Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXIII, S. I a, f. 2, 1977

ON THE GLOBAL PSEUDOLOCALITY OF PSEUDO-DIFFERENTIAL OPERATORS

BY

T. JUCAN

Some papers on pseudo-differential operators ([1], [3], [4]) deal with ρ-hypoanalyticity and ρ-hypolocality, both of these properties being investigated in a local sense, i.e. on compact sets.

The notion of global analytic-hypoellipticity for pseudo-differential operators, introduced by Taniguchi in [2], gives new posibilities to investigate pseudo-differential operators.

In connection with these posibilities, the purpose of this paper is the study of global ρ-pseudolocality and global ρ-pseudolocality with respect to a group of variables, for pseudo-differential operators.

§ 1. Global ρ-pseudolocality of pseudo-differential operators. Let $x, \xi \in R^n$, a function $a \in C^\infty (R^n \times R^n)$ and the pseudo-differential operator A with symbol a, given by

$$Au(x) = \int e^{i(x,\xi)} a(x, \xi) \tilde{u}(\xi)\, d\xi. \tag{1.1}$$

Definition 1.1. *$G^\rho_{L^2}(R^n)$ is the set of all functions $u \in C^\infty (R^n)$ which satisfy*

$$\| D^\alpha u(x) \|_{L^2} \leq M^{|\alpha|+1} (\alpha !)^\rho \tag{1.2}$$

where $\rho \geq 1$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in N^n$ and $\| . \|_{L^2}$ means the norm in $L^2(R^n)$ M is a positive constant which doesn't depend on α.

Definition 1.2. *The pseudo-differential operator A is global ρ-pseudolocal if $u \in G^\rho_{L^2}(R^n)$ implies $Au \in G^\rho_{L^2}(R^n)$.*

Theorem 1.1. *Let a be the symbol of a pseudo-differential operator A, such that*

$$\iint | D^\beta_x a(x, \xi) |^2\, dx\, d\xi \leq M_1^{|\beta|} (\beta !)^{2\rho}, \tag{1.3}$$

where $\beta \in N^n$ and M_1 is a positive constant which doesn't depend on β.

Then A is global ρ-pseudolocal.

Proof. From (1.1) we formally get:

$$D_x^\alpha Au(x) = \sum_{|\alpha'|\leq|\alpha|} \frac{\alpha!}{\alpha'!\,\alpha''!}\int \xi^{\alpha'} e^{i(x,\xi)} D_x^{\alpha''} a(x,\xi)\,\tilde{u}(\xi)\,d\xi, \tag{1.4}$$

where $\alpha'' = \alpha - \alpha'$.

By (1.4), it results:

$$\| D_x^\alpha Au(x)\|_{L^2}^2 \leq \int\Bigl(\sum_{|\alpha'|\leq|\alpha|}\frac{\alpha!}{\alpha'!\,\alpha''!}\Bigl|\int \xi^{\alpha'}e^{i(x,\xi)}D_x^{\alpha''}a(x,\xi)\tilde{u}(\xi)\,d\xi\Bigr|^2\Bigr)\,dx. \tag{1.5}$$

Applying Schwartz's inequality for the integral with respect to ξ, it follows:

$$\| D_x^\alpha Au(x)\|_{L^2}^2 \leq$$

$$\leq \int\Bigl(\sum_{|\alpha'|\leq|\alpha|}\frac{\alpha!}{\alpha'!\,\alpha''!}\Bigl(\int | D_x^{\alpha''} a(x,\xi)|^2\,d\xi\Bigr)^{1/2}\Bigl(\int|\xi^{\alpha'}\tilde{u}(\xi)|^2\,d\xi\Bigr)^{1/2}\Bigr)^2 dx \leq$$

$$\leq \int\Bigl(\sum_{|\alpha'|\leq|\alpha|}\frac{\alpha!}{\alpha'!\,\alpha''!}\Bigl(\int | D_x^{\alpha''} a(x,\xi)|^2\,d\xi\Bigr)^{1/2} M^{|\alpha'|+1}(\alpha'!)^{\rho}\Bigr)^2 dx \leq$$

$$\leq \sum_{|\alpha'|\leq|\alpha|}\Bigl(\frac{\alpha!}{\alpha'!\,\alpha''!}\Bigr)^2 M^{2(|\alpha'|+1)}(\alpha'!)^2.\iint | D_x^{\alpha''} a(x,\xi)|^2\,d\xi\,dx.$$

Taking into account (1.3) we obtain:

$$\| D_x^\alpha Au(x)\|_{L^2}^2 \leq 2\sum_{|\alpha'|\leq|\alpha|}\Bigl(\frac{\alpha!}{\alpha'!\,\alpha''!}\Bigr)^2 M_1^{2|\alpha''|}(\alpha''!)^{2\rho} M^{2(|\alpha'|+1)}(\alpha'!)^{2\rho} =$$

$$= 2\sum_{|\alpha'|\leq|\alpha|}(\alpha!)^{2\rho}\Bigl(\frac{\alpha'!\,\alpha''!}{\alpha!}\Bigr)^{2(\rho-1)} M_1^{2|\alpha''|}M^{2(|\alpha'|+1)}.$$

Since $\alpha'!\,\alpha''!/\alpha! \leq 1$ it holds:

$$\| D_x^\alpha Au(x)\|_{L^2}^2 \leq 2\sum_{|\alpha'|\leq|\alpha|}(\alpha!)^{2\rho} M_1^{2|\alpha''|}M^{2(|\alpha'|+1)}.$$

Choosing $M_1 \geq 2M$ we have:

$$\| D_x^\alpha Au(x)\|_{L^2}^2 \leq 2(\alpha!)^{2\rho} M_1^{2(|\alpha|+1)}\sum_{|\alpha'|\leq|\alpha|}\Bigl(\frac{M}{M_1}\Bigr)^{2(|\alpha'|+1)} \leq$$

$$\leq 2(\alpha!)^{2\rho} M_1^{2(|\alpha|+1)}\Bigl(\sum_{|\alpha'|\leq|\alpha|}\Bigl(\frac{M}{M_1}\Bigr)^{|\alpha'|+1}\Bigr)^2 \leq$$

$$\leq (\alpha!)^{2\rho} M_1^{2(|\alpha|+1)}\frac{1}{2}\Bigl(\sum_{|\alpha'|\leq|\alpha|}\Bigl(\frac{1}{2}\Bigr)^{|\alpha'|}\Bigr)^2 \leq$$

$$\leq \frac{1}{2}(\alpha!)^{2\rho} M_1^{2(|\alpha|+1)}\Bigl|\Bigl(\sum_{\alpha_1'\leq\alpha_1}\Bigl(\frac{1}{2}\Bigr)^{\alpha_1'}\Bigr)\Bigl(\sum_{\alpha_2'\leq\alpha_2}\Bigl(\frac{1}{2}\Bigr)^{\alpha_2'}\Bigr)\cdots\Bigl(\sum_{\alpha_n'\leq\alpha_n}\Bigl(\frac{1}{2}\Bigr)^{\alpha_n'}\Bigr)\Bigr|^2$$

$$2^{2n}(\alpha!)^{2\rho} M_1^{2(|\alpha|+1)}.$$

Taking a constant M_2 such that $M_2 \geq 2^n M_1$, it results:

$$\| D_x^\alpha Au(x)\|_{L^2}^2 \leq M_2^{2(|\alpha|+1)}(\alpha!)^{2\rho}. \tag{*}$$

From (*) it follows

$$\| D_x^\alpha Au(x)\|_{L^2} \leq M_2^{|\alpha|+1}(\alpha!)^{\rho}, \tag{**}$$

and this means that $Au \in G_{L^2}^{\rho}(R^n)$.

Theorem 1.2. *Let $a \in C^\infty(R^n \times R^n)$ be such that*

$$\int | D_x^\beta a(x,\xi)|^2\,dx \leq M_1^{2|\beta|}(\beta!)^{2\rho}(1+|\xi|^2)^m,$$

where m is a positive integer and $M > 0$ doesn't depend on β.

Then the pseudo-differential operator A is global ρ-pseudolocal.

Proof. Estimating the L^2-norm of $D_x^\alpha Au(x)$, we have:

$$\| D_x^\alpha Au(x)\|_{L^2}^2 \leq \int\Bigl(\sum_{\alpha'\leq\alpha}\frac{\alpha!}{\alpha'!\,\alpha''!}\Bigr)\Bigl|\int \xi^{\alpha'}e^{i(x,\xi)}D_x^{\alpha-\alpha'}a(x,\xi)\tilde{u}(\xi)\,d\xi\Bigr|^2 dx$$

$$\leq 2\sum_{\alpha'\leq\alpha}\Bigl(\frac{\alpha!}{\alpha'!\,\alpha''!}\Bigr)^2\int\Bigl(\int(1+|\xi|)^{-m-[n/2]-1}(1+|\xi|)^{m+[n/2]+1}\,\times$$

$$|D_x^{\alpha''}a(x,\xi)|\,|\xi^{\alpha'}\tilde{u}(\xi)|\,d\xi\Bigr)^2 dx \leq 2\sum_{\alpha'\leq\alpha}\Bigl(\frac{\alpha!}{\alpha'!\,\alpha''!}\Bigr)^2\times$$

$$\int\Bigl(\int(1+|\xi|^2)^{-m-[n/2]-1}|D_x^{\alpha''}a(x,\xi)|^2\,d\xi\int(1+|\xi|^2)^{m+[n/2]+1}\,\times$$

$$|\xi^{\alpha'}\tilde{u}(\xi)|^2\,d\xi\Bigr)dx = 2\sum_{\alpha'\leq\alpha}\Bigl(\frac{\alpha!}{\alpha'!\,\alpha''!}\Bigr)^2\|(1-\Delta)^{m+[n/2]+1}D_x^{\alpha'}u\|_{L^2}^2\,\times$$

$$\times\iint(1+|\xi|^2)^{-m-[n/2]-1}|D_x^{\alpha''}a(x,\xi)|^2\,d\xi\,dx \leq$$

$$\leq 2\sum_{\alpha'\leq\alpha}\Bigl(\frac{\alpha!}{\alpha'!\,\alpha''!}\Bigr)^2 CM^{2(|\alpha'|+1)}(\alpha'!)^{2\rho}C_1M_1^{2|\alpha''|}(\alpha''!)^{2\rho} =$$

$$= 2CC_1\sum_{\alpha'\leq\alpha}\Bigl(\frac{\alpha'!}{\alpha'!\,\alpha''!}\Bigr)^2(\alpha'!)^{2\rho}(\alpha''!)^{2\rho}M^{2(|\alpha'|+1)}M^{2|\alpha''|} \leq$$

$$\leq 2C_1C(\alpha!)^{2\rho}\sum_{\alpha'\leq\alpha}M^{2(|\alpha'|+1)}M_2^{2|\alpha''|}.$$

Taking $M_1 \geq 2M$, we obtain :

$$\| D_x^\alpha Au(x) \|_{L^2}^2 \leq 2C_1 C (\alpha !)^{2\rho} M_1^{2(|\alpha|+1)} \sum_{\alpha' < \alpha} \left(\frac{M}{M_1} \right)^{2(|\alpha'|+1)} \leq$$

$$2^n C_1 C (\alpha !)^{2\rho} M_1^{2(|\alpha|+1)} \leq M_2^{2(|\alpha|+1)} (\alpha !)^{2\rho} ,$$

where M_2 is chosen so that $M_2 \geq 2^n C_1 CM$.

Thus we get :

$$\| D_x^\alpha Au(x) \|_{L^2}^2 \leq M_2^{2(|\alpha|+1)} (\alpha !)^{2\rho} .$$

It follows :

$$\| D_x^\alpha Au(x) \|_{L^2} \leq M_2^{|\alpha|+1} (\alpha !)^\rho ,$$

which concludes the proof.

§ 2. Global ρ-pseudolocality with respect to x'. Let us consider $x \in R^n$ of the form $x = (x', x'')$, where $x' \in R^k$, $x'' \in R^{n-k}$ and $1 \leq k < n$.

Definition 2.1. *A function $u \in C^\infty(R^n) \cap L^2(R^n)$ belongs to $G_{L^2}^\rho(R^n)$ if and only if*

$$\| D_{x'}^\alpha u(x) \|_{L^2} \leq M^{|\alpha|+1} (\alpha !)^\rho \qquad \rho \geq 1,$$

where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $\alpha_i \in N$, holds.

Definition 2.2. *The pseudo-differential operator A is global ρ-pseudolocal with respect to x' if $Au \in G_{L^2}^\rho(R^n)$ when $u \in G_{L^2}^\rho(R^n)$.*

Theorem 2.1. *Let $a \in C^\infty(R^n \times R^n)$ be such that :*

$$\iint | D_{x'}^\beta a(x, \xi) |^2 \, dx \, d\xi \leq M_1^{2|\beta|} (\beta !)^{2\rho} .$$

Then A is global ρ-pseudolocal with respect to x'.

Proof. Let be $\xi = (\xi', \xi'')$, $\xi' = (\xi_1, \xi_2, \ldots, \xi_k)$, $\xi'' = (\xi_{k+1}, \ldots, \xi_n)$. We have the following estimate :

$$\| D_{x'}^\alpha Au(x) \|_{L^2}^2 \leq \int \left(\sum_{\alpha' < \alpha} \frac{\alpha !}{\alpha' ! \, \alpha'' !} \left| \int \xi'^{\alpha'} e^{i(x,\xi)} D_{x'}^{\alpha''} a(x, \xi) \tilde{u}(\xi) \, d\xi \right| \right)^2 dx \leq$$

$$\leq 2 \sum_{\alpha' < \alpha} \left(\frac{\alpha !}{\alpha' ! \, \alpha'' !} \right)^2 \| D_{x'}^{\alpha'} u(x) \|_{L^2}^2 \iint | D_{x'}^{\alpha''} a(x, \xi) |^2 \, d\xi \, dx \leq$$

$$\leq 2 \sum_{\alpha' < \alpha} \left(\frac{\alpha !}{\alpha' ! \, \alpha'' !} \right)^2 M^{2(|\alpha'|+1)} (\alpha' !)^{2\rho} M_1^{2|\alpha''|} (\alpha'' !)^{2\rho} \leq M_2^{2(|\alpha|+1)} (\alpha !)^{2\rho} ,$$

where M_2 is a suitable constant.

It follows :

$$\| D_{x'}^\alpha Au(x) \|_{L^2} \leq M_2^{|\alpha|+1} (\alpha !)^\rho .$$

Thus the proof is complete.

Theorem 2.2. *Let $a \in C^\infty(R^n \times R^n)$ be such that*

$$\int | D_{x'}^\beta a(x, \xi) |^2 \, dx \leq M_1^{2|\beta|} (\beta !)^{2\rho} (1 + | \xi' |^2)^m (1 + | \xi'' |^2)^{-n-1}$$

where $m \in N$ and $M_1 > 0$ are constants which do not depend on β.

Proof. We estimate the L^2-norm of $D_{x'}^\alpha Au(x)$ and we obtain :

$$\| D_{x'}^\alpha Au(x) \|_{L^2}^2 \leq \int \left(\sum_{\alpha' < \alpha} \frac{\alpha !}{\alpha' ! \, \alpha'' !} \left| \int \xi'^{\alpha'} e^{i(x,\xi)} D_{x'}^{\alpha''} a(x, \xi) \tilde{u}(\xi) \, d\xi \right| \right)^2 dx \leq$$

$$\leq 2 \sum_{\alpha' < \alpha} \left(\frac{\alpha !}{\alpha' ! \, \alpha'' !} \right)^2 \int \left(\int (1 + | \xi' |)^{-m-[k/2]-1} | D_{x'}^{\alpha''} a(x, \xi) | (1 + | \xi' |)^{m+[k/2]+1} \times \right.$$

$$\left. \times | \xi'^{\alpha'} \tilde{u}(\xi) | \, dx \right)^2 dx \leq 2 \sum_{\alpha' < \alpha} \left(\frac{\alpha !}{\alpha' ! \, \alpha'' !} \right)^2 \| (1 - \Delta_{x'})^{m+[k/2]+1} D_{x'}^{\alpha'} u \|_{L^2}^2 \times$$

$$\iint (1 + | \xi' |^2)^{-m-[k/]-1} | D_{x'}^{\alpha''} a(x, \xi) |^2 \, d\xi \, dx \leq C 2^k M_1^{2(|\alpha|+1)} (\alpha !)^{2\rho} \leq$$

$$\leq M_2^{2(|\alpha|+1)} (\alpha !)^{2\rho} ,$$

with M_2 chosen so that $M_2 > 2M_1 C$.

Finally we have :

$$\| D_{x'}^\alpha Au(x) \|_{L^2} \leq M_2^{|\alpha|+1} (\alpha !)^\rho .$$

which means global ρ-pseudolocality of A with respect to x'.

BIBLIOGRAPHY

1. Barbu, V. — *Partially hypoanalytic distributions and pseudo-differential operators.* Acad. Naz. dei Lincei, vol. XLV (1968), p. 200—206.
2. Taniguchi, K. — *On the hypoellipticity and the global analytic-hypoellipticity of pseudo-differential operators.* Osaka. J. Math., 11, (1974), p. 221—238.
3. Trèves, F. — *Analytic-hypoelliptic partial differential equations of principal type.* Comm. pure appl. math. 24 (1971), p. 537—570.
4. Volevič, L. R. — *Ocenki operatorov tipa svertki v klassah Gevrey.* U.M.N., 25 (1970), p. 269—280.

Received 6.XII.1977

Faculty of Mathematics
University of Iaşi
Romania