greater than the one carried by S^n, then such a structure does not exist. The inequality is :

$$2^c + 8\left[\frac{m}{4}\right] - 1 < i \cdot \frac{2^m - 1}{k}.$$

In particular, the most convenient case is $i = [k/2]$; which is (by reasons of dimension) $i = (k-1)/2$.

The last expression can be reduced to :

$$2^c + 8\left[\frac{m}{4}\right] - 1 < \frac{k-1}{k} \cdot \frac{2^m - 1}{2}.$$

By using the argument made in 4.6, and again considering reasons of dimension, we can suppose $k \geqslant 3$. So, it is verified $2/3 \leqslant (k-1)/k$. Then, we only must prove :

$$\frac{2 \cdot 2^c + 16\left[\frac{m}{4}\right] - 2}{2^m - 1} < \frac{2}{3}.$$

In fact, keeping in mind $[m/4] = (m-c)/4$, the various values of c, and several simplifications ; the last expression is equivalent to :
$6(m-c) + h < 2^m - 1$; where $h = 0, 3, 9, 9$, when c is respectively $c = 0, 1, 2, 3$. The last expression is true for $m \geqslant 5$.

We conclude with the following

Theorem 4.12. a) *The sphere S^n does not admit almost tangent structures of any order, except $n = 3, 7$ or 15.*

b) *S^3 admits an almost tangent structure of order two ($J^3 = 0$).*

c) *admits an almost tangent structure of order six ($J^7 = 0$).*

S^{15} could only admit structures of the type $J^3 = 0$, $J^5 = 0$.

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ALMOST QUATERNAL STRUCTURES

BY

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1. **Introduction.** The *almost quaternal* structures are G-structures for which the structural group is $GL(m, \mathbf{H}) \cdot Sp(1)$ where $GL(m, \mathbf{H})$ is the linear group of the $m \times m$ nondegenerate matrices with entries quaternions and $Sp(1) \cong SO(3)$ is the (multiplicative) group of unit quaternions. If the structural group is only $GL(m, \mathbf{H})$, we obtain the almost quaternion structures. The almost quaternion and almost quaternal structures were intensively studied in the last years, especially in the case where there exist Kaehlerian metrics for them [1, 3, 4], i.e. the structural group is further reduced to $Sp(m)$ or to $Sp(m) \cdot Sp(1)$ and it is satisfied a supplementary condition. In this paper we intend to begin a study of the almost quaternal structures from the point of view of the general theory of G-structures. The first problem touched is that of integrability. Looking for the almost quaternal connections, we obtain the obstructions to integrability given by the structure tensors of this structure. Let us recall that the first structure tensor is the Chern-Bernard invariant.

It is known that the integrable almost quaternion structures are affinely flat, i.e. the manifold which carries an integrable almost quaternion structure has an atlas defining its structure of differentiable manifold by affine coordinate transformations.

The typical example of integrable almost quaternal structure is that of the quaternion projective spaces.

2. **Definitions and algebra.** Let $\mathbf{H}^m$ be the m-dimensional right module over the quaternions $\mathbf{H}$. Its elements are columns of m quaternions. If we think $\mathbf{H}^m$ as a vector space over the reals $\mathbf{R} \subset \mathbf{H}$ then it can be represented as a $4m$-dimensional real vector space $\mathbf{R}^{4m}$ as follows. Let ${}^tq = (q^1, \ldots, q^m) \in \mathbf{H}^m$ where $q^h = u^h + iv^h + jx^h + ky^h$; $h = 1, \ldots, m$; i, j, k — the standard units of $\mathbf{H}$. If ${}^tU = (u^1, \ldots, u^m)$, ${}^tV = (v^1, \ldots, v^m)$, ${}^tX = (x^1, \ldots, x^m)$, ${}^tY = (y^1, \ldots, y^m)$, where t denotes the transposition, then ${}^tQ = {}^tU + i\,{}^tV + j\,{}^tX + k\,{}^tY$ and the corresponding vector of $\mathbf{R}^{4m}$ is $(u^1, \ldots, u^m, v^1, \ldots, v^m, x^1, \ldots, x^m, y^1, \ldots, y^m) = ({}^tU, {}^tV, {}^tX, {}^tY)$, considered as a column. The linear group $GL(m, \mathbf{H})$ acts on $\mathbf{H}^m$ from the left by matrix multiplication while $\mathbf{H}$ acts on $\mathbf{H}^m$ from the right by the usual vector space multiplication. The resulting action of $GL(m, \mathbf{H})$ and $\mathbf{H}$ on $\mathbf{H}^m$ gives a

new linear group of transformations. Since any quaternion can be written as the product of a unit quaternion with a real number and the multiplication in $\mathbf{H}^m$ with real numbers is commutative, we can restrict the right action to the unit quaternions. Thus the new group acting on $\mathbf{H}^m$ is $GL(m, \mathbf{H}).\,Sp(1)$ where $Sp(1)$ is the group of unit quaternions.

Using the real representation of $\mathbf{H}^m$, we get the real representation of $GL(m, \mathbf{H})$ and $GL(m, \mathbf{H}).\,Sp(1)$. Let $M = A + iB + jC + kD$ be an element of $GL(m, \mathbf{H})$, where A, B, C, D are real $m \times m$ matrices. Then, writing the real representation of the vector

$$MQ = (A + iB + jC + kD)(U + iV + jX + kY),$$

we obtain the following $4m \times 4m$ real matrix corresponding to M:

$$M = \begin{bmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{bmatrix}. \tag{1}$$

Remark that we denoted by the same letter quaternion matrix and its real representation.

Thus the group $GL(m, \mathbf{H})$ has a real representation as a subgroup of $GL(4m, \mathbf{R})$, consisting of matrices of type (1).

Now, let $\lambda = a + ib + jc + kd \in \mathbf{H}$. Then:

$$Q\lambda = (U + iV + jX + kY)(a + ib + jc + kd)$$

gives the following representation of $\mathbf{H}$, considered as a linear transformation group on $\mathbf{H}^m$:

$$\lambda = \begin{bmatrix} aI & -bI & -cI & -dI \\ bI & aI & dI & -cI \\ cI & -dI & aI & bI \\ dI & cI & -bI & aI \end{bmatrix} \tag{2}$$

where I is the $m \times m$ identity matrix.

The real representation of the group $GL(m, \mathbf{H}).\,Sp(1)$ consists from $4m \times 4m$ matrices of the form:

$$\begin{bmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{bmatrix} \begin{bmatrix} aI & -bI & -cI & -dI \\ bI & aI & dI & -cI \\ cI & -dI & aI & bI \\ dI & cI & -bI & aI \end{bmatrix} = \tag{3}$$

$$= \begin{bmatrix} aI & -bI & -cI & -dI \\ bI & aI & dI & -cI \\ cI & -dI & aI & bI \\ dI & cI & -bI & aI \end{bmatrix} \begin{bmatrix} A & -B & -C & -D \\ B & A & -D & C \\ C & D & A & -B \\ D & -C & B & A \end{bmatrix}$$

Remark. According to the above remark we can restrict ourselves to the right vector multiplication by unit quaternions. Since the left matrix multiplication commutes with the right vector space multiplication on $\mathbf{H}^m$, it follows that the matrices which are real images in $GL(4m, \mathbf{R})$ of the elements of $GL(m, \mathbf{H})$ and $Sp(1)$ do commute.

Taking $\lambda = i, j, k$, the corresponding matrices are:

$$F_1 = \begin{bmatrix} 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \\ 0 & 0 & 0 & I \\ 0 & 0 & -I & 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 & 0 & -I & 0 \\ 0 & 0 & 0 & -I \\ I & 0 & 0 & 0 \\ 0 & I & 0 & 0 \end{bmatrix}, \quad F_3 = \begin{bmatrix} 0 & 0 & 0 & -I \\ 0 & 0 & I & 0 \\ 0 & -I & 0 & 0 \\ I & 0 & 0 & 0 \end{bmatrix}.$$

Since $i^2 = j^2 = k^2 = -1$; $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$, it follows:

$$F_1^2 = F_2^2 = F_3^2 = -I, \quad F_1F_2 = -F_2F_1 = -F_3, \tag{4}$$
$$F_2F_3 = -F_3F_2 = -F_1, \qquad F_3F_1 = -F_1F_3 = -F_2.$$

Then the (real representations of the) matrices of $GL(m, \mathbf{H})$ commute with the matrices F_1, F_2, F_3. Conversely, it follows by a straightforward calculation that all matrices of $GL(4m, \mathbf{R})$ commuting with F_1, F_2, F_3 are of the form (1). Thus we have:

$$GL(m, \mathbf{H}) = \{M \in GL(4m:\mathbf{R});\ MF_1 = F_1M;\ MF_2 = F_2M, MF_3 = F_3M\}.$$

Let $Q \in \mathbf{H}^m$, $M \in GL(m, \mathbf{H})$, $\lambda \in \mathbf{H}$ with $|\lambda| = 1$. If $P = MQ\lambda$, then:

$$Pi = MQ\lambda i = M(Q\lambda i\bar{\lambda})\lambda; \quad Pj = MQ\lambda j = M(Q\lambda j\bar{\lambda})\lambda;$$
$$Pk = MQ\lambda k = M(Q\lambda k\bar{\lambda})\lambda.$$

Thus, the multiplication of P by the units i, j, k implies the multiplication of Q by the following units of $\mathbf{H}$:

$$i' = \lambda i\bar{\lambda}; \qquad j' = \lambda j\bar{\lambda}; \qquad k' = \lambda k\bar{\lambda}.$$

If $\lambda = a + ib + jc + kd$ with $a^2 + b^2 + c^2 + d^2 = 1$, then, writing the explicit expressions for i', j', k', we get the following change for the matrices F_1, F_2, F_3, corresponding to the action of an element of $GL(m, \mathbf{H}).\,Sp(1)$:

$$F'_h = \sum_{l=1}^{3} s_{hl} F_l ; \qquad h = 1, 2, 3.$$

where :

$$S = \begin{bmatrix} s_{11} & s_{12} & s_{13} \\ s_{21} & s_{22} & s_{23} \\ s_{31} & s_{32} & s_{33} \end{bmatrix} =$$

$$= \begin{bmatrix} a^2 + b^2 - c^2 - d^2 & 2(ad + bc) & 2(bd - ac) \\ 2(bc - ad) & a^2 - b^2 + c^2 - d^2 & 2(ab + cd) \\ 2(ac + bd) & 2(cd - ab) & a^2 - b^2 - c^2 + d^2 \end{bmatrix}$$

is a matrix from $SO(3)$ (isomorphic to $Sp(1)$).

The Lie algebra $gl(m, \mathbf{H})$ of $GL(m, \mathbf{H})$ consists from all $4m \times 4m$ real matrices of the type (1), or : $gl(m, \mathbf{H}) = \{M \in gl(4m, \mathbf{R}) ; MF_h = F_h M ; h = 1, 2, 3\}$.

Using the expression (3) of the elements of $GL(m, \mathbf{H}) . Sp(1)$, it results that any matrix from the Lie algebra $\mathfrak{g}$ of $GL(m, \mathbf{H}) . Sp(1)$ can be written as a sum of a matrix from $gl(m, \mathbf{H})$ and a matrix of the type (2). For our purposes it is more convenient the following definition of $\mathfrak{g}$:

$$\mathfrak{g} = \{M \in gl(4m, \mathbf{R}) ; MF_1 - F_1 M = cF_3 - dF_2 ; MF_2 - F_2 M =$$
$$= dF_1 - bF_3 ; MF_3 - F_3 M = bF_2 - cF_1\},$$

where b, c, d are real numbers. The equivalence of this definition with the above one is proved by a straightforward computation.

We are interested in finding the prolongations of the Lie algebras of the groups $GL(m, \mathbf{H})$ and $GL(m, \mathbf{H}) . Sp(1)$. The result of this section is that these Lie algebras are of finite type.

First, let us recall the definition of the first prolongation of a Lie algebra [2]. Let $\mathbf{V}$, $\mathbf{W}$ be two real finite dimensional vector spaces and let $\mathbf{L}(\mathbf{V} ; \mathbf{W})$ be the vector space of the linear mappings from $\mathbf{V}$ to $\mathbf{W}$. We shall denote by $\mathbf{S}_p(\mathbf{V} ; \mathbf{W})$ the vector space of the symmetric p-linear mappings from $\mathbf{V}$ to $\mathbf{W}$. For any $u \in \mathbf{V}$ and $A \in \mathbf{S}_p(\mathbf{V} ; \mathbf{W})$ we denote by $A_u \in \mathbf{S}_{p-1}(\mathbf{V} ; \mathbf{W})$ the symmetric $(p-1)$-linear mapping from $\mathbf{V}$ to $\mathbf{W}$ defined by :

$$A_u(v_1, \ldots, v_{p-1}) = A(u, v_1, \ldots, v_{p-1}) ; \; v_1, \ldots, v_{p-1} \in \mathbf{V}.$$

Let $\mathfrak{g}$ be a vector subspace of $\mathbf{L}(\mathbf{V} ; \mathbf{W})$.

Definition. *The first prolongation* $\mathfrak{g}^{(1)}$ *of* $\mathfrak{g}$ *is the vector subspace of* $\mathbf{S}_2(\mathbf{V} ; \mathbf{W})$ *defined by :*

$$\mathfrak{g}^{(1)} = \{A \in \mathbf{S}_2(\mathbf{V} ; \mathbf{W}) ; A_u \in \mathfrak{g} \text{ for any } u \in \mathbf{V}\}.$$

The prolongations of higher order are defined inductively.

If $\mathbf{V} = \mathbf{W} = \mathbf{R}^n$ and $\mathfrak{g}$ is a Lie subalgebra of the linear Lie algebra $gl(n, \mathbf{R})$ we obtain the prolongations of a Lie algebra.

Proposition 1. *The first prolongation of the Lie algebra* $gl(m, \mathbf{H})$ *is trivial.*

Proof. $(gl(m, \mathbf{H}))^{(1)} = \{A \in \mathbf{S}_2(\mathbf{R}^{4m} ; \mathbf{R}^{4m}) ; A_u \in gl(m, \mathbf{H}), u \in R^{4m}\}$ $= \{A \in \mathbf{S}_2(\mathbf{R}^{4m} ; \mathbf{R}^{4m}) ; A_u F_a = F_a A_u ; a = 1, 2, 3, u \in \mathbf{V}\}$.

Let $A \in (gl(m, \mathbf{H}))^{(1)}$; $u, v \in \mathbf{R}^{4m}$. Then, using the relations (4) :

$$A(u, v) = A_u(v) = -F_2 A_u(F_2 v) = -F_2 A(u, F_2 v) = -F_2 A(F_2 v, u) =$$
$$= -F_2 A_{F_2 v}(u) = F_2 F_1 A_{F_2 v}(F_1 u) = -F_1 F_2 A(F_1 u, F_2 v) =$$
$$= -F_1 F_2 A_{F_1 u}(F_2 v) = F_1 A_{F_1 u}(v) = F_1 A(v, F_1 u) = F_1 A_v(F_1 u) =$$
$$= -A_v(u) = -A(u, v).$$ Thus $A = 0$ and, hence $(gl(m, \mathbf{H}))^{(1)} = 0$.

Proposition 2. *The first prolongation of the Lie algebra* $\mathfrak{g}$ *of* $GL(m, \mathbf{H}) . Sp(1)$ *is isomorphic to* $(\mathbf{R}^{4m})^*$. *The second prolongation is trivial.*

Proof. Let $A \in \mathfrak{g}^{(1)}$ and $u, v \in \mathbf{R}^{4m}$. Then $A(u, v) = A(v, u)$ and $A_u \in \mathfrak{g}$, i.e. there exist $\alpha, \beta, \gamma \in (\mathbf{R}^{4m})^*$ such that :

$$A_u F_1 - F_1 A_u = \gamma(u) F_2 - \beta(u) F_3 ; \; A_u F_2 - F_2 A_u = -\gamma(u) F_1 +$$
$$+ \alpha(u) F_3 ; \; A_u F_3 - F_3 A_u = \beta(u) F_1 - \alpha(u) F_2 .$$

Then :

$$A(u, v) = -F_1 A(F_1 u, v) + \beta(v) F_2 u + \gamma(v) F_3 u,$$
$$A(u, v) = -F_2 A(F_2 u, v) + \alpha(v) F_1 u + \gamma(v) F_3 u,$$
$$A(u, v) = -F_3 A(F_3 u, v) + \alpha(v) F_1 u + \beta(v) F_2 u,$$

and, hence :

$$A(u, v) = -F_1 A(F_1 u, v) + \beta(v) F_2 u + \gamma(v) F_3 u =$$
$$= -F_1 \{-F_2 A(F_2 v, F_1 u) + \alpha(F_1 u) F_1 v + \gamma(F_1 u) F_3 v\} + \beta(v) F_2 u +$$
$$+ \gamma(v) F_3 u = -F_2 F_1 A(F_1 u, F_2 v) + \alpha(F_1 u) v + \gamma(F_1 u) F_2 v +$$
$$+ \beta(v) F_2 u + \gamma(v) F_3 u = -F_2 \{-A(u, F_2 v) + \beta(F_2 v) F_2 u +$$
$$+ \gamma(F_2 v) F_3 u\} + \alpha(F_1 u) v + \gamma(F_1 u) F_2 v + \beta(v) F_2 u + \gamma(v) F_3 u =$$
$$= F_2 A(F_2 v, u) + \beta(F_2 v) u - \gamma(F_2 v) F_1 u + \alpha(F_1 u) v + \gamma(F_1 u) F_2 v +$$
$$+ \beta(v) F_2 u + \gamma(v) F_3 u = -A(u, v) + \alpha(u) F_1 v + \gamma(u) F_3 v +$$
$$+ \beta(F_2 v) u - \gamma(F_2 v) F_1 u + \alpha(F_1 u) v + \gamma(F_1 u) F_2 v + \beta(v) F_2 u + \gamma(v) F_3 u.$$

Let us replace α, β, γ by αF_1, βF_2, γF_3, respectively, where $(\alpha F_1)(u) = \alpha(F_1 u)$, etc. Then :

$$2A(u,v) = \alpha(F_1 u)F_1 v - \alpha(u)v - \beta(v)u + \beta(F_2 v)F_2 u +$$
$$+ \gamma(F_3 u)F_3 v + \gamma(F_3 v)F_3 u + \gamma(F_2 u)F_2 v + \gamma(F_1 v)F_1 u.$$

Interchanging u with v and using the symmetry of A, it folows:

$$\alpha(u)v - \alpha(v)u - \alpha(F_1 u)F_1 v + \alpha(F_1 v)F_1 u - \beta(u)v + \beta(v)u +$$
$$+ \beta(F_2 u)F_2 v - \beta(F_2 v)F_2 u + \gamma(F_1 u)F_1 v - \gamma(F_1 v)F_1 u -$$
$$- \gamma(F_2 u)F_2 v + \gamma(F_2 v)F_2 u = 0.$$

From this relation it follows by a usual contraction (that means to calculate the trace of the mapping $v \to$ the left hand side of the last relation):

$$(4m - 2)(\alpha(u) - \beta(u)) = 0.$$

Thus $\alpha = \beta$. Then, by a similar computation it is obtained $\beta = \gamma$. The final result is the following expression for a generic element of $g^{(1)}$.

$$A(u,v) = \alpha(F_1 u)F_1 v + \alpha(F_1 v)F_1 u + \alpha(F_2 u)F_2 v + \tag{5}$$
$$+ \alpha(F_2 v)F_2 u + \alpha(F_3 u)F_3 v + \alpha(F_3 v)F_3 u - \alpha(u)v - \alpha(v)u.$$

where α is an arbitrary linear form on $\mathbf{R}^{4m}$. Thus the formula (5) establishes an isomorphism between $(\mathbf{R}^{4m})^*$ and $g^{(1)}$.

We shall now prove that $g^{(2)} = 0$. Let $A \in g^{(2)}$ and $u, v, w \in \mathbf{R}^{4m}$. Then $A(u,v,w) = A(v,u,w) = A(u,w,v)$ and $A_w \in g^{(1)}$. Thus:

$$A_w(u,v) = A(u,v,w) = \alpha(F_1 u, w)F_1 v + \alpha(F_1 v, w)F_1 u +$$
$$+ \alpha(F_2 u, w)F_2 v + \alpha(F_2 v, w)F_2 u + \alpha(F_3 u, w)F_3 v + \alpha(F_3 v, w)F_3 u -$$
$$- \alpha(u,w)v - \alpha(v,w)u.$$

Similarly, writing the expression $A_v(u,w) = A(u,v,w)$, one obtains by usual contractions: $\alpha(u,v) = \alpha(v,u)$; $(2n + 1)\alpha(u,v) + \alpha(F_1 u, F_1 v) + \alpha(F_2 u, F_2 v) + \alpha(F_3 u, F_3 v) = 0$.
Replacing in this relation u, v by $F_1 u$, $F_1 v$, respectively, and substracting, it follows:

$$\alpha(u,v) = \alpha(F_1 u, F_1 v).$$

Similarly:

$$\alpha(u,v) = \alpha(F_2 u, F_2 v) = \alpha(F_3 u, F_3 v).$$

Then: $(2n + 4)\alpha(u,v) = 0$. Hence, for any $A \in g^{(2)}$, $A = 0$ and this proves that $g^{(2)} = 0$.

Using the prolongations one can define the Spencer cohomology $H^{l,k}(g)$ of a linear Lie algebra [2]. The obstructions to integrability of a given G structure having the Lie algebra of the structural group g, take values in $H^{l,2}(g)$; $l = 0, 1, 2, \ldots$ Since for $gl(m, \mathbf{H})$ we have $(gl(m,\mathbf{H}))^{(1)} = 0$ it follows $H^{l,2}(gl(m,\mathbf{H})) = 0$ for $l > 1$. Thus the obstructions to

integrability of almost quaternion structures are only in $H^{0,2}$ and $H^{1,2}$ and they are represented by the torsion and the curvature of an almost quaternion connection.

Similarly, for the almost quaternal structure, the obstructions to integrability are only in $H^{0,2}(g)$, $H^{1,2}(g)$, $H^{2,2}(g)$ since $H^{k,2}(g) = 0$ $k > 2$ for the Lie algebra g of the group $GL(m, \mathbf{H}).Sp(1)$.

3. **Almost quaternal connections.** Let M be a $4m$-dimensional real manifold. An almost quaternion structure on M is given by a reduction of the structural group of the principal frame bundle of M to $GL(m, \mathbf{H})$. The frames of this reduced bundle can be used in order to map the matrices F_1, F_2, F_3, which can be considered as tensors of type (1,1) on $\mathbf{R}^{4m}$, in three tensor field, globally defined on M. For convenience we shall change the sign of the matrix F_3 and we shall denote also by F_1, F_2, F_3 the tensor fields corresponding to the matrices $F_1, F_2, -F_3$. Then:

$$F_1^2 = F_2^2 = F_3^2 = -I; \quad F_1 F_2 = -F_2 F_1 = F_3, \quad F_2 F_3 = \tag{6}$$
$$= -F_3 F_2 = F_1, \quad F_3 F_1 = -F_1 F_3 = F_2.$$

An almost quaternal structure on M is given by a reduction of the structural group of the principal frame bundle of M to $GL(m, \mathbf{H}).Sp(1)$. In this case, the above tensor fields F_1, F_2, F_3 can be defined only locally. Namely, let U, U' be two domains of trivializing charts for the reduced bundle. To the matrices F_1, F_2, F_3 corresponding to the right multiplications by $i, j, -k \in \mathbf{H}$, will correspond the tensor fields F_1, F_2, F_3 on U and F_1', F_2', F_3' on U'. As the trivializing sections on U, U' of the reduced bundle are related by a $GL(m, H).Sp(1)$-valued mapping, it follows that on the intersection $U \cap U'$, the tensor fields F_1, F_2, F_3; F_1', F_2', F_3' are related by the formula:

$$\begin{aligned} F_1' &= s_{11} F_1 + s_{12} F_2 + s_{13} F_3, \\ F_2' &= s_{21} F_1 + s_{22} F_2 + s_{23} F_3, \\ F_3' &= s_{31} F_1 + s_{32} F_2 + s_{33} F_3, \end{aligned} \tag{7}$$

where s_{ab}; $a, b = 1, 2, 3$ are functions defined on $U \cap U'$ such that the matrix $S = (s_{ab})$; $a, b = 1, 2, 3$, has its values in $SO(3) = Sp(1)$. Thus, the giving of an almost quaternal structure on M implies the existence of a vector subbundle V of the vector bundle of tensors of type (1,1) on M, defined by a vector atlas such that for the vector charts of this vector atlas there are canonical bases, i.e. local bases consisting of three tensor fields F_1, F_2, F_3 satisfying the conditions (6). Conversely, the giving of such a vector subbundle implies the existence of an almost quaternal structure on M. Any two canonical bases of V are related by formulae of type (7).

Definition. *An almost quaternal connection on the almost quaternal manifold M is a linear connection on M which preserves by paralel transport the vector bundle V defining the almost quaternal structure on M.*

Remark. It can be proved in the usual way that giving an almost quaternal connection on M is equivalent to giving an infinitesimal connection on the reduced principal frame bundle corresponding to the almost quaternal structure defined by V.

Let ∇ be a linear connection on M. Then ∇ is almost quaternal if and only if the covariant derivatives of the local canonical bases of V are expressed as follows:

$$\nabla F_a = \sum_{b=1}^{3} \eta_{ab} \otimes F_b ; \qquad a = 1, 2, 3 \tag{8}$$

where η_{ab} are 1-forms defined on the domain of F_1, F_2, F_3.

Since F_1, F_2, F_3 satisfy the relations (6) it follows that the relations: $\eta_{ab} = -\eta_{ba}$ are satisfied by the 1-forms η_{ab}. We shall use the following notations $\eta_{12} = -\eta_{21} = \eta_3$; $\eta_{23} = -\eta_{32} = \eta_1$; $\eta_{31} = -\eta_{13} = \eta_2$. Then the condition (8) can be written as:

$$\nabla F_1 = \eta_3 \otimes F_2 - \eta_2 \otimes F_3 ; \quad \nabla F_2 = -\eta_3 \otimes F_1 + \eta_1 \otimes F_3 \tag{8'}$$
$$\nabla F_3 = \eta_2 \otimes F_1 - \eta_1 \otimes F_2 .$$

Let $\eta_1', \eta_2', \eta_3'$ be the 1-forms defined by the connection ∇ with respect to the canonical base F_1', F_2', F_3'. Then, using the relations (7), we obtain the following change formulae:

$$\eta_a' = \sum_{b=1}^{3} s_{ab}\, \eta_b + \lambda_a' ; \qquad a = 1, 2, 3. \tag{9}$$

The forms λ_a' are defined by:

$$\lambda_a' = s_{c1}\, ds_{b1} + s_{c2}\, ds_{b2} + s_{c3}\, ds_{b3} ,$$

where (a, b, c) is a circular permutation of the numbers $(1, 2, 3)$, i.e., (a, b, c) can be $(1, 2, 3)$, $(2, 3, 1)$, $(3, 1, 2)$.

Remark. From the formulae (9) it follows that the matrix:

$$\eta = [\eta_{ab}]$$

can be considered as the connection matrix of a linear connection defined by ∇ on the vector bundle V.

This connection is called the connection on V associated with ∇.

Let us look for the general family of the almost quaternal connections on M, i.e. the family of the linear connections on M satisfying (8), with $\eta = (\eta_{ab})$ the connection matrices of a linear connection on V. Let $\dot\nabla$ be an arbitrary fixed connection on M and let us write:

$$\nabla = \mathring{\nabla} + P, \tag{10}$$

where P is a tensor field of type $(1, 2)$ which must be determined. From condition (8′), it follows:

$$P_X F_a - F_a P_X = -\dot\nabla_X F_a + \eta_c(X) F_b - \eta_b(X) F_c ; \; a = 1, 2, 3, \tag{11}$$

where X is a vector field on M, P_X is the tensor field of type (1,1) defined by $P_X Y = P(X, Y)$, and (a, b, c) is a circular permutation of $(1, 2, 3)$.

Applying the method used by Obata for the determination of the almost quaternion connections [6] we get the general family of the almost quaternal connections:

$$\nabla_X = \mathring{\nabla}_X + \sum_{a=1}^{3} \left\{ \frac{1}{4} (\mathring{\nabla}_X F_a) F_a + \frac{1}{2}\, \eta_a (X) F_a \right\} + \frac{1}{4} (I \otimes I - \Lambda) Q_X, \tag{12}$$

where Q is an arbitrary tensor field of type $(1, 2)$,

$$\Lambda = \sum_{a=1}^{3} F_a \otimes F_a ; \tag{13}$$

and the operation $(F_a \otimes F_a) Q_X$ is given by:

$$[(F_a \otimes F_a) Q_X] (Y) = (F_a \otimes F_a)_Y\, Q_X = F_a Q_X F_a Y .$$

Remark. While F_a defined locally, the tensor field Λ is defined globally on M. Similarly, the expression:

$$\sum_{a=1}^{3} \left\{ \frac{1}{4} (-\mathring{\nabla}_X F_a) F_a + \frac{1}{2}\, \eta_a (X) F_a \right\}$$

s globally defined on M.

Remark. If the initial linear connection $\mathring{\nabla}$ is almost quaternal with the connection associated in V of connection matrix $\mathring{\eta}$, formula (12) becomes:

$$\nabla_X = \mathring{\nabla}_X + \frac{1}{2} \sum_{a=1}^{3} (\eta_a - \mathring{\eta}_a)(X) F_a + \frac{1}{4} (I \otimes I - \Lambda) Q_X . \tag{12'}$$

Thus an almost quaternal connection is essentially defined by its associated connection on V and the tensor field $(I \otimes I - \Lambda) Q$.

4. The Chern-Bernard invariant of the almost quaternal structures

Let N_a be the Nijenhuis tensor field of F_a, i.e.:

$$N_a (X, Y) = [F_a X, F_a Y] - F_a [F_a X, Y] - F_a [X, F_a Y] - [X, Y]. \tag{14}$$

Substituting the brackets from the expression of the torsion tensor field of a linear connection:

$$T (X, Y) = \nabla_X Y - \nabla_Y X - [X, Y], \tag{15}$$

one obtains:

$$(16)\qquad N_a(X,Y) = T(X,Y) - T(F_aX, F_aY) + F_a T(F_aX, Y) +$$
$$+ F_aT(X, F_aY) + (\nabla_{F_aX}F_a)Y - (\nabla_{F_aY}F_a)X - F_a(\nabla_X F_a)Y + F_a(\nabla_Y F_a)X.$$

If ∇ is an almost quaternal connection with the associated connection matrix, γ_{ij}, then:

$$(17)\qquad N_a(X,Y) = T(X,Y) - T(F_aX, F_aY) + F_a T(F_aX, Y) +$$
$$+ F_a T(X, F_aY) + (\gamma_{ic}F_a - \gamma_{ib})(X) F_bY - (\gamma_{ic}F_a - \gamma_{ib})(Y) F_bX -$$
$$- (\gamma_{ib}F_a + \gamma_{ic})(X) F_cY + (\gamma_{ib}F_a + \gamma_{ic})(Y) F_cX,$$

where (a, b, c) is a circular permutation of $(1, 2, 3)$. Calculating the tensor ield S defined by $S = N_1 + N_2 + N_3$, we obtain:

$$S(X,Y) = 3T(X,Y) + \sum_{a=1}^{3} \{-T(F_aX, F_aY) +$$
$$(18)\qquad + F_a T(F_aX, Y) + F_aT(X, F_aY)\} - \sum_{(a,b,c)} \{(2\gamma_{ia} + \gamma_{ic}F_b -$$
$$\gamma_{ib}F_c)(Y) F_aX - (2\gamma_{ia} + \gamma_{ic}F_b - \gamma_{ib}F_c)(X) F_aY\}$$

where $\sum_{(a,b,c)}$ means that the summation is extended over the circular permutations (a, b, c) of $(1, 2, 3)$.

On the other hand, from the expression (13) of A we get:

$$\sum_{a=1}^{3} F_aT(X, F_aY) = T(X,Y) + (I \otimes I - A)_Y T_X,$$

$$\sum_{a=1}^{3} F_a T(F_aX, Y) = T(X,Y) - (I \otimes I - A)_X T_Y,$$

$$\sum_{a=1}^{3} T(F_aX, F_aY) = \frac{1}{2}\sum_{a=1}^{3} \{(I \otimes I - A)_Y (T_{F_aX}F_a) - (I \otimes I -$$
$$- A)_X(T_{F_aY} F_a) - F_a T(X, F_aY) - F_a T(F_aX, Y)\}.$$

Then:

$$S(X,Y) = 6T(X,Y) - \frac{1}{4}(I \otimes I - A)_Y \left(6T_X + 2\sum_{a=1}^{3} T_{F_aX}F_a\right) +$$
$$(19)\qquad + \frac{1}{4}(I \otimes I - A)_X\left(6T_Y + 2\sum_{a=1}^{3} T_{F_aY} F_a\right) - \sum_{(a,b,c)} (2\gamma_{ia} -$$
$$- \gamma_{ib}F_c + \gamma_{ic}F_b)(X) F_aY + \sum_{(a,b,c)} (2\gamma_{ia} - \gamma_{ib}F_c + \gamma_{ic}F_b)(Y) F_aX.$$

The tensor field S is not globally defined on M. Using some contractions of S we shall construct a new tensor field globally defined on M,

related to the torsion tensor field of an almost quaternal connection. Let us denote:

$$\alpha_a(X) = \frac{1}{4n-2}\ \text{trace}\ (Y \to F_a S(X,Y)),$$

$$\rho_a(X) = \frac{1}{2n-1}\ \text{trace}\ (Y \to F_aT(X,Y)).$$

From (18) we get:

$$(20)\qquad \alpha_a(X) = (2\gamma_{ia} - \gamma_{ib}F_c + \gamma_{ic}F_b) + (2\rho_a - \rho_bF_c + \rho_cF_b).$$

Combining (19) and (20) we have:

$$\frac{1}{6}\left\{S(X,Y) + \sum_{a=1}^{3} (\alpha_a(X) F_aY - \alpha_a(Y) F_aX)\right\} =$$
$$= T(X,Y) + \frac{1}{6}\sum_{(a,b,c)} \{(2\rho_a - \rho_bF_c + \rho_cF_b)(X) F_aY - (2\rho_a - \rho_bF_c +$$
$$(21)\qquad + \rho_cF_b)(Y) F_aX\} - \frac{1}{4}(I \otimes I - A)_Y \left(T_X + \frac{1}{3}\sum_{a=1}^{3} T_{F_aX}F_a\right) +$$
$$+ \frac{1}{4}(I \otimes I - A)_X \left(T_Y + \frac{1}{3}\sum_{a=1}^{3} T_{F_aY} F_a\right).$$

It can be verified by a straightforward calculation, using (7) that:

$$\sum_{(a,b,c)} (2\rho_a - \rho_bF_c + \rho_cF_b) \otimes F_a \quad \text{and} \quad \sum_{a=1}^{3} T_{F_aX} F_a$$

are tensor fields defined globally on M. Then the right hand side of the formula (21) is defined globally on M and, thus, the left hand side is defined globally on M. Considering the connection:

$$(22)\qquad \tilde{\nabla}_X = \nabla_X + \frac{1}{6}\sum_{(a,b,c)} (2\rho_a - \rho_bF_c + \rho_cF_b)(X) F_a -$$
$$- \frac{1}{4}(I \otimes I - A)\left(T_X + \sum_{a=1}^{3} T_{F_aX} F_a\right)$$

which is almost quaternal, its torsion tensor field is $\tilde{S}$, where:

$$(23)\qquad \tilde{S}(X,Y) = \frac{1}{6}\left\{S(X,Y) + \sum_{a=1}^{3} (\alpha_a(X) F_aY - \alpha_a(Y) F_aX)\right\}.$$

The tensor field $\tilde{S}$ considered in (23) is a representative of the Chern-Bernard invariant of the almost quaternal structure. Moreover, the

Chern-Bernard invariant of this structure vanishes if and only if $\tilde{S}$ vanishes. In fact, if $\tilde{S}$ vanishes there exists a torsion free almost quaternal connection, hence the Chern-Bernard invariant vanishes. Conversely, if the Chern-Bernard invariant vanishes, there exists a torsion free almost quaternal connection and, from (21), it follows that $\tilde{S}$ vanishes.

Suppose that for the considered almost quaternal structure the tensor field $\tilde{S}$ vanishes. Then:

$$S = -\sum_{a=1}^{3} \{\alpha_a(X) F_a Y - \alpha_a(Y) F_a X\}. \tag{24}$$

On the other hand, it can be verified by a straightforward calculation the following formula:

$$6N_a(X, Y) = S(X, Y) + F_a S(F_a X, Y) + F_a S(X, F_a Y) - S(F_a X, F_a Y). \tag{25}$$

If S is given by (24), then:

$$3N_a(X, Y) = -\alpha_b(X) F_b Y + \alpha_b(Y) F_b(X) - \alpha_c(X) F_c Y + \alpha_c(Y) F_c X - \alpha_b(F_a X) F_c Y + \alpha_b(F_a Y) F_c X + \alpha_c(F_a X) F_b Y - \alpha_c(F_a Y) F_b X. \tag{26}$$

Thus, the vanishing of the Chern-Bernard invariant of the almost quaternal structure does not imply the vanishing of the Nijenhuis tensor fields of the vector fields F_a; $a = 1, 2, 3$.

The integrability problem for the almost quaternal structures related to the curvature of the almost quaternal connections will be studied in a forthcoming paper.

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AN ADAPTATION OF THE MOVING FRAME METHOD TO THE STUDY OF GEOMETRICAL OBJECT FIELDS

BY

ALEXANDRU NEAGU

In a recent paper [5], we have described the algorithm of the moving frame for any distribution (involutive or not). Here we adapt this algorithm to the study of geometrical object fields.

§1. Let X and Y be the differentiable G-space, where G is a compact Lie group which acts proper and free on X [1]. Let $f: X \to Y$ be an equivariant map. A *MRM-configuration* is a set (X, f, Y, G). If every orbit of Y is of type G/H and there exists an equivariant retraction λ of Y to an orbit $G \cdot y_0$ (see [1]), then (X, f, Y, G) is called *reducible*.

Theorem 1. *If (X, f, Y, G) is a reducible MRM-configuration, then the structure group G of the principal fibre bundle $\pi: X \to X/G$ (X/G = the orbit space of X) is reducible to the subgroup H.*

Proof. Let $\lambda: Y \to G \cdot y_0$ be an equivariant retraction to the orbit $G \cdot y_0$ of $y_0 \in Y$. Then $\lambda^{-1}(y_0) = S$ is a global slice of Y, [1]. We put $X_1 = (\lambda \circ f)^{-1}(y_0)$ and by an usual argument we prove that X_1 is a principal fibre subbundle of X with the structure group H.

Corollary 1. *In the condition of Theorem 1, the restriction of f to the reduced bundle $X_1 \subset X$ defines a map J of X/G to the slice $S \subset Y$.*

Proof. Let $x \in X_1$ and $x^* = \pi(x)$. We put $J(x^*) = f(x)$ and since f is constant on the orbits of the H-space X_1, it follows that J is well defined.

The map J is called *invariant* of the reducible *MRM*-configuration (X, f, Y, G). If G acts transitivelly on Y then the *MRM*-configuration (X, f, Y, G) is called *singular*.

Theorem 2. *If (X, f, Y, G) is a singular MRM-configuration, then the invariant J is constant.*

Proof. Since G acts transitivelly on Y, then the retraction λ is the identity of Y and hence J is the restriction of $f \circ \lambda$ to X_1.

The process described in Theorem 1 and Corollary 1, by which to a *MRM*-configuration corresponds a subbundle X_1 and an invariant J is called R-algorithm. If Y is a trivial G-space, then we say that (X, f, Y, G) is *irreducible*.

§2. Let $\pi: X \to X^* = X/G$ be a principal fibre bundle with the structure group G and $\dim X^* = m$. Let $\sigma: Y \to X^*$ be an associate fibre bundle with the fibre F and $\overline{K}_p(Y)$ $(p < m)$ the space of contact p^1 — regular elements on Y, which are transversal to σ. $\overline{K}_p(Y)$ is the associate fibre bun-