, \quad \lambda = 1, \dots, m + 1 - \varkappa,$$

where

$$a^{1}_{\lambda,\mu} = \begin{cases} a_{\lambda,\mu} & \text{for } \lambda \neq \mu \\ 1 - a_{\lambda,\mu} & \text{for } \lambda = \mu \end{cases}, \quad \lambda,\ \mu = 1,\dots,m, \text{ and}$$

$$a^{\varkappa+1}_{\lambda,\mu} = \begin{cases} a^{\varkappa}_{1,1}\, a^{\varkappa}_{\lambda+1,\mu+1} + a^{\varkappa}_{\lambda+1,1}\, a^{\varkappa}_{1,\mu+1} & \text{for } \lambda \neq \mu \\ a^{\varkappa}_{1,1}\, a^{\varkappa}_{\lambda+1,\mu+1} - a^{\varkappa}_{\lambda+1,1}\, a^{\varkappa}_{1,\mu+1} & \text{for } \lambda = \mu \end{cases}, \quad \varkappa = 1,\dots,m-1, \quad \lambda,\ \mu = 1,\dots,m-\varkappa$$

(see [10], Lemma 1.2).

Proof. Similarly as in the proof of Theorem 3 for every $\varphi_i : X \to Y_i$ such that (10) holds, where μ is a real, define the constants $\mu(\varphi_i)$ by

$$\mu(\varphi_i) = \sup\left\{\frac{\sigma_i(\varphi_i(x),\ \varphi_i(\xi))}{\rho(x,\xi)^{\alpha}} : x \in X/\{\xi\}\right\}, \quad i = 1,\dots,m.$$

Thus, if $\varphi_i : X \to Y$ fulfil (9), (10) and $\varphi_i(\xi) \in Y_{i,0}$, then

$$\sigma_i(\varphi_i(x),\ \varphi_i(\xi)) \leqslant \left(a_i + \sum_{j=1}^{m} a_{i,j}\,\mu(\varphi_j)\right)\rho(x,\xi)^{\alpha}, \quad x \in X, \quad i = 1,\dots,m$$

which implies the inequalities

$$\mu(\varphi_i) \leqslant a_i + \sum_{j=1}^{m} a_{i,j}\mu(\varphi_j), \quad i = 1,\dots,m.$$

It follows from Matkowski's lemma [8] (also [10] Lemma 1.1) (see proof of the Theorem 3 in [1]) that there exist positive constants M_i, $i = 1,\dots,m$, such that

$$a_i + \sum_{j=1}^{m} a_{i,j}M_j \leqslant M_i, \quad i = 1,\dots,m$$

and consequently

$$\mu(\varphi_i) - M_i \leqslant \sum_{j=1}^{m} a_{i,j}(\mu(\varphi_j) - M_j), \quad i = 1,\dots,m,$$

whence by Lemma 1.3 in [10]

$$\mu(\varphi_i) \leqslant M_i, \quad i = 1,\dots,m.$$

Thus, there exist positive constants M_i, $i = 1,\dots,m$, and a $\theta \in (0,1)$ such that

$$\sum_{j=1}^{m} a_{i,j}\,M_j \leqslant \theta M_i, \quad i = 1,\dots,m$$

and for every solution $\varphi_i : X \to Y_i$ of (9) which fulfils (10) with a real μ and $\varphi_i(\xi) \in Y_{i,0}$

$$\sigma_i(\varphi_i(x),\ \varphi_i(\xi)) \leqslant M_i\rho(x,\xi)^{\alpha}, \quad x \in X, \quad i = 1,\dots,m.$$

By this and the induction principle we get

$$\sigma_i(\varphi_{i,1}(x),\ \varphi_{i,2}(x)) \leqslant$$
$$\leqslant \sum_{j_1,\dots,j_{\nu+1}=1}^{m}\ \sum_{k_1,\dots,k_{\nu+1}=1}^{n} a_{i,j_1,k_1}\,a_{j_1,j_2,k_2}\cdots a_{j_\nu,j_{\nu+1},k_{\nu+1}}\,\sigma_{j_{\nu+1}}(\varphi_{j_{\nu+1},1}(\xi),\ \varphi_{j_{\nu+1},2}(\xi)) +$$
$$+ 2\theta^{\nu+1}M_i\rho(x,\xi)^{\alpha}, \quad x \in X, \quad \nu = 0, 1, 2,\dots$$

for every two solutions $\varphi_{i,\omega} : X \to Y_i$ of (9) which fulfil (14) with reals μ_ω and $\varphi_{i,\omega}(\xi) \in Y_{i,0}$, $i = 1,\dots,m$, $\omega = 1, 2$.

Now, let us take an arbitrary positive real number ε and choose a positive integer ν such that

$$2M_i\,d^{\alpha}\,\theta^{\nu+1} < \varepsilon, \quad i = 1,\dots,m$$

where d is the diameter of X and put

$$\delta = \min\left[\frac{\varepsilon - 2M_i d\theta^{\nu+1}}{1 + \sum\limits_{j_1,\dots,j_{\nu+1}=1}^{m}\ \sum\limits_{k_1,\dots,k_{\nu+1}=1}^{n} a_{i,j_1,k_1}\,a_{j_1,j_2,k_2}\cdots a_{j_\nu,j_{\nu+1},k_{\nu+1}}} : i = 1,\dots,m\right].$$

This completes the proof since for every two solutions $\varphi_{i,\omega} : X \to Y_i$ of (9) such that (14) and (15) hold with reals μ_ω, δ defined as above and such that $\varphi_{i,\omega}(\xi) \in Y_{i,0}$, $i = 1,\dots,m$, $\omega = 1, 2$, we have

$$\sigma_i(\varphi_{i,1}(x),\ \varphi_{i,2}(x)) \leqslant$$
$$\leqslant \delta \sum_{j_1,\dots,j_{\nu+1}=1}^{m}\ \sum_{k_1,\dots,k_{\nu+1}=1}^{n} a_{i,j_1,k_1}\,a_{j_1,j_2,k_2}\cdots a_{j_\nu,j_{\nu+1},k_{\nu+1}} + 2M_i d^{\alpha}\theta^{\nu+1} < \varepsilon$$

for every $x \in X$

Remark 3. In the case where

$$\sum_{j=1}^{m} a_{i,j} \leqslant 1, \quad i = 1,\dots,m,$$

the assumption of boundedness of X may be omitted (cf. Remark 1).

The following theorem yields an analogue of Theorem 4 concerning systems of functional equations.

Theorem 6. *Suppose that* $h_i : X \times Y_1^n \times \dots \times Y_m^n \to Y_i$, $i = 1,\dots,m$, $f_K : X \to X$, $k = 1,\dots,n$, *and* $P_i : X \to Y_i$, $i = 1,\dots,m$, *where* (X, ρ) *and* $(Y_i, \|.\|_i)$ *are metric and normed linear spaces, respectively, fulfil*

$$\|h_i(x;\ y_{1,1},\dots,y_{1,n};\ \dots;\ y_{m,1},\dots,y_{m,n}) - h_i(x;\ \bar{y}_{1,1},\dots,\bar{y}_{1,n};\ \dots;\bar{y}_{m,1},\dots,\bar{y}_{m,n})\|_i \leqslant$$
$$\leqslant \sum_{j=1}^{m}\sum_{k=1}^{n} a_{i,j,k}\|y_{j,k} - \bar{y}_{j,k}\|_j, \quad x \in X,\ y_{j,k},\ \bar{y}_{j,k} \in Y_j,\ i,\ j = 1,\dots,m,\ k = 1,\dots,n,$$

$$h_i(x;\ \eta_1 + y_{1,1},\dots,\eta_1 + y_{1,n};\ \dots;\ \eta_m + y_{m,1},\dots,\eta_m + y_{m,n}) =$$
$$= h_i(x;\ y_{1,1},\dots,y_{1,n};\ \dots;\ y_{m,1},\dots,y_{m,n}) + \eta_i$$
$$x \in X,\ y_{j,k} \in Y_j,\ i,\ j = 1,\dots,m,\ k = 1,\dots,n,$$

whenever (11) *holds with* $Y_{i,0}$ *being subsets of* Y_i, $i = 1,\dots,m$, (12) *and* $P_i(\xi) = 0$, $i = 1,\dots,m$, *respectively, with* $a_{i,j,k}$ *and* s_k, i, $j = 1,\dots,m$, $k = 1,\dots,n$, *being positive real constants and a* $\xi \in X$. *If all the characteristic roots of the matrix* $[a_{i,j}]_{i,j=1,\dots,m}$ *given by* (13) *are less than one in absolute value, then for every two solutions* $\varphi_{i,\omega} : X \to Y_i$ *of* (9) *which admit a representation of the form*

$$\varphi_{i,\omega}(x) = \eta_{i,\omega} + P_i(x) + \rho(x,\xi)^{\alpha}\,\Phi_{i,\omega}(x), \quad x \in X, \quad i = 1,\dots,m \tag{16}$$

with $\eta_{i,\omega} \in Y_{i,0}$ *and* $\Phi_{i,\omega} : X \to Y_i$ *being bounded,* $i = 1,\dots,m$, $\omega = 1, 2$, *the relation*

$$\varphi_{i,1}(x) - \varphi_{i,2}(x) = \varphi_{i,1}(\xi) - \varphi_{i,2}(\xi), \quad x \in X$$

occurs.

Proof. (cf. the proof of Theorem 1 in [7]). We perceive easily that he functions $\mathcal{H}_i : (X \setminus \{\xi\}) \times Y_1^n \times \dots \times Y_m^n \to Y_i$ defined by

$$\mathcal{H}_i(x\,;\; y_{1,1},\dots, y_{1,n}\,;\dots;\, y_{m,1},\dots, y_{m,n}) =$$

$$= \frac{1}{\rho(x,\ \xi)^{\alpha}}\,[h_i(x\,; P_1[f_1(x)] + \rho(f_1(x),\xi)^{\alpha}\, y_{1,1},\dots, P_1[f_n(x)] + \rho(f_n(x),\ \xi)^{\alpha}\, y_{1,n}\,;\dots;$$

$$\dots; P_m[f_1(x)] + \rho(f_1(x),\xi)^{\alpha}\, y_{m,1},\dots, P_m[f_n(x)]\ +\ \rho(f_n(x),\ \xi)^{\alpha}\, y_{m,n}) - P_i(x)],$$

$$x \in X\setminus\xi,\ y_{i,h} \in Y_j,\ \ i,\ j = 1,\dots, m,\ \ k = 1,\dots, n,$$

fulfil

$$\|\mathcal{H}_i(x\,;\; y_{1,1},\dots, y_{1,n}\,;\dots;\, y_{m,n}) - \mathcal{H}_i(x\,;\; y_{1,1},\dots, \overline{y}_{1,n}\,;\dots;\, \overline{y}_{m,1},\dots, \overline{y}_{m,n})\|_i \leqslant$$

$$\leqslant \sum_{j=1}^{m}\sum_{j=1}^{n} a_{i,j,k}\, s_k^{\alpha}\, \| y_{j,k} - \overline{y}_{j,k}\|_j,\quad x \in X\setminus\{\xi\},\ y_{j,k},\ \overline{y}_{j,k} \in Y_j,\ i,\ j = 1,\dots, m,\ k = 1,\dots, n$$

Moreover, if $\varphi_{i,\omega} : X \to Y_i$ fulfil (9) and (16) hold with $\eta_{i,\omega} \in Y_{i\,0}$ and bounded $\Phi_{i,\omega} : X \to$ $\to Y_i$, $i = 1,\dots, m$, $\omega = 1, 2$, then

$$\Phi_{i,\omega}(x) = \mathcal{H}(x\,;\ \Phi_1[f_1(x)],\dots, \Phi_{i,\omega}[f_n(x)]\,;\dots;\, \Phi_{m,\omega}[f_1(x)],\dots, \Phi_{m,\omega}[f_n(x)]),\ x \in X\setminus\{\xi\}.$$

Therefore

$$\sup\{\|\Phi_{i,1}(x) - \Phi_{i,2}(x)\|_i : x \in X\setminus\{\xi\}\} \leqslant$$

$$\leqslant \sum_{j=1}^{m} a_{i,j} \sup\{\|\Phi_{j,1}(x) - \Phi_{j,2}(x)\|_j : x \in X\setminus\{\xi\}\},\quad i = 1,\dots, m.$$

Recalling Lemma 1.3 from [10] once more we obtain

$$\Phi_{i,1}(x) = \Phi_{i,2}(x),\quad x \in X\setminus\{\xi\},\ i = 1,\dots, m$$

and the proof is finished.

REFERENCES

1. Baron K. — *A few observations regarding continuous solutions of a system of functiona equations*, Publ. Math. Debrecen 21 (1974), 185—191.
2. Baron K. and Matkowski J. — *On the solutions fulfilling a Lipschitz condition of a nonlinear functional equation of order n*, Rev. Roumaine Math. Pures Appl. 17 (1972), 1149—1154.
3. Baron K. and Matkowski J. — *On the continuous solutions of a functional equation*, submitted to Publ. Math. Debrecen.
4. Czaja-Pospiech D. and Kuczma M. — *Continuous solutions of some functional in the indeterminate case*, Ann. Polon. Math. 24 (1970), 9—20.
5. Kuczma M. — *Special solutions of a functional equation*, Ann. Polon. Math. 27 (1972), 29—32.
6. Kuczma M. and Matkowski J. — *Solutions of a functional equation in a special class of functions*, Ann. Polon. Math. 26 (1972), 287—293.
7. Matkowski J. — *The uniqueness of solutions of a system of functional equations in some classes of functions*, Aequationes Math. 8 (1972), 233—237.
8. Matkowski J. — *Some inequalities and a generalization of Banach's principle*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21 (1973), 323—324.
9. Matkowski J. — *On Lipschitzian solutions of a functional equation*, Ann. Polon. Math. 28 (1973), 135—139.
10. Matkowski J. — *Integrable solutions of functional equations*, Dissertationes Math. Rozprawy Mat. (to appear).

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ON A GENERALIZATION OF CAUCHY-BUNIAKOWSKI-SCHWARZ INEQUALITY

BY

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Let X be a real-valued inner product space. U. Richard [3] has established the following simple inequality

$$\text{(R)}\qquad \frac{-\|a\|\,\|b\| + < a, b >}{2}\,\|x\|^2 \leqslant < x, a > < x, b > \leqslant \frac{\|a\|\,\|b\| + < a, b >}{2}\,\|x\|^2,$$

for any $a,\ b,\ x \in X$. It is clear that for $x = a$ or $x = b$ or $a = b$ we obtain the famous *Cauchy-Buniakowski-Schwarz inequality*.

This Note is concerned with the boundedness problem for "quadratic form in inner products", generalizing the inequality (*R*). So, we obtain a simple proof for Richard's inequality, and at the same time, we precise the conditions in which we have just equality. Also, we prove that the main results of M. Moore [2] are direct consequences of (R). In this way, we also ameliorate Moore's boundedness constant.

Theorem 1. *For any $a \in X$, $x, y \in X\setminus\{O\}$ we have the inequality*

$$\text{(1)}\qquad \frac{-\|a\|\,\|b\| + < a, b >}{2} \leqslant \frac{< x, a > < x, b >}{\|x\|^2} + \frac{< y, a > < y, b >}{\|y\|^2} -$$

$$- 2\,\frac{< x,\ a > < y,\ b > < x,\ y >}{\|x\|^2\,\|y\|^2} \leqslant \frac{\|a\|\ \|b\| + < a, b >}{2}.$$

In the right-hand side or in the left-hand side we have even equality if and only i there are $\lambda,\ \mu \in R$ such that

$$\lambda\frac{< x,\ a >}{\|x\|^2}\,x + \mu\frac{< y,\ b >}{\|y\|^2}\,y = 1/2\,[\lambda a + \mu b].$$

Proof. According to the norm definition in an inner product space, it is easy to prove that

$$\text{(2)}\qquad \left\| 2\frac{< u,\ v >}{\|u\|^2}\,u - v \right\| = \|v\|,\quad u \in X\setminus\{O\},\ v \in X.$$

By Cauchy-Buniakowski-Schwarz inequality we have

$$-\|a\|\,\|b\| \leqslant < 2\frac{< x,\ a >}{\|x\|^2}\,x - a,\ b - 2\frac{< y,\ b >}{\|y\|^2}\,y > \leqslant \|a\|\,\|b\|,$$

which is just the inequality (1). The second part of the theorem is obvious.

Special cases. 1. If $x \perp a$ or $y \perp b$ we obtain the inequality (R). Hence, in (R) it can be equality in the right-hand side or in the left-hand side if and only if there are $\lambda,\ \mu \in R$ such that

$$2\lambda < x, a > x = (\lambda a + \mu b)\,\|x\|^2$$