$$\mathcal{H}_i(x\,;\,y_{1,1},\dots,y_{1,n}\,;\dots;\,y_{m,1},\dots,y_{m,n}) =$$

$$= \frac{1}{\rho(x,\xi)^{\alpha}}\,[h_i(x\,;\,P_1[f_1(x)] + \rho(f_1(x),\xi)^{\alpha}\,y_{1,1},\dots,P_1[f_n(x)] + \rho(f_n(x),\xi)^{\alpha}\,y_{1,n}\,;\dots;$$

$$\dots;\,P_m[f_1(x)] + \rho(f_1(x),\xi)^{\alpha}\,y_{m,1},\dots,P_m[f_n(x)] + \rho(f_n(x),\xi)^{\alpha}\,y_{m,n}) - P_i(x)],$$

$$x \in X\setminus\xi,\ y_{l,k} \in Y_j,\ i,\ j = 1,\dots,m,\ k = 1,\dots,n,$$

fulfil

$$\|\mathcal{H}_i(x\,;\,y_{1,1},\dots,y_{1,n}\,;\dots;\,y_{m,n}) - \mathcal{H}_i(x\,;\,y_{1,1},\dots,\overline{y}_{1,n}\,;\dots;\,\overline{y}_{m,1},\dots,\overline{y}_{m,n})\|_i \leqslant$$

$$\leqslant \sum_{j=1}^{m}\sum_{j=1}^{n} a_{i,j,k}\,s_k^{\alpha}\,\|y_{j,k} - \overline{y}_{j,k}\|_j,\quad x \in X\setminus\{\xi\},\ y_{j,k},\ \overline{y}_{j,k} \in Y_j,\ i,\ j = 1,\dots,m,\ k = 1,\dots,n$$

Moreover, if $\varphi_{i,\omega} : X \to Y_i$ fulfil (9) and (16) hold with $\eta_{i,\omega} \in Y_{i\,0}$ and bounded $\Phi_{i,\omega} : X \to \to Y_i$, $i = 1,\dots,m$, $\omega = 1, 2$, then

$$\Phi_{i,\omega}(x) = \mathcal{H}(x\,;\,\Phi_1[f_1(x)],\dots,\Phi_{i,\omega}[f_n(x)]\,;\dots;\,\Phi_{m,\omega}[f_1(x)],\dots,\Phi_{m,\omega}[f_n(x)]),\ x \in X\setminus\{\xi\}.$$

Therefore

$$\sup\{\|\Phi_{i,1}(x) - \Phi_{i,2}(x)\|_i : x \in X\setminus\{\xi\}\} \leqslant$$

$$\leqslant \sum_{j=1}^{m} a_{i,j}\sup\{\|\Phi_{j,1}(x) - \Phi_{j,2}(x)\|_j : x \in X\setminus\{\xi\}\},\quad i = 1,\dots,m.$$

Recalling Lemma 1.3 from [10] once more we obtain

$$\Phi_{i,1}(x) = \Phi_{i,2}(x),\quad x \in X\setminus\{\xi\},\ i = 1,\dots,m$$

and the proof is finished.

REFERENCES

1. Baron K. — *A few observations regarding continuous solutions of a system of functiona equations*, Publ. Math. Debrecen 21 (1974), 185—191.
2. Baron K. and Matkowski J. — *On the solutions fulfilling a Lipschitz condition of a nonlinear functional equation of order n*, Rev. Roumaine Math. Pures Appl. 17 (1972), 1149—1154.
3. Baron K. and Matkowski J. — *On the continuous solutions of a functional equation*, submitted to Publ. Math. Debrecen.
4. Czaja-Pospiech D. and Kuczma M. — *Continuous solutions of some functional in the indeterminate case*, Ann. Polon. Math. 24 (1970), 9—20.
5. Kuczma M. — *Special solutions of a functional equation*, Ann. Polon. Math. 27 (1972), 29—32.
6. Kuczma M. and Matkowski J. — *Solutions of a functional equation in a special class of functions*, Ann. Polon. Math. 26 (1972), 287—293.
7. Matkowski J. — *The uniqueness of solutions of a system of functional equations in some classes of functions*, Aequationes Math. 8 (1972), 233—237.
8. Matkowski J. — *Some inequalities and a generalization of Banach's principle*, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 21 (1973), 323—324.
9. Matkowski J. — *On Lipschitzian solutions of a functional equation*, Ann. Polon. Math. 28 (1973), 135—139.
10. Matkowski J. — *Integrable solutions of functional equations*, Dissertationes Math. Rozprawy Mat. (to appear).

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ON A GENERALIZATION OF CAUCHY-BUNIAKOWSKI-SCHWARZ INEQUALITY

BY

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Let X be a real-valued inner product space. U. Richard [3] has established the following simple inequality

$$\text{(R)}\qquad \frac{-\|a\|\|b\| + \langle a, b\rangle}{2}\|x\|^2 \leqslant \langle x, a\rangle\langle x, b\rangle \leqslant \frac{\|a\|\|b\| + \langle a, b\rangle}{2}\|x\|^2,$$

for any a, b, $x \in X$. It is clear that for $x = a$ or $x = b$ or $a = b$ we obtain the famous *Cauchy-Buniakowski-Schwarz inequality*.

This Note is concerned with the boundedness problem for "quadratic form in inner products", generalizing the inequality (*R*). So, we obtain a simple proof for Richard's inequality, and at the same time, we precise the conditions in which we have just equality. Also, we prove that the main results of M. Moore [2] are direct consequences of (R). In this way, we also ameliorate Moore's boundedness constant.

Theorem 1. *For any $a \in X$, $x, y \in X\setminus\{O\}$ we have the inequality*

$$\text{(1)}\qquad \frac{-\|a\|\|b\| + \langle a, b\rangle}{2} \leqslant \frac{\langle x, a\rangle\langle x, b\rangle}{\|x\|^2} + \frac{\langle y, a\rangle\langle y, b\rangle}{\|y\|^2} - 2\frac{\langle x, a\rangle\langle y, b\rangle\langle x, y\rangle}{\|x\|^2\|y\|^2} \leqslant \frac{\|a\|\|b\| + \langle a, b\rangle}{2}.$$

In the right-hand side or in the left-hand side we have even equality if and only i there are λ, $\mu \in R$ such that

$$\lambda\frac{\langle x, a\rangle}{\|x\|^2}x + \mu\frac{\langle y, b\rangle}{\|y\|^2}y = 1/2\,[\lambda a + \mu b].$$

Proof. According to the norm definition in an inner product space, it is easy to prove that

$$\text{(2)}\qquad \left\|2\frac{\langle u, v\rangle}{\|u\|^2}u - v\right\| = \|v\|,\quad u \in X\setminus\{O\},\ v \in X.$$

By Cauchy-Buniakowski-Schwarz inequality we have

$$-\|a\|\|b\| \leqslant \left\langle 2\frac{\langle x, a\rangle}{\|x\|^2}x - a,\ b - 2\frac{\langle y, b\rangle}{\|y\|^2}y\right\rangle \leqslant \|a\|\|b\|,$$

which is just the inequality (1). The second part of the theorem is obvious.

Special cases. 1. If $x \perp a$ or $y \perp b$ we obtain the inequality (R). Hence, in (R) it can be equality in the right-hand side or in the left-hand side if and only if there are λ, $\mu \in R$ such that

$$2\lambda\langle x, a\rangle x = (\lambda a + \mu b)\|x\|^2$$

2. For $a = b$, we obtain the inequality

$$0 \leqslant \frac{<x, a>^2}{\| x \|^2} + \frac{<y, a>^2}{\| y \|^2} - 2\frac{<x, a> <y, a> <x, y>}{\| x \|^2 \| y \|^2} \leqslant \| a \|^2. \tag{3}$$

From this it follows:

$$\frac{<x, y>}{\| x \| \; \| y \|} \geqslant \frac{1}{2 \| a \|^2} \left[\frac{<x, a>}{\| x \|} + \frac{<y, a>}{\| y \|} \right]^2 - 3/2. \tag{4}$$

The last inequality gives a lower bound of the inner product. A similar problem was considered by M. Moore [2]. Actually, a relation of the same kind follows from (R). Indeed, the inequality (R) can be equivalently rewritten as

$$2\frac{<x, a> <x, b>}{\| x \|^2} - \| a \| \| b \| \leqslant <a, b> \leqslant 2\frac{<x, a> <x, b>}{\| x \|^2} + \| a \| \| b \| \tag{R'}$$

for any $a, b \in X$, $x \in X/\{0\}$.

Thus, we obtain an easy extension of M. Moore's result.

Theorem 2. *If $a, b, x \in X$ and $0 < \varepsilon_1 \leqslant \varepsilon_2$ are such that*

$$\varepsilon_1 \| x \| \| a \| \leqslant <x, a> \leqslant \varepsilon_2 \| x \| \| a \|,$$
$$\varepsilon_1 \| x \| \| b \| \leqslant <x, b> \leqslant \varepsilon_2 \| x \| \| b \|,$$

then

$$\gamma_1(\varepsilon_1) \| a \| \| b \| \leqslant <a, b> \leqslant \gamma_2(\varepsilon_2) \| a \| \| b \|, \tag{5}$$

where $\gamma_1(\varepsilon) = 2\varepsilon^2 - 1$ *and* $\gamma_2(\varepsilon) = 2\varepsilon^2 + 1$.

Remark 1. One can easily verify that the inequality (R) implies the inequality

$$| <x, a> <x, b> | \leqslant \frac{\| a \| \| b \| + | <a, b> |}{2} \| x \|^2. \tag{R''}$$

In this form it is true also in the complexe case. Hence, if $| <x, a> | > (1 - \varepsilon) \| x \| \| a \|$ and $| <x, b> | > (1 - \varepsilon) \| x \| \| b \|$, it follows

$$| <a, b> | > (1 - 4\varepsilon + 2\varepsilon^2) \| a \| \| b \|. \tag{6}$$

In same conditions M. Moore establishes the relation

$$<a, b> \; > \gamma(\varepsilon) \| a \| \| b \|, \tag{M}$$

where $\gamma(\varepsilon) = \max \{1 - \varepsilon - \sqrt{2\varepsilon},\ 1 - 4\varepsilon, 0\}$.

We observe that the boundedness constant of (6) is an improvement of Moore's constant for $\varepsilon < 1 - \sqrt{2}/2$. The inequality (R) can also be considered as a consequence of the fact that (see [4], §10.5) *for every self-adjoint operator $T : X \to X$, which has only two eigen-values $\lambda_1 \leqslant \lambda_2$, it holds the inequality*

$$\lambda_1 \| x \|^2 \leqslant <Tx, x> \leqslant \lambda_2 \| x \|^2, \text{ for any } x \in X. \tag{7}$$

Indeed, it is sufficient to consider the operator $Tx = <x, a> b + <x, b> a$. This operator is self-adjoint and has the only eigen-values $\lambda_1, \lambda_2 = <a, b> \pm \| a \| \| b \|$. In this way we can obtain another generalization of Richard's inequality.

Theorem 3. *For any $a, b, x \in X$ and $\alpha, \beta, \gamma \in R$ we have the inequality*

$$\lambda_1 \| x \|^2 \leqslant \alpha <x, a>^2 + 2\beta <x, a> <x, b> + \gamma <x, b>^2 \leqslant \lambda_2 \| x \|^2, \tag{8}$$

where

$$\lambda_{1,2} = \frac{\alpha \| a \|^2 + 2\beta <a, b> + \gamma \| b \|^2 \pm \sqrt{\Delta}}{2}, \tag{9}$$

with $\Delta = (\alpha \| a \|^2 - \gamma \| b \|^2)^2 + 4(\beta \| a \|^2 + \gamma <a, b>)(\alpha <a, b> + \beta \| b \|^2)$.

Proof. It suffices to observe that the operator $T : X \to X$, defined by

$$Tx = \alpha <x, a> a + \beta(<x, a> b + <x, b> a) + \gamma <x, b> b, \tag{11}$$

for every $x \in X$, is self-adjoint and has only the eigen-values λ_1, λ_2 gived by (9).

Special cases. 1. For $\alpha = \gamma = 0$ we obtain (R).

2. For $\alpha = 1$, $\beta = 0$ and $\gamma = \pm 1$ we obtain

$$\frac{\| a \|^2 \pm \| b \|^2 - \sqrt{(\| a \|^2 \mp \| b \|^2) \pm 4 <a, b>^2}}{2} \| x \|^2 \leqslant <x, a>^2 \pm <x, b>^2 \leqslant \frac{\| a \|^2 \pm \| b \|^2 + \sqrt{(\| a \|^2 \mp \| b \|^2)^2 \pm 4 <a, b>^2}}{2} \| x \|^2. \tag{12}$$

This inequality is comparable with (3). In particular, if $\| a \| = \| b \| = 1$ we obtain

$$(1 - |<a, b>|) \| x \|^2 \leqslant <x, a>^2 + <x, b>^2 \leqslant (1 + |<a, b>|) \| x \|^2. \tag{13}$$

Remark 2. The inequality (R) can be obtained also as solution of the *optimization problem*

$$\inf \text{ or } \sup <x, a> <x, b>, \text{ with constraint } \| x \| \leqslant 1.$$

To solve this problem we can use a theorem of Kuhn-Tucker type or the caracterization with saddle-points of Lagrangian (see [1], Chapter III).

REFERENCES

1. Barbu V., Precupanu T. — *Convexitate și optimizare în spații Banach.* Ed. Acad. RSR, București, 1975.
2. Moore M. H. — *An inner product inequality.* SIAM J. Math. Anal., 4, 1973, p. 514—518.
3. Richard U. — *Sur des inégalités du type Wirtinger et leurs application aux équations différentielles ordinaires.* Colloquium of Analysis held in Rio de Janeiro, August, 1972.
4. Вулих Б. З. — *Введение в функциональный анализ.* Издательство „Наука", Москва, 1967.

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