Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXIII, S. I a, f. 2, 1977

INVARIANT SOLUTIONS OF DIFFERENTIAL EQUATIONS

BY

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To Prof. Gh. Gheorghiev and Prof. Ilie Popa on the occasion of their 70-th birthdays.

Starting with the fact that a periodical solution of a differential equation (as, in general for a function of a real variable) is a solution invariant under translations, the purpose of this paper is to study differential equations, which admit solutions invariant under transformations, more general than translations, as, for instance, transformations of the form :

$$\text{(a)} \qquad \tilde{x} = \varphi(x) \qquad \varphi \in C^1[(-\infty, +\infty)].$$

It is obvious that the case of equations admitting periodical solutions will be a particular case of the above one.

A more general case of invariantness was investigated in [1], but under other points of view.

A first generalization of the periodical functions will be the solutions invariant under the affine transformations

$$\text{(b)} \qquad \tilde{x} = ax + b, \quad a \neq 1.$$

But, it is easy to prove that a continuous function $R \to R$ invariant under the transformation (b) is a constant. Indeed, the identity

$$f(x) = f(ax + b), \quad f \in C[(-\infty, +\infty)],$$

leads, after n iterations, to

$$f(a^n x + b(a^{n-1} + a^{n-2} + \ldots + 1)) = f(x),$$

and then to

$$\lim_n f\left(a^n x + b\frac{a^n - 1}{a - 1}\right) = f(x).$$

Since f is continuous, it follows

$$f(x) = f\left(\frac{b}{1-a}\right) \text{ for } |a| < 1,$$

and such a function is also a constant if $|a| > 1$, $b \neq 0$.

Therefore, to include the possibility of affine transformations, we will state a more general problem, that of differential equations a solution of which satisfies — under (a) — the relation

$$f(x) = mf(\varphi(x)),\ m = \text{const.} \tag{c}$$

A. Let φ be a function defined on R with values on R, satisfying the following conditions:

Aa) φ is continuous and has continuous first derivative.

Ab) φ is strong monotonic on R,

Ac) $\varphi(0) \neq 0$,

B. Let f be a vector function $R \times R^n \to R^n$, so that:

Ba) f is continuous on $R \times R^n$,

Bb) f has bounded first derivatives with regard to the variables y,

Bc) f satisfies the condition that equation

$$y' = f(x, y)$$

has a solution defined on a certain domain of R — depending on (x_0, y_0) — passing through (x_0, y_0), for all $(x_0, y_0) \in R \times R^n$;

Bd) f satisfies the relation:

$$f(\varphi(x), my)\,\varphi'(x) = mf(x, y).$$

C. The functions f, φ and the constant m satisfy one of the conditions:

C$_1$. $$|\varphi^{-1}(0)| \left| \frac{1}{m\varphi^{-1}(0)} - \frac{\partial f}{\partial y}(s, y) \exp \int_0^s \frac{\partial f}{\partial y}(t, y)\, dt \right| < 1,$$

C$_2$. $$m \neq 1, \left|\frac{m}{m-1}\right| |\varphi^{-1}(0)| \left| \frac{\partial f}{\partial y}(s, y) \exp \int_0^s \frac{\partial f}{\partial y}(t, y)\, dt \right| < 1.$$

We can prove now the following

Theorem 1. *If conditions* Aa — Ac, Ba — Bd, C$_1$ *are satisfied, the equation*

$$y' = f(x, y) \tag{1}$$

has a solution $y = y(x)$, *satisfying the relation*

$$y(\varphi(x)) = my(x). \tag{2}$$

Proof. Equation (1) is obviously equivalent to

$$y(x) = y_0 + \int_0^x f(t, y(t))\, dt, \tag{3}$$

y_0 being a vector — for the moment arbitrary. Replacing x by $\varphi(x)$, we get:

$$y(\varphi(x)) = y_0 + \int_0^{\varphi(x)} f(t, y(t))\, dt.$$

By a change of variable

$$t = \varphi(s).$$

supposing that the problem has a solution satisfying (2), taking into account the monotonicity of φ, we have

$$my(x) = y_0 + \int_{\varphi^{-1}(0)}^x f(\varphi(s), my(s))\,\varphi'(s)\, ds. \tag{4}$$

or, using Bd:

$$y(x) = \frac{1}{m} y_0 + \int_{\varphi^{-1}(0)}^x f(s, y(s))\, ds; \tag{4'}$$

(3) and (4′) give:

$$y_0\left(1 - \frac{1}{m}\right) + \int_0^{\varphi^{-1}(0)} f(s, y(s))\, ds = 0, \tag{5}$$

and also

$$y_0 = \int_0^{\varphi^{-1}(0)} \left| -f(s, y(s)) + \frac{1}{m\varphi^{-1}(0)} y_0 \right| ds. \tag{5'}$$

Conversely (3) and (5′), lead to (3) and (4). Then, substituting in the integral in (4) $t = \varphi(s)$, we get:

$$my(x) = y_0 + \int_0^{\varphi(x)} f(t, my(\varphi^{-1}(t)))\, dt$$

and from this

$$my(\varphi^{-1}(x)) = y_0 + \int_0^x f(t, my(\varphi^{-1}(t))\, dt. \tag{3'}$$

Taking now into account that the solution of (3) is unique, it follows

$$y(x) = my(\varphi^{-1}(x)).$$

or

$$y(\varphi(x)) = my(x),$$

i.e. (2).

It remains to prove that it exists y_0 so that (3) and (5′) be satisfied.

To this end, remark first that from (3) we deduce that the solution $y(x, y_0)$ of (3), satisfies also

$$\frac{\partial y(x, y_0)}{\partial y_0} = I + \int_0^x \frac{\partial f}{\partial y} \frac{\partial y}{\partial y_0}(t, y_0)\, dt, \quad I\text{-identity},$$

i.e.

$$\frac{\partial y}{\partial y_0}(x, y_0) = \exp\left[\int_0^x \frac{\partial f}{\partial y}(t, y(t, y_0))\, dt\right]. \tag{6}$$

With this expression of the matrix $\partial y / \partial y_0$, we will prove now that the mapping

$$\tilde{y}_0 = T y_0 = \int_0^{\varphi^{-1}(0)} \left[\frac{1}{m \varphi^{-1}(0)} y_0 - f(s, y(s, y_0))\right] ds$$

admits a fixed point, if the relation C_1 is satisfied. Indeed we have:

$$\tilde{y}_0 - \tilde{y}_0' = T y_0 - T y_0' = \int_0^{\varphi^{-1}(0)} \left[\frac{1}{m \varphi^{-1}(0)} - \frac{\partial f}{\partial y}(s, y) \frac{\partial y}{\partial y_0}\right] (y_0 - y_0')\, ds,$$

or also:

$$T y_0 - T y_0' = \int_0^{\varphi^{-1}(0)} \left\{\frac{1}{m \varphi^{-1}(0)} - \left|\frac{\partial f}{\partial y}(s, y) \exp\left(\int_0^s \frac{\partial f}{\partial y}(t, y)\right) dt\right|\right\} (y_0 - y_0')\, ds.$$

As one easily sees, supposing — which follows from C_1 —

$$\int_0^{\varphi^{-1}(0)} \left\{\frac{1}{m \varphi^{-1}(0)} - \left[\frac{\partial f}{\partial y}(s, y) \exp \int_0^s \frac{\partial f}{\partial y}(t, y)\, dt\right]\right\} ds \leqslant \alpha < 1,$$

the above mapping admits a fixed point, i.e. a value y_0 which satisfies (5).

Theorem 2. *Under hypotheses* Aa — Ac, Ba — Bd *and* C_2, *equation (1) has a solution* $y = y(x)$ *satisfying* (2).

Proof. The proof is similar to the preceding one, the only difference consisting in the fact that the operator T is replaced by S, defined by:

$$\tilde{y}_0 = S y_0 = \frac{m}{1 - m} \int_0^x f(s, y(s, y_0))\, ds.$$

Remark. An important particular case of the above considerations is given by

$$\varphi(x) = ax + b.$$

If $a \neq 1$, theorems 1 and 2 are satisfied, if:

$$a f(ax + b, my) = m f(x, y).$$

For instance, the function

$$f(x, y) = y^2\left(x + \frac{b}{a - 1}\right)$$

satisfies the relation

$$f(ax + b, my)\, a = m f(x, y), \quad m = \frac{1}{a^2},$$

and the equation

$$y' = y^2\left(x + \frac{b}{a - 1}\right)$$

has a solution

$$y(x) = \frac{-2}{\left(x + \frac{b}{a - 1}\right)^2},$$

satisfying

$$y(ax + b) = \frac{1}{a^2} y(x) = m y(x).$$

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Received 18. XII. 1976

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