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LIAPOUNOFF STABILITY OF FUZZY SETS

BY

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In 1965 L.A. Zadeh [7] has introduced the notion of fuzzy set. J.A. Goguen in [3] has generalized this notion by the idea of L-fuzzy set and C. L. Chang in [2] has investigated the fuzzy topologies.

The purpose of this paper is to establish the relation between the L-fuzzy topologies and the approximation structure and to extend for the case of generalized L-fuzzy systems the Liapounoff stability in the sense of Bushaw [1] generalized by us in [4] and [6].

1. L-fuzzy sets and L-fuzzy topologies. Let X be a set in the classical sense and L a partial ordered set with first element 0 and last element 1.

Definition 1. [3, 7] *A mapping $\omega : X \to L$ is said to be a L-fuzzy set in X.*

For $L = \{0, 1\}$ we obtain the characteristic functions of the subsets in the classical sense, for $L = [0, 1] \subset R$, one obtains the notion of Zadeh fuzzy sets [7]: if L is a complete distributive lattice we have the generalization of the characteristic function given by Goguen [3].

In the following we denote by $\omega_{\emptyset}$ and ω_X the void and the total sets defined by $\omega_{\emptyset}(x) = 0 \ \forall x \in X$, $\omega_X(x) = 1 \ \forall x \in X$.

Definition 2. [2] *A family $\mathcal{T} = \{\omega\}$ of L-fuzzy sets in X such that or every $\omega \neq \omega_{\emptyset}$ it exists an element $x \in X$ $\omega(x) = 1$ will be defined as L-fuzzy topologic if*

i) $\omega_{\emptyset}, \omega_X \in \mathcal{T}$;

ii) $\forall \omega_1, \omega_2 \in \mathcal{T}, \exists \omega_{12} \in \mathcal{T}, \omega_{12}(x) = \min \{\omega_1(x), \omega_2(x)\} \ \forall x \in X$;

iii) $\forall \varphi \subset \mathcal{T}, \exists \omega_{\Gamma\varphi} \in \mathcal{T}, \omega_{\Gamma\varphi}(x) = \sup\limits_{\omega \in \varphi} \{\omega(x)\} \ \forall x \in X$.

This definition has been given by Chang for $L = [0, 1] \subset R$.

2. The relation between fuzzy topologies and approximation structures
In [5] we have introduced the approximation structure on a set X directed by A as a subset

$$\mathcal{A} \subset X \times \mathcal{R}(A, X),$$

where $\mathcal{R}(A, X)$ is the family of all binary relations $\alpha \subset A \times X$ satisfying:

(1) $\quad \forall x \in X \quad \exists \alpha \in \mathcal{R}(A, X) \quad (x, \alpha) \in \mathcal{A}$.

Let us remark that by associating the family $\mathcal{R}(A, X)$ one can conceive that the set X is endowed with a structure of $\mathcal{P}(A)$-fuzzy sets. Indeed a binary relation $\alpha \in \mathcal{R}(A, X)$ can be considered as a $\mathcal{P}(A)$-fuzzy set on X for which the characteristic function with values in $\mathcal{P}(A)$ is defined by

$$\partial(\alpha, x_0) = \{a \in A \mid (a, x_0) \in \alpha\}. \tag{2}$$

This is a special case of definition 1 where $L = \mathcal{P}(A)$ is a Boole algebra

Particularly we can consider the approximation structure associated with a $\mathcal{P}(A)$-fuzzy topology $\mathcal{T}$ on X defined by

$$(x, \alpha) \in \mathcal{A}_{\mathcal{T}} \Leftrightarrow \partial(\alpha, .) \in \mathcal{T}, \qquad \partial(\alpha, x) = A. \tag{3}$$

One can immediately prove :

Proposition 1. *Let $\mathcal{A} \subset X \times \mathcal{R}(A, X)$ be an approximation structure on X with the following properties ,*

$$\forall (x, \alpha) \in \mathcal{A} \quad \partial(\alpha, x) = A ; \tag{4}$$

$$\forall x : (x, \quad A \times X) \in \mathcal{A} ; \tag{5}$$

$$\forall (x, \alpha) \in \mathcal{A}, \ (y . \beta) \in \mathcal{A} : (\exists (z, \gamma) \in \mathcal{A} : \gamma = \alpha \cap \beta) \vee (\alpha \cup \beta = \emptyset) ; \tag{6}$$

$$\forall (x_i, \alpha_i) \in \mathcal{A}, \ i \in I : (x_j \bigcup_{i \in I} \alpha_i) \in \mathcal{A}, \ \forall j \in I. \tag{7}$$

Then the family $\mathcal{T} = \{\alpha \in \mathcal{R}(A, X) : \exists x : (x, \alpha) \in \mathcal{A}\} \cup \{\emptyset\}$ is a $\mathcal{P}(A)$-fuzzy topology on X.

Conversely given a $\mathcal{P}(A)$-fuzzy topology on X the approximation structure defined by (3) satisfies conditions (4)—(7).

The such established relation between the $\mathcal{P}(A)$-fuzzy topologies and the approximation structures allowes to formulate a Liapounoff set stability result with respect to a $\mathcal{P}(A)$-fuzzy preordered relation and a $\mathcal{P}(A)$ fuzzy topology, by using a suitable modification of a general stability theorem given in [6].

Let σ be a $\mathcal{P}(A)$-fuzzy preorder relation on X i.e. a subset of the cartezian product $\sigma \subset X \times X \times A$ with the following properties :

$$\forall a \in A, \ \forall x \qquad (x, x, a) \in \sigma ; \tag{8i}$$

$$\forall a \ \forall x, y, z \text{ such that } (x, y, a) \in \sigma, \ (y, z, a) \in \sigma \rightarrow (x, z, a) \in \sigma. \tag{8ii}$$

We denote in the following by

$$\sigma(u) = \{(a, z) \in A \times X \ (u, z, a) \in \sigma\}$$

the $\mathcal{P}(A)$-fuzzy „trajectory" of the „dynamical system" σ through u.

Let ω be a $\mathcal{P}(A)$-fuzzy set on X and M the collection of all $x \in X$ such that $\omega(x) = A$ and

$$\sigma(\omega) = \bigcup_{x} \{\sigma(x) \ x \in M\}.$$

Let also $\{X, G, A, K, \rho\}$ be an approximative generalized metric (a.g.m.) on X (see [6]) i.e. a mapping

$$\rho : X \times X \rightarrow G^A \text{ and a family } K \subset \mathcal{R}(A, G).$$

As it results from proposition 1 of [6] it always exists an a.g.m. $\{X, G, A, \mathcal{K}, \rho\}$ so that

$$(x, \alpha) \in \mathcal{A}_{\mathcal{T}} \Leftrightarrow \exists \varkappa \in \mathcal{K}, \ \forall a \ (a . y) \in \alpha, \ \rho(x . y)[a] \in \varkappa[a]. \tag{9}$$

Moreover from (6) it results that the a.g.m. is filtral i.e.

$$\forall \varkappa_1, \varkappa_2 \in K, \qquad \exists x_{12}, \qquad \varkappa_{12} \subset \varkappa_1 \cap \varkappa_2 .$$

Let ω be a $\mathcal{P}(A)$ fuzzy set on X. We shall say that

$$\mathcal{A}(\omega ; \varkappa) = \{(a, y) \in A \times X : \exists x, \ \omega(x) = A, \ \rho(x, y)[a] \in \varkappa[a]\} \tag{10}$$

is a k-order approximation of ω.

Definition 3. *We shall say that the $\mathcal{P}(A)$-fuzzy set ω is stable with respect tot the generalized preorder σ and the a.g.m. $\{X, G, A, K, \rho\}$, briefly (σ, ρ)-stable if*

$$\forall \varkappa \mathcal{K}, \ \exists \varkappa' \in K : \sigma[\mathcal{A}(\omega, \varkappa')] \subset \mathcal{A}(\omega, \varkappa). \tag{11}$$

Definition 4. [6] *Let $\{\mathcal{H}, \leqslant, h_0\}$ be a partial ordered set with first element h_0, $\mathcal{H}^+ = \mathcal{H} / \{h_0\} \neq \emptyset$ and U a $\mathcal{P}(A)$-fuzzy set so that $\sigma(U) = U$, and there is $\varkappa \in \mathcal{K}$, $\mathcal{A}(\omega, \varkappa) \subset U$.*

The mapping $\mathcal{L} : U \rightarrow \mathcal{H}$ is said a Liapounoff function for ω if

$$\forall (a, x) \in U, \ (x, y, a) \in \sigma \Rightarrow \mathcal{L}(a, y) \leqslant \mathcal{L}(a, \varkappa) ; \tag{L I}$$

$$\forall h \in \mathcal{H}^+, \exists \varkappa \in K, \ \forall (a, x) \in \mathcal{A}(\omega, \varkappa) \cap U \Rightarrow h \nless \mathcal{L}(a, \varkappa) ; \tag{L II}$$

$$\forall \varkappa \in \mathcal{K}, \exists L \in \mathcal{H}^+, \forall (a, x) \in U / \mathcal{A}(\omega, \varkappa) \Rightarrow h < \mathcal{L}(a, \varkappa). \tag{L III}$$

This is the stated generalization of Bushaw's definition.

Let $\mathcal{C}$ be a collection of subsets of Y ($\mathcal{C} \subset \mathcal{P}(Y)$). The set

$$\mathcal{E}(\mathcal{C}) = \{\Psi \subset \mathcal{C} : M \in \Psi, \ N \in \mathcal{C}, M \subset N \Rightarrow N \in \Psi\}$$

ordered by

$$\Psi < \chi \Leftrightarrow \chi \subset \Psi.$$

is called the scale of $\mathcal{C}$.

With these, by the same argument as for the proof of proposition 4 in [6] one can prove.

Proposition 2. *The $\mathcal{P}(A)$-fuzzy set ω on X is (σ, ρ)-stable iff there exists a Liapounoff function for ω with values on the scale $\mathcal{E}(K)$.*

Remark. By suitable particularizations of the approximation structure $\mathcal{A}$ associated with the fuzzy set structure and a fuzzy topology on X one can obtain a fuzzy stability theorem in the Zadeh's sense ($L = [0, 1] \subset \mathbf{R}$) for the fuzzy trajectories of a dynamical generalized system defined by the preorder relation σ.

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LINEAR DIFFERENTIAL EQUATIONS WITH PERIODIC COEFFICIENTS IN THE CRITICAL CASE

BY

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1. In [3], [4] and [5] we dealt with

$$\text{(q)} \qquad y'' = q(t)\, y,\ q \in C^{\infty}(\mathbf{R}) = C^{\infty},\ q(t+\pi) = q(t) \text{ on } \mathbf{R} = (-\infty, \infty),$$

we derived a criterion for periodicity of all solutions of (q), and gave an explicit form of all such equations (q). The present paper is devoted to equations (q) having some (i.e. in the critical case) but not all periodic solutions. With respect to Liapunoff and Haupt theory of Hill's equations $y'' = [\lambda + p(t)]\, y$, $p(t+\pi) = p(t)$, those equations are in close relation to the „problem of coexistence" of periodic solutions (cf., e.g., [1] or [2]).

2. For a positive integer n, $0 \neq A_i \in \mathbf{R}$ for $i = 1, \ldots, n$, and $0 \leqq a_1 < a_2 < \ldots < a_n < \pi$, define $p_n : \mathbf{R} \to \mathbf{R}$ by

$$\text{(1)} \qquad p_n(t) := \left(\sum_{i=1}^{n} \frac{A_i^2}{\sin^2(t-a_i)}\right)^{-1/2} \cdot \operatorname{sign} \prod_{i=1}^{n} \sin(t-a_i),\quad p_n(a_i) := 0.$$

This $p_n \in C^2$, $p_n(t+\pi) = (-1)^n p_n(t)$, and $p_n(t) \neq 0$ for $t \neq a_i$. Recall a result of [5, Th. 5.1] that explicitly gives all equations (q) having some, but not necessarily all, periodic solutions.

Lemma 1. *A differential equation* (q) *has a nontrivial periodic or half-periodic* ($y(t+\pi) = -y(t)$) *solutions with period* π *and with* n *zeros* $a_i \in [0, \pi)$, $i = 1, \ldots, n$, $n \geqq 1$, *iff*

$$q = f'' + f'^2 + 2 \cdot f' \cdot p_n'/p_n + p_n''/p_n,$$

where $f \in C^2$, $f(t+\pi) = f(t)$, $f(a_i) = f'(a_i) = 0$ *for* $i = 1, \ldots, n$.

The function $t \to p_n(t) \cdot \exp f(t)$ *is a periodic* (n *even*) *or half-periodic* (n *odd*) *solution of the differential equation* (q) *that vanishes just at* a_i *having* $y'(a_i) = (-1)^{n-i}|A_i|$, $i = 1, \ldots, n$.

For $n = 1$ we get

Corollary 1. *A differential equation* (q) *has a nontrivial half-periodic solution* y *with period* π *and just one zero* a *on* $[0, \pi)$ *iff*

$$q = f'' + f'^2 + 2 \cdot f' \cdot \cot(t-a) - 1,$$

where $f \in C^2$, $f(t+\pi) = f(t)$, $f(a) = f'(a) = 0$.

The function $\exp f(t) \cdot \sin(t-a)$ *is a half-periodic solution of* (q) *that vanishes at* a.