$$\sum_{i=j+1}^{j+n} \int_{b_{i-1}}^{b_i} \left(y_2^{-2}(s) - (y_2'(a_i) \sin(s-a))^{-2} \right) ds =$$

$$\sum_{i=j+1}^{j+n} \left\{ - \int_{b_j}^{b_{i-1}} (y_2'(a_i) \sin(s-a_i))^{-2}\, ds - \int_{b_i}^{b_j+\pi} (y_2'(a_i) \sin(s-a_i))^{-2}\, ds \right.$$

$$\left. + \int_{b_{i-1}}^{b_i} \left(y_2^{-2}(s) - (y_2'(a_i) \sin(s-a))^{-2} \right) ds \right\} =$$

$$\int_{b_j}^{b_j+\pi} \left(y_2^{-2}(s) - \sum_{i=j+1}^{j+n} \frac{A_i^2}{\sin^2(s-a_i)} \right) ds.$$

Because of π-periodicity of the integrand, the last expression is equal to $(-1)^{n+1} K = -\varepsilon K$. Hence $c_{j+n} - c_j = -\varepsilon K$ and $y_1 := -y_3/K$ satisfies $y_1(t+\pi) = \varepsilon y_1(t) + y_2(t)$. Now det $W(y_1, y_2) = -1/K$, det $W(y_3, y_2) = 1/K$, since det $W(y_3, y_2) = -1$. Due to Lemma 2, Wronskian of any pair satisfying (2) is of the same sign as $1/K$ or as the expression (3).

Now the problem imposed in the beginning of the Section can be solved. Since f in (3) is restricted only by $f \in C^2$, $f(a_i) = f'(a_i) = 0$ (Lemma 1), then:
in the oscillatory case we can always choose f such that (3) is of prescribed sign and construct according to Lemma 1 the corresponding oscillatory differential equation (q), $q = f'' + f'^2 + 2 \cdot f' \cdot p_n'/p_n + p_n''/p_n$, *with prescribed sign of Wronskian of its pair of solutions satisfying the relation (2), whereas in the nonoscillatory case the Wronskians are always negative.*

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ON BOUNDEDNESS RESULT OF A VOLTERRA EQUATION

BY

STIG-OLOF LONDEN

1. Introduction. In this paper we state and prove a boundedness result on the Volterra equation

$$x(t) + \int_0^t a(t-\tau)\, g(x(\tau))\, d\tau = f(t), \quad t \in R^+ = [0, \infty). \tag{1.1}$$

where a, g, f are given real valued functions and $x(t)$ is the solution whose existence on R^+ we postulate.

Define $A(t) = \int_0^t a(\tau)\, d\tau$, $t \in R^+$. Our result is then the following

Theorem. *Let*

$$a(t) \geqslant 0, \quad a(t) \text{ nonincreasing}, \quad t \in R^+, \tag{1.2}$$

$$0 < a(0) < \infty, \tag{1.3}$$

$$g \in C(-\infty, \infty), \tag{1.4}$$

and assume that

$$f \text{ is locally absolutely continuous on } R^+, \tag{1.5}$$

with

$$f' A^{-1/2} \in L^1(1, \infty). \tag{1.6}$$

Moreover, suppose that there exists a positive, nondecreasing function $b(t)$, $t \in R^+$, satisfying

$$ab^{-1} \in L^1(1, \infty), \quad |f'|^2 A^{-1} b \in L^1(1, \infty), \tag{1.7}$$

and such that if $c(t) \overset{\text{def}}{=} \int_t^\infty (ab^{-1})(\tau)\, d\tau$, $t \geqslant 0$, *then*

$$\widehat{da}(\omega) \geqslant \widehat{dc}(\omega), \quad \omega \in R. \tag{1.8}$$

Finally let $x \in C[0, \infty)$ satisfy (1.1) and suppose

$$\liminf_{x \to \infty} G(x) > -\infty. \tag{1.9}$$

Then

$$\sup_{t\in R^+} Q_a(g, t) < \infty, \qquad (1.11) \quad \sup_{t\in R^+} G(x(t)) < \infty. \tag{1.10}$$

In (1.8)—(1.11) we use the notations

$$\widehat{da}(\omega) = a(0) + \int_0^\infty \cos(\omega\tau)\, da(\tau), \quad G(x) = \int_0^x g(u)\, du,$$

$$Q_a(g, t) = \int_0^t g(x(\tau)) \left[a(0)\, g(x(\tau)) + \int_0^\tau g(x(\tau - s))\, da(s)\right] d\tau.$$

The first part of (1.7) shows that $c(t)$ is well-defined.

It will become obvious from the proof below that the bounds in (1.10), (1.11) can be explicity computed and only depend on $G(x)$, on $x(0)$ and $\int_0^1 f'(\tau)\, g(x(\tau))\, d\tau$, and on the quantities

$$K_1 \overset{\text{def}}{=} \int_1^\infty |f'(\tau)|\, A^{-1/2}(\tau)\, d\tau, \quad K_2^2 \overset{\text{def}}{=} \int_1^\infty |f'(\tau)|^2 A^{-1}(\tau)\, b(\tau)\, d\tau. \tag{1.12}$$

Observe also that having (1.2), (1.10), one can without great difficulties deduce that

$$\sup_{t\in R^+} \int_t^{t+1} |g(x(\tau))|^2\, d\tau < \infty. \tag{1.13}$$

If $G(x)$ in addition to (1.9) satisfies $\limsup_{|x|\to\infty} G(x) = \infty$, then (1.11) of course yields

$$\sup_{t\in R^+} |x(t)| < \infty. \tag{1.14}$$

As a guide to the conditions (1.6)—(1.8) consider the following example. Assume $a(t) = t^{-p}$, for all sufficiently large t and some fixed p satisfying $0 < p \leqslant 1$. The choice $b(t) = \alpha\,[1 + t]$, with a suitable chosen positive constant α, will then make the first part of (1.7) and (1.8) hold. The condition (1.6) and the second part of (1.7) do in this case reduce to

$$\int_1^\infty |f'(\tau)|\, \tau^{\frac{p-1}{2}}\, d\tau < \infty, \quad \int_1^\infty |f'(\tau)|^2\, \tau^p\, d\tau < \infty. \tag{1.15}$$

If for example $p = 1/2$ we are therefore allowed to pick any nonhomogeneous term satisfying $|f'(\tau)| \leqslant \tau^{-3/4-\delta}$, where $\delta > 0$ can be taken arbitrarily small. This indicates that the size conditions (1.6), (1.7) are quite weak. In particular they do not preclude an unbounded nonhomogeneous term; the existence of $\lim_{t\to\infty} f(t)$ is not assumed. On the other hand, note that a necessary prerequisite for (1.8) to hold is $\widehat{da}(\omega) > 0$, $\omega \neq 0$, which of course excludes a large class of kernels satisfying (1.2), (1.3).

The proof below not tied to R^1 but can easily be carried over to a Hilbert space setting.

Boundedness results on the eq. (1.1) under conditions comparable to those above have earlier been obtained by, among others, Levin [2], Londen [3], and more recently by Gripenberg [1] who generalized the results of [2]. The above Theorem partially extends all these results. It should also be remarked that the proof of [2] (which is also used in [1]) does not seem to admit a straightforward extension from R^1 to a Hilbert space setting. Note finally the recent results by Staffans [4] who considers a more general equation than (1.1) but imposes somewhat stronger conditions on the nonhomogeneous term than what is done here.

2. *Proof.* Differentiate (1.1), multiply the resulting equation by $g(t) = g(x(t))$ and integrate. This gives

$$G(x(t)) - G(x(0)) + Q_a(g, t) = \int_0^t f'(\tau)\, g(\tau)\, d\tau. \tag{2.1}$$

The primary problem is now to estimate the right side of (2.1). This we do in what follows.

For $\tau > 0$ we have the identity

$$g(\tau) = [A(\tau)]^{-1} \int_0^\tau a(\tau - s)\, g(s)\, ds - [A(\tau)]^{-1} \int_0^\tau a(\tau - s)\,[g(s) - g(\tau)]\, ds \tag{2.2}$$

and so, by elementary estimates and (1.2),

$$\begin{aligned} |g(\tau)| \leqslant {} & [A(\tau)]^{-1/2} \left[\int_0^\tau a(\tau - s)\, g^2(s)\, ds\right]^{1/2} + \\ & + [A(\tau)]^{-1/2} \left[\int_0^\tau a(\tau - s)\, [g(s) - g(\tau)]^2\, ds\right]^{1/2}. \end{aligned} \tag{2.3}$$

Recall that the following identity is valid, see [3, p. 110],

$$\begin{aligned} 2Q_a(g, \tau) = {} & \int_0^\tau a(\tau - s)\, g^2(s)\, ds + \int_0^\tau a(s)\, g^2(s)\, ds - \\ & - \int_0^\tau \left\{\int_0^s [g(s) - g(s - v)]^2\, da(v)\right\} ds. \end{aligned} \tag{2.4}$$

An application of (2.3) gives the first inequality in (2.5). The second uses the fact that b is positive and nondecreasing and the first part of (1.12). The third follows by the second part of (1.12), the definition of c and an obvious estimate. The last inequality in (2.5) is a consequence of (1.2) and (2.4).

$$
\begin{aligned}
\int_1^t f'(\tau)\, g(\tau)\, d\tau &\leqslant \int_1^t |f'(\tau)|\, A^{-1/2}(\tau) \left[\int_0^\tau a(\tau - s)\, g^2(s)\, ds\right]^{1/2} d\tau + \\
&+ \int_1^t |f'(\tau)|\, A^{-1/2}(\tau) \left|\int_0^\tau a(\tau - s)\, [g(s) - g(\tau)]^2\, ds\right|^{1/2} d\tau \leqslant \\
&\leqslant K_1 \sup_{0 \leqslant \tau \leqslant t} \left[\int_0^\tau a(\tau - s)\, g^2(s)\, ds\right]^{1/2} + \\
(2.5) \qquad &+ \int_1^t |f'(\tau)|\, A^{-1/2}(\tau)\, b^{1/2}(\tau) \left[\int_0^\tau \frac{a(\tau - s)}{b(\tau - s)}\, [g(s) - g(\tau)]^2\, ds\right]^{1/2} d\tau \leqslant \\
&\leqslant K_1 \sup_{1 \leqslant \tau \leqslant t} \left|\int_0^\tau a(\tau - s)\, g^2(s)\, ds\right|^{1/2} + \\
&+ K_2 \left[\int_0^t \int_0^\tau |c'(\tau - s)|\, [g(s) - g(\tau)]^2\, ds\, d\tau\right]^{1/2} \leqslant \\
&\leqslant 2K_1 \sup_{0 \leqslant \tau \leqslant t} [Q_a(g, \tau)]^{1/2} + 2K_2\, [Q_c(g, t)]^{1/2}.
\end{aligned}
$$

By (1.2), (1.8) one has

$$
(2.6) \qquad 0 \leqslant Q_c(g, t) \leqslant Q_a(g, t), \quad t \geqslant 0.
$$

After using (2.5) and the second inequality of (2.6) in (2.1) we arrive at

$$
\begin{aligned}
&G(x(t)) - G(x(0)) + \frac{1}{2} Q_a(g, t) + \frac{1}{2} Q_c(g, t) \leqslant \\
(2.7) \qquad &\leqslant 2K_1 \sup_{0 \leqslant \tau \leqslant t} [Q_a(g, \tau)]^{1/2} + 2K_2\, [Q_c(g, t)]^{1/2} + \int_0^1 f'(\tau)\, g(\tau)\, d\tau.
\end{aligned}
$$

Employing (1.9) and the first inequality of (2.6) in (2.7) one now gets (1.10), (1.11).

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ADMISSIBILITY AND PERTURBATION THEORY OF SOME LINEAR VOLTERRA INTEGRAL AND INTEGRODIFFERENTIAL SYSTEMS

BY

N. LUCA

1. Introduction. The concept of admissibility introduced by O. Perron for ordinary differential equations was extended for more general functional equations by J. L. Massera and J. J. Schäffer [12].

The admissibility of the integral operators in various functional spaces was studied systematically by C. Corduneanu [2, 5], R. K. Miller [13], D. Petrovanu [16] and G. Bantaş [1]. On the basis of these results some theorems of existence, uniqueness and behaviour for the solutions of integral or integro-differential equations were obtained in the previous papers. In [7,14] S. I. Grossman and R. K. Miller have studied such kinds of problems for systems of the form

$$
(N) \quad \dot{x}(t) = f(t) + A(t)\, x(t) + \int_0^t B(t, s)\, x(s)\, ds + (gx)(t),
$$

$$
x(0) = x_0,
$$

or of the form

$$
(N_1) \quad \dot{x}(t) = Ax(t) + \int_0^t B(t - s)\, x(s)\, ds + (hx)(t) + F(t - r), \; t \geqslant r \geqslant 0,
$$

$$
x(t) = \alpha(t), \; 0 \leqslant t \leqslant r,
$$

where A and B are known square matrices of order n, f, g, h, and F are given vector-valued functions with n components, r is a given positive number, and $x(t)$ is the unknown vector function.

The proofs of the theorems in the previous papers are based upon a variation of constants formula and on some main theorems of functional analysis.

Analogously, other results were obtained (see, for instance [15]).