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SUBDIFFERENTIALS OF COMPOSED FUNCTIONS AND APPLICATIONS IN OPTIMAL CONTROL

BY

DAN TIBA

1. Introduction and main results. Let V be a reflexive Banach space with dual V'. The norms in V, V' will be denoted by $\|\cdot\|$, $\|\cdot\|^*$ and the duality between V and V' by $(\ldots)$.

Let $\varphi : V \to]-\infty, +\infty]$ be a convex, lower semicontinuous, proper function.

We suppose familiarity with the definitions and basic results in theory of convex functions defined on infinite dimensional spaces. However, for easy references we remember some facts about subdifferentials.

We denote by $\partial\varphi(x)$ the set of all $z \in V'$ such that

$$\varphi(x) \leqslant \varphi(y) + (x - y, z), \quad \forall y \in V,$$

and we call it the subdifferential of φ at x.

For every $x \in V$, $\partial\varphi(x)$ is a closed convex subset (possibly empty) of V'.

Now, let $A : V \to V$ be a linear bounded operator and A^* its adjoint operator. Consider the composed mapping $\psi : V \to]-\infty, +\infty]$ given by

$$\psi(x) = \varphi(Ax), \qquad \forall x \in V. \tag{1.1}$$

Function ψ is convex, lower semicontinuous, proper if

$$R(A) \cap \mathrm{Dom}\,\varphi \neq \emptyset. \tag{1.2}$$

The main result of this paper may be stated as follows.

Theorem 1. *Assume V can be decomposed into the direct sum of its subspaces:*

$$V = V_1 \otimes V_2,$$

such that

$$R(A) \cap \mathrm{int}\, V_1\,[\mathrm{Dom}\,\varphi \cap V_1] \neq \emptyset \tag{1.3}$$

$$A^*/V_1' : V_1' \to V' \tag{1.4}$$

has bounded inverse. Then we have the formula:

$$\partial\psi(x) = A^*\partial\psi(Ax), \qquad \forall x \in V. \tag{1.5}$$

Here V_1' denotes the polar set of V_1 and int V_1 [Dom $\varphi \cap V_1$] means the interior of Dom $\varphi \cap V_1$ in the topology of V_1.

Theorem 1 is a generalization to infinite dimensional spaces of a well-known result given in the finite dimensional case in R o c k a f e l l a r [6]. Some similar results appear in [4], but the hypotheses are much more restrictive. Our theorem can be remarked by the unusual weak assumptions made and by the large class of applications, especially to control problems.

In Section 2 we prove Theorem 1 and Section 3 contains applications to abstract control problems and an example.

2. Proof of Theorem 1. First we notice that ψ, as defined by (1.1) verifies the relation :

$$A^*\partial\varphi(Ax) \subseteq \partial\psi(x), \quad \forall x \in V. \tag{2.1}$$

Consider the operator $S : V \to V'$ given by :

$$Sx = A^*\partial\varphi(Ax), \qquad \forall x \in V. \tag{2.2}$$

S is monotone (from (2.1)). We must show that S is maximal monotone, i.e., the equation

$$(J + S)\,x \ni f \tag{2.3}$$

has at least one solution for every f in V'. Here $J : V \to V'$ is the duality mapping.

Consider the approximating equations :

$$Jx_\lambda + A^*\partial\varphi_\lambda(Ax) \ni f, \quad \lambda > 0, \tag{2.4}$$

where φ_λ is defined by

$$\varphi_\lambda(x) = \inf_{y \in V}\left\{\frac{\|x - y\|^2}{2\lambda} + \varphi(y)\right\}, \quad x \in V.$$

For every $\lambda > 0$ equation (2.4) has a solution in V because operator $A^*\partial\psi_\lambda(A)$ is monotone and demicontinuous and therefore maximal monotone ([2], p. 38).

Suppose first

$$0 \in R(A) \cap \text{int } V_1 \,[\text{Dom } \psi \cap V_1]. \tag{1.3'}$$

We multiply by x_λ in (2.4) and apply the definition of the subdifferential :

$$\|x_\lambda\|^2 + \varphi_\lambda(Ax_\lambda) - \varphi_\lambda(0) \leqslant (f, x_\lambda). \tag{2.5}$$

We use the inequalities :

$$\varphi_\lambda(0) \leqslant \varphi(0), \qquad \lambda > 0, \tag{2.6}$$

$$\varphi_\lambda(Ax_\lambda) \geqslant -\|\alpha\|_* \, m \, \|x_\lambda\| + \beta, \tag{2.7}$$

where $\alpha \in V'$, m, $\beta \in R$ are such that $(\alpha, x) + \beta$ is the affine mapping which majorizes from below function φ and m is the norm of operator A. From (2.5), (2.6) and (2.7) we deduce that $\{x_\lambda\}$, is bounded in V. Now we

write $\partial\varphi_\lambda(Ax_\lambda) = \partial_1\varphi_\lambda(Ax_\lambda) + \partial_2\varphi_\lambda(Ax_\lambda)$ where $\partial_1\varphi_\lambda(Ax_\lambda) \in V_1'$ and $\partial_2\varphi_\lambda(Ax_\lambda) \in V_2'$. Here V_1', V_2' are the polar spaces of V_1, V_2 respectively and, of course, we have :

$$V' = V_1' \oplus V_2'.$$

Equations (2.4) can be rewritten as :

$$Jx_\lambda + A^*\partial_1\varphi_\lambda(Ax_\lambda) + A^*\partial_2\varphi_\lambda(Ax_\lambda) = f. \tag{2.8}$$

From (1.3)' one can find $\bar\rho > 0$, C such that :

$$\varphi(\rho h) \leqslant C \quad |\rho| \leqslant \bar\rho, \quad h \in V_1, \quad \|h\| = 1. \tag{2.9}$$

From the definition of subdifferential, (2.7), (2.8), (2.9) and the boundedness of $\{x_\lambda\}$ we obtain :

$$(\partial_2\varphi_\lambda(Ax_\lambda), \rho h) \leqslant (\partial\varphi_\lambda(Ax_\lambda), Ax_\lambda) + \varphi_\lambda(\rho h) - \varphi_\lambda(Ax_\lambda) \leqslant (f, x_\lambda) - \|x_\lambda\|^2 +$$
$$+ \varphi(\rho h) - \varphi_\lambda(Ax_\lambda) \leqslant M, \quad |\rho| \leqslant \bar\rho, \; h \in V_1, \; \|h\| = 1.$$

From the reflexivity assumption we can identify V_1 with the dual, of V_2', by the duality $(\ldots)$. So $\{\partial_2\varphi_\lambda(Ax_\lambda)\}$, is bounded in V_2' and this implies its boundedness in V'. Equations (2.8) and (1.4) give $\{\partial\varphi_\lambda(Ax_\lambda)\}$, $\lambda > 0$, bounded in V'.

Next we pass to the limit :

$$x_\lambda \to x \text{ weakly in } V, \tag{2.10}$$

$$\partial\varphi_\lambda(Ax_\lambda) \to y \text{ weakly in } V'. \tag{2.11}$$

From Equations (2.4) one gets

$$\|x_\lambda - x_\mu\|^2 + (\partial\varphi_\lambda(Ax_\lambda) - \partial\varphi_\mu(Ax_\mu), Ax_\lambda - Ax_\mu) = 0 ;$$

therefore

$$\limsup_{\lambda,\mu \to 0} (\partial\varphi_\lambda(Ax_\lambda) - \partial\varphi_\mu(Ax_\mu), Ax_\lambda - Ax_\mu) \leqslant 0.$$

A well-known result then implies $y \in \partial\varphi(Ax)$ i.e. x verifies (2.3), ([2], p. 42).

To conclude the proof, we note that from hypotheses (1.3) one obtains $\xi \in V$ such that

$$A\xi \in \text{int } V_1 \,[\text{Dom}(\varphi) \cap V_1].$$

Consider functions $\varphi_1 : V \to]-\infty, +\infty]$; $\varphi_1(x) = \varphi(x + A\xi)$ and

$$\psi_1 : V \to]-\infty, +\infty]; \; \psi_1(x) = \varphi_1(Ax) = \psi(x + \xi_1).$$

They verify relation (1.3)', so

$$\partial\psi_1(x) = A^*\partial\varphi_1(Ax).$$

But $\partial\psi_1(x) = \partial\psi(x + \xi)$ and $\partial\varphi_1(x) = \partial\varphi(x + A\xi)$ $\forall x \in V$ and this proves Theorem 1.

3. Applications to abstract control problems. Let X be another reflexive Banach space with norm $|\cdot|$ and dual X'. We consider a convex lower semicontinuous, proper mapping $L : V \times X \to]-\infty, +\infty]$ and a linear bounded operator $A : X \to V$. For every $[y, u] \in V \times X$, $\partial L(y, u)$, the subdifferential of L at point $[y, u]$, is a closed, convex subset of $V' \times X'$, which we denote

$$\partial L(y, u) = [\partial_v L(y, u), \partial_x (L(y,u))],$$

where $[\ldots]$ means ordered pair and $\partial_x L(y, u) \in X'$, $\partial_v L(y, u) \in V'$.

Suppose that $G^{-1}(A) \cap \mathrm{Dom}\, L \neq \emptyset$, where $G(A)$ denotes the graph of operator A. Then the function $\Gamma : X \to]-\infty, +\infty]$

(3.1) $$\Gamma(u) = L(Au, u)$$

is convex, lower semicontinuous and proper.

Theorem 2. *Suppose there is $y \in X$ such that*

(3.2) $$G^{-1}(A) \cap [\mathrm{int\, Dom}\, L(\cdot, y), y] \neq \emptyset.$$

Then we have:

(3.3) $$\partial\Gamma(u) = A^* \partial_v L(Au, u) + \partial_x L(Au, u).$$

Proof. Suppose first

(3.2)' $$G^{-1}(A) \cap [\mathrm{int\, Dom}\, L(\cdot, 0), 0] \neq \emptyset.$$

Consider the operator $\mathcal{A} : V \times X \to V \times X$:

$$\mathcal{A}(v, x) = [Ax, x], \qquad [v, x] \in V \times X;$$

$\mathcal{A}$ is a linear bounded operator and

$$\mathcal{A}^*(u, z) = [0, A^*u + z], \quad \forall [u, z] \in V' \times X'.$$

Remark that, taking $V_1 = V$ and $V_2 = X$, hypotheses of Theorem 1 are fulfilled for $\mathcal{A}$, so we obtain the desired conclusion.

In the case $y \neq 0$, consider the function

$$L_1 : V \times X \to]-\infty, +\infty] \quad L_1(v, x) = L(v + Ay, x + y),$$

which verifies (3.2)' and use the argument from the final of the proof of Theorem 1.

Theorem 2 can be used in finding necessary and sufficient conditions for optimality, in subdifferential form, in control problems of the following abstract type:

(3.4) $$\inf\{L(y, u) ; [y, u] \in V \times X\},$$

subject to the contraints:

(3.5) $$\Lambda y = Bu,$$

where $B : X \to V$ is linear, continuous and $\Lambda : \mathrm{Dom}(\Lambda) \subset V \to V$ is linear, closed, with continuous inverse.

Problem (3.4) – (3.5) includes, by redefining L, more general constrained of the type:

(3.4)' $$\inf\{L(y, u) ; [y, u] \in K \times X_0\},$$

(3.5)' $$\Lambda y = Bu + f,$$

where f is a fixed element and K, X_0 are closed, convex subsets of V, X.

Usually L is an integral convex cost functional and Λ a differential operator.

We shall say that a given pair $[x, u] \in V \times X$ is extremal for problem (3.4) – (3.5) if there exists $p \in V'$ such that

(3.5) $$\Lambda x = Bu,$$

(3.6) $$\Lambda^* p \in -\partial_v L(x, u),$$

(3.7) $$B^* p \in \partial_x L(x, u).$$

Corollary. *Assume the existence of functions and operators satisfying the above hypotheses. Then, in order that the pair $[x, u]$ be optimal in the problem (3.4)–(3.5) it is necessary and sufficient that it is extremal.*

Now, let Ω be a domain in R^n, the n dimensional Euclidean space and $g : \Omega \times R \times R \to R$ a function such that $g(x, \ldots)$ is convex, lower semicontinuous, proper for every x in Ω.

Consider the problem

(3.8) $$\inf\left\{\int_\Omega g(x, y(x), u(x))\, dx : y \in L^P(\Omega),\ u \in L^P(\Omega)\right\},$$

subject to the constraints:

(3.9) $$\Delta y = Bu \text{ a.e. } \Omega,$$

(3.10) $$y|_\Gamma = 0,$$

where operator $B : L^P(\Omega) \to L^P(\Omega)$ is linear continuous, Γ denotes the boundary of Ω and $1 < P < +\infty$.

Moreover we suppose that there is $u_0 \in R$ such that:

(3.11) $$|g(x, y, u_0)| \leq \alpha(x) |y|^P + \beta(x) \text{ a.e. } \Omega.$$

Here $\alpha \in L^\infty(\Omega)$ and $\beta \in L^P(\Omega)$.

To find necessary and sufficient conditions for optimality for the problem (3.8) — (3.10) we shall apply Corollary 1.

We take $V = X = L^P(\Omega)$, $\Lambda = \Delta$ with $\mathrm{Dom}(\Lambda) = W^{2,P}(\Omega) \cap W_0^{1,P}(\Omega)$, and the functional

$$L : L^P(\Omega) \times L^P(\Omega) \to]-\infty, +\infty]$$

$$L(y, u) = \int_\Omega g(x, y(x), u(x))\, dx.$$

Here $W^{2,P}(\Omega)$, $W_0^{1,P}(\Omega)$ are the usual Sobolev spaces.

Operator Λ is well-known to have bounded inverse and from (3.11) one easily infer that

$$\mathrm{Dom}\ L(.,\ u_0) = L^r(\Omega).$$

and in consequence hypothesis (3.2) is evidently fulfilled.

To obtain the optimality principle we have only to transcript (3.5) — —(3.7), taking into account that ([2], p. 61)

$$\partial L(y,\ u) = \{[h,\ v] \in L^{P'}(\Omega) \times L^{P'}(\Omega) \mid [h(x).\ v(x)] \in$$
$$\in \partial g(x,\ y(x),\ u(x))\ \text{a.e.}\ \Omega\}.$$

Corollary 2. *Suppose* (3.11) *hold. Then in order that pair* $[y,\ u]$ *be optimal for problem* (3.8) — (3.10) *it is necessary and sufficient that there is* $p \in L^{P'}(\Omega)$ *such that* $(1/P + 1/P' = 1)$:

(3.11) $$\Delta y = Bu,$$

(3.12) $$[-\Delta p(x),\ B^*p(x)] \in \partial g(x,\ y(x),\ u(x))\ \text{a.e.}\ \Omega,$$

(3.13) $$y/\Gamma = 0,$$

(3.14) $$p/\Gamma = 0.$$

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SUR LES SYSTÈMES DE PFAFF DE CARACTÈRE s_1 DONT LE SYSTÈME DÉRIVÉ A MOINS DE $s - s_1$ ÉQUATIONS

PAR

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Dans cette Note, nous nous occupons de la forme du système dérivé d'un système d'équations de Pfaff de caractère s_1. Le principal résultat est le théorème à la fin de l'ouvrage.

§1. Partout dans ce qui suit, nous allons utiliser la conventions de sommation d'Einstein.

Soit

(1.1) $$\omega^i = 0 \qquad (i = 1, \ldots, s)$$

un système de Pfaff dont le premier caractère est s_1 et supposons que $s > s_1 + 1$. Son système dérivé a tout au plus $s - s_1$ équations indépendantes, c'est-à-dire les différentielles $d\omega^i$ sont liées par moins de $s - s_1$ relations linéaires (mod ω^i). Donc, entre n'importe quelles $s_1 + 1$ formes parmi les $d\omega^i$, il ne peut pas y avoir de relations linéaires qui soient nulles (mod ω^i). Pour fixer les idées, nous allons supposer que l'on n'a aucune congruence de la forme

(1.2) $$u_\varkappa\, d\omega^\varkappa = 0\ (\text{mod}\ \omega^i), \qquad (\varkappa = 1, \ldots, s_1 + 1).$$

Soit h le genre minimum de l'expression :

(1.3) $$u_\varkappa\, d\omega^\varkappa\ (\text{mod}\ \omega^i).$$

Nous pouvons alors supposer que l'on a

(1.4) $$[d\omega^1]^{h+1} = 0\ (\text{mod}\ \omega^i)$$

et que pour aucun système de valeurs de u_k on n'a pas

(1.5) $$[u_\varkappa\, d\omega^\varkappa]^h \equiv 0\ (\mathit{mod}\ \omega^i).$$

§2. Supposons d'abord $h = 1$. Grâce aux hypothèses faites jusqu'à présent, on peut mettre le système obtenu en différentiant extérieurement le système (1.1) sous la forme [1]

*) Cet article a été rédigé par Alexandru Neagu d'après un manuscrit de I. Moisini, dont la mort prématurée est advenue à Jassy le 7 Décembre 1969.