Analele ştiinţifice ale Universităţii „Al. I. Cuza" din Iaşi
Tomul XXIII, S. I a, f. 2, 1977

ON A CLASS OF FOURIER-TYPE INTEGRAL TRANSFORMS

BY

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To Professor Gh. Gheorghiev on the occasion of his 70-th birthday

§ 0. **Introduction.** The concept of state function, as a complex valued map ψ on R^3, appears in the study of the discrete spectrum in quantum mechanics. The meaning of $|\psi|^2$ is that of density of probability, which involves the condition

$$\int_{R^3} |\psi|^2 \, dx \, dy \, dz = 1, \tag{0.1}$$

so that $\psi \in L^2(R^3, C)$.

The set of all linear combinations $c_1\psi_1 + c_2\psi_2$ of arbitrary state functions ψ_1, ψ_2, with $c_1, c_2 \in C$ (the complex field) is a linear space S and obviously

$$S \subset L^2. \tag{0.2}$$

To establish if S is a proper subspace of L^2, we shall turn to the space of linear operators $\mathcal{L}(S, S)$ from S to S; we admit, as customary in quantum mechanics, that

$$x, y, z, -i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}, -i\frac{\partial}{\partial z} \in \mathcal{L}(S, S) \tag{0.3}$$

(here $x \in \mathcal{L}(S, S)$ means that if $\psi \in S$, then $x\psi \in S$, etc.). Consequently, it is just fit to choose for S the space of rapidly decreasing C^∞ functions $\psi: R^3 \to C$, endowed with the inner product and the norm from the Hilbert space L^2.

A much more difficult problem is to get a convenient mathematical approach to the study of the continuous spectrum. To illustrate this, it will suffice to mention O. M. Nikodym's works [1]; in this case, the wave function does not belong to L^2.

Our considerations are an attempt to give a natural mathematical basis to the treatment of the continuous spectrum which presents a similarity as manifest as possible to the mathematical frame of the study of the continuous spectrum. We shall take into account the striking fact that the wave function of the free particle

$$\varphi(x, p) = e^{ipx} \tag{0.4}$$

plays a representative part in the study of the continuous spectrum; by means of φ one defines the Fourier transform operator and this has a very weight in quantum mechanics.

We shall consider an extended set of functions with analogous properties, more precisely a linear space Φ with a certain structure similar to that of the space L^2. As previously mentioned, we shall have also

$$x, y, z, -i\frac{\partial}{\partial x}, -i\frac{\partial}{\partial y}, -i\frac{\partial}{\partial z} \in \mathcal{L}(\Phi, \Phi). \tag{0.5}$$

§ 1. **Preliminaries.** Let S be the space of rapidly-decreasing complex-valued C^∞-functions, i.e. the space of those infinitely continuously differentiable functions $f: R^n \to C$ which satisfy the inequalities

$$\sup_{x \in R^n} |x^\alpha D^\beta f(x)| \leqslant k_{\alpha,\beta} < +\infty, \text{ for all } \alpha, \beta \in N_0^n. \tag{1.1}$$

Here we use the symbol N_0^n to denote the set of non-negative integers and,

$$x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}, \; D^\beta = \frac{\partial^{\beta_1 + \beta_2 + \ldots + \beta_n}}{\partial x_1^{\beta_1} \partial x_2^{\beta_2} \ldots \partial x_n^{\beta_n}} \tag{1.2}$$

for $x = (x_1, x_2, \ldots, x_n) \in R^n$, $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n) \in N_0^n$, $\beta = (\beta_1, \beta_2, \ldots, \beta_n) \in N_0^n$.

The seminorms

$$|f|_{\alpha,\beta} = \sup_{x \in R^n} |x^\alpha D^\beta f(x)|, \; \alpha, \beta \in N_0^n, \tag{1.3}$$

induce on S the usual separated locally convex topology.

S can also be considered as a subspace of the space of all Lebesgue (measurable) square integrable functions on R^n; for the sake of correctness and in order to use in S the inner product and the norm from the Hilbert space $L^2(R^n, C)$, namely

$$(f, g) = \int_{R^n} f(x)\overline{g}(x)\, dx, \tag{1.4}$$

$$\|f\| = (f, \overline{f})^{1/2}, \tag{1.5}$$

we have to identify an element $f \in S$ with the equivalence class of all functions equal to f almost everywhere (that is, excepting eventually on a zero measure set in the sense of Lebesgue); note that two functions from S belong to the same class if and only if they coincide (everywhere).

Thus the space S endowed with the norm (1.5) is prehilbertian; it is not complete, as it is well-known that the C^∞-functions with compact support (contained in S) are dense in $L^2(R^n, C)$ [2]. Though the prehilbertian topology is weaker than the locally convex one, we prefer the norm (1.5) to the seminorms (1.3), as it is handy in our considerations.

In the following paragraphs we will use the notations $\mathcal{L}_c(S, S)$, $\mathcal{L}_c(L^2, L^2)$, for the space of all continuous linear operators from S to S, respectively from $L^2(R^n, C)$ to $L^2(R^n, C)$; this notation should be interpreted to imply that the continuous linear operators are normed as usual by

$$\|L\| = \sup_{\|f\| \leqslant 1} \|Lf\|. \tag{1.6}$$

In the following, all integrals are extended over the entire n-dimensional Euclidean space R^n.

§2. **The Spaces Φ_c and Φ.** Let $\mathcal{M}$ be the set of all complex-value *continuous* functions $\psi(x, p)$, defined on the cartesian product $R^n \times R^n$, which are *uniformly tempered* with respect to p and x, as follows:

(2.1) for a fixed compact $K \subset R^n$, there exist $a \in R$, $M_K > 0$, such that for sufficiently large $|p|$ (more accurately, for $p \in R^n - S_K$, where S_K is an adequate compact neighbourhood of 0), one has $|\psi(x, p)| < M_K |p|^a$, $\forall x \in K$;

(2.2) for a fixed compact $K \subset R^n$, there exist $b \in R$, $M_K > 0$ and an adequate compact neighbourhood S_K of 0, such that $|\psi(x, p)| < M_K |x|^b$ for all $x \in R^n - S_K$, $p \in K$;

(2.3) there exist also $c, d \in R$, $M > 0$ such that for sufficiently large $|x|$ and $|p|$ ($x \in R^n - C_1$, $p \in R^n - C_2$, C_1, C_2 compacts), one has $|\psi(x, p)| < M |x|^c |p|^d$.

Remark 2.1. The assumption (2.3) cannot be inferred from (2.1) and (2.2); (2.3) means that ψ is totally tempered on $R^n \times R^n$ (constants M_K are in tempered dependence upon K), which is essential in the proof of Theorem 2.3. Note also that (2.1) and (2.2) are independent from (2.3).

Remark 2.2. All *bounded* continuous functions $\psi: R^n \times R^n \to C$ belong to $\mathcal{M}$, as they verify (2.1), (2.2) and (2.3).

For $\psi \in \mathcal{M}$ let us define *the integral operators* ψ_x and ψ_p

$$\langle \psi_x, f \rangle (x) = \int \psi(x, p) f(p)\, dp, \; f \in S, \tag{2.4}$$

$$\langle \psi_p, f \rangle (p) = \int \psi(x, p) f(x)\, dx, \; f \in S. \tag{2.5}$$

Remark 2.3. ψ_x (as well as ψ_p) is correctly defined on S, since

$$|\langle \psi_x, f \rangle (x)| \leqslant \int |\psi(x, p)|\,|f(p)|\, dp \leqslant$$

$$\leqslant M_K \int_{R^n - S_K} |p|^a |f(p)|\, dp + \int_{S_K} |\psi(x, p)|\,|f(p)|\, dp < +\infty,$$

according to (1.1) and (2.1).

ψ_x, ψ_p are obviously linear mappings on S.

Definition 2.1. *Let Φ be the set of those (continuous complex-valued) functions $\psi \in \mathcal{M}$ (satisfying (2.1), (2.2), (2.3)) and*

$$\psi_x \in \mathcal{L}(S, S), \quad \psi_p \in \mathcal{L}(S, S). \tag{2.6}$$

Definition 2.2. *Let Φ_c be the subset of those $\psi \in \Phi$ for which we have also*

$$\psi_x \in \mathcal{L}_c(S, S), \quad \psi_p \in \mathcal{L}_c(S, S). \tag{2.7}$$

Theorem 2.1. *The sets Φ and Φ_c are non-trivial complex linear spaces.*

Proof. The bounded function

$$\varphi(x, p) = e^{ixp}, \; p \in R^n, \; x \in R^n, \; px = p_1 x_1 + \ldots + p_n x_n, \tag{2.8}$$

belongs to Φ_c. Indeed, in this case the mapping φ_x and φ_p are the Fourier transform operators with regard to the variable x, respectively p, and it is known (see e.g. [2]) that S is mapped into S under Fourier transform.

Further, we recall that by Parseval's formula for L^1-functions (applied to S), the adjoint operator of φ_x is $\overline{\varphi_p}$; by the Fourier inversion theorem for continuous L^1-functions, the inverse operator of φ_x is $(2\pi)^{-n}\overline{\varphi_p}$. Thus we have

$$(<\varphi_x, f_1>, <\varphi_x, f_2>) = (f_1, <\overline{\varphi_p}\varphi_x, f_2>) =$$

$$(f_1, <(2\pi)^n \varphi_x^{-1}\varphi_x, f_2>) = (2\pi)^n (f_1, f_2);$$

this proves that φ_x is a continuous operator (with norm $(2\pi)^{n/2}$), hence (2.7) holds.

The linearity of Φ and Φ_c is an immediate consequence of the conditions (2.1), (2.2), (2.3) and of the definitions (2.4), (2.5).

Remark 2.4. Moreover, it is to be remarked that Φ contains also $D_x^\alpha(e^{ixp})$, $D_p^\alpha(e^{ixp})$. D_x^α and D_p^α are the partial differential operators of order $|\alpha| = \alpha_1 + \ldots + \alpha_n$, for the n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n) \in N_o^n$, with respect to $x = (x_1, \ldots, x_n)$, respectively $p = (p_1, \ldots, p_n)$. Although $D_x^\alpha(e^{ixp})$, $D_p^\alpha(e^{ixp})$ are not bounded, they verify (2.1), (2.2), (2.3) and it is not difficult to show that (2.6) is true (see also § 5).

Theorem 2.2. *The spaces Φ_c and Φ are algebraically isomorphic to certain subspaces of $\mathcal{L}_c(L^2, L^2)$, respectively $\mathcal{L}(S, S)$.*

Proof. We proceed to shaw that Φ_c is algebraically embedded in $\mathcal{L}_c(L^2, L^2)$ by the mapping

$$T_x : \Phi_c \to \mathcal{L}_c(L^2, L^2), \quad T_x(\psi) = \hat{\psi}_x, \tag{2.9}$$

where the symbol $\hat{\psi}_x$ stands for the extension of ψ_x from S to L^2; recalling that S is dense in L^2, it is clear that there is a unique such extension of ψ_x to preserve its liniarity, continuity and norm (see e.g. [3], chap. 2, p. 100).

It is easy to see that T_x is a linear mapping. Now, to prove the injectivity of T_x, it suffices to show that $T_x(\psi) = 0$ implies $\psi = 0$.

Suppose $\hat{\psi}_x = 0$; then $\psi_x = 0$, that is $\int \psi(x, p) f(p)\, dp = 0$, for $f \in S$, $p \in R^n$. We fix an arbitrary $x_0 \in R^n$; $\psi(x_0, p)$ is locally integrable on R^n (as it is continuous) so that it defines a distribution on the space C_o^∞ of C^∞ - complex function with compact support, by

$$f \to \int \psi(x_0, p) f(p)\, dp, \; f \in C_o^\infty \subset S.$$

Since this distribution is identically zero on C_o^∞ (in fact $\psi(x_0, p)$ defines the null tempered distribution on S), it follows (see e.g. [3], chap. 4, p. 174) that $\psi(x_0, p) = 0$, a.e. for $p \in R^n$. Finally, noting that x_0 was arbitrarily chosen and that ψ is continuous, we must obviously have $\psi \equiv 0$.

A similar argument is valid for the embedding operator

$$T_p : \Phi_c \to \mathcal{L}_c(L^2, L^2), \; T_p(\psi) = \hat{\psi}_p \tag{2.10}$$

where $\hat{\psi}_p$ denotes the extension of ψ_p from S to L^2.

In consequence it is found that Φ_c is algebraically isomorphic with both $T_x(\Phi_c)$ and $T_p(\Phi_c)$.

It is easy to see now that the mappings $\psi \xrightarrow{T_x} \psi_x$, $\psi \xrightarrow{T_p} \psi_p$ establish isomorphisms from Φ onto subspaces of $\mathcal{L}(S, S)$.

Theorem 2.3. *For each $\psi \in \Phi$, the adjoint operator of ψ_x is $\overline{\psi}_p$ and the adjoint operator of ψ_p is $\overline{\psi}_x$.*

Proof. If $f, g \in S$ then

$$(<\psi_x, f>, g) = \int \left\{ \int \psi(x, p) f(p)\, dp \right\} \bar{g}(x)\, dx,$$

$$(f, <\overline{\psi}_p, g>) = \int f(p) \left\{ \int \overline{\overline{\psi}(x, p) g(x)\, dx} \right\} dp =$$

$$= \int f(p) \left\{ \int \psi(x, p) \bar{g}(x)\, dx \right\} dp.$$

Now, $\psi(x, p)$ is uniformly and totally tempered, so that the assumptions (2.1), (2.2) and (2.3) lead to the following sequence of inequalities:

$$\int \left\{ \int |\psi(x, p) f(p) \bar{g}(x)|\, dp \right\} dx =$$

$$= \int_{S_{C_1}} \left\{ \int_{C_1} |\psi f(p) \bar{g}(x)|\, dp \right\} dx + \int_{R^n - S_{C_1}} \left\{ \int_{C_1} |\psi f(p) \bar{g}(x)|\, dp \right\} dx +$$

$$+ \int_{R^n - C_1} \left\{ \int_{R^n - C_1} |\psi f(p) \bar{g}(x)|\, dp \right\} dx + \int_{C_1} \left\{ \int_{S_{C_1} \cap (R^n - C_1)} |\psi f(p) \bar{g}(x)|\, dp \right\} dx +$$

$$+\int\limits_{C_1}\Big\{\int\limits_{(R^n - S_{C_1})\cap(R^n - C_2)} |\psi f(p)\bar{g}(x)|\,dp\Big\}\,dx \leqslant$$

$$\leqslant I_1 + M_{C_2}\int\limits_{R^n - S_{C_2}} |x|^b\,|\bar{g}(x)|\,dx \int\limits_{C_2}|f(p)|\,dp +$$

$$+ M\int\limits_{R^n - C_1} |x|^c\,\bar{g}(x)|\,dx \int\limits_{R^n - C_2} |p|^d\,|f(p)|\,dp + I_2 +$$

$$+ M_{C_1}\int\limits_{C_1}|p|^a\,|f(p)|\,dp \int\limits_{R^n - S_{C_1}} |\bar{g}(x)|\,dx.$$

But $f, g \in S \subset L^1$, therefore the last form shaws that $\int\Big\{\int |\psi f(p)\bar{g}(x)|\,dp\Big\}\,dx < +\infty$. Hence, we are guaranteed that the above iterated integral exists; then $|\psi(x,p)f(p)\bar{g}(x)|$ is integrable over $R^n \times R^n$ and so must be $\psi(x,p)f(p)\bar{g}(x)$. Therefore by Fubini's theorem the iterated integrals

$$\int\Big\{\int \psi(x,p)f(p)\,dp\Big\}\bar{g}(x)\,dx \quad\text{and}\quad \int f(p)\Big\{\int \psi(x,p)\bar{g}(x)\,dx\Big\}\,dp$$

(exist and) are equal (see [4], chap. V, p. 314, or [5], chap. VIII, p. 145). Thus we have

$$(<\psi_x, f>, g) = (f, <\bar{\psi}_p, g>).$$

We conclude that $\psi_x^* = \bar{\psi}_p$.

The equality $\psi_p^* = \bar{\psi}_x$ can be proved in a similar way or straightforwardly

$$\psi_p^* = (\bar{\psi}_p)^* = (\overline{\psi_x^*})^* = \bar{\psi}_x .$$

§3. **The Fourier Product. Definition 3.1.** *The Fourier product on Φ (with respect to p or x) is defined as a mapping from $\Phi \times \Phi$ to $\mathcal{L}(S, S)$, denoted by $[\,,\,]_p$ or $[\,,\,]_x$, such that*

(3.1) $$<[\psi_1, \psi_2]_p, f>(p) = \int \psi_1(x,p)\Big\{\int \bar{\psi}_2(x,q)f(q)\,dq\Big\}\,dx,$$

(3.2) $$<[\psi_1, \psi_2]_x, f>(x) = \int \psi_1(x,p)\Big\{\int \bar{\psi}_1(y,p)f(y)\,dy\Big\}\,dp$$

for $\psi_1, \psi_2 \in \Phi$.

Remark 3.1. We are allowed to state that the Fourier product $[\psi_1, \psi_2]$ defined for $\psi_1, \psi_2 \in \Phi(\Phi_c)$ by means of (3.1) or (3.2), is a linear (continuous) operator from S to S, since (with (2.4) and (2.5) in mind) we assert that

(3.3) $$[\psi_1, \psi_2]_p = \psi_{1p}\bar{\psi}_{2x} .$$

(3.4) $$[\psi_1, \psi_2]_x = \psi_{1x}\bar{\psi}_{2p} .$$

Theorem 3.1. *The Fourier product satisfies the following related properties:*

(3.5) $$[\psi_1, \psi_2]_p = \hat{\psi}_{1p}(\hat{\psi}_{2p})^*/_S = (\hat{\psi}_{1x})^*\hat{\bar{\psi}}_{2x}/_S ; \ \forall\, \psi_1, \psi_2 \in \Phi_c .$$

(3.6) $$[\psi_1, \psi_2]_x = \hat{\psi}_{1x}(\hat{\psi}_{2x})^*/_S = (\hat{\psi}_{1p})^*\hat{\bar{\psi}}_{2p}/_S ; \ \forall\, \psi_1, \psi_2 \in \Phi_c .$$

the sign $/_S$ means the restriction of an operator to S).

(3.7) $$[\psi, \psi] \geqslant 0 .$$

(3.8) $$[\psi_1, \psi_2]^* = [\psi_2, \psi_1] .$$

(3.9) $$[c_1\psi_1 + c_2\psi_2, \psi_3] = c_1[\psi_1, \psi_3] + c_2[\psi_2, \psi_3] ,$$

for all $\psi, \psi_1, \psi_2, \psi_3 \in \Phi$; $c_1, c_2 \in C$. ($[\,,\,]$ means either $[\,,\,]_p$ or $[\,,\,]_x$).

Proof. According to (3.3) we have $[\psi_1, \psi_2]_p = \psi_{1p}\bar{\psi}_{2x}$; since $\bar{\psi}_{2x}$ is the adjoint operator on S of ψ_{2p} (Theorem 2.3), its extension to L^2 (which preserves continuity) is the unique adjoint of $\hat{\psi}_{1p} = T_p(\psi_2)$; therefore

$$[\psi_1, \psi_2]_p = T_p(\psi_1)(T_p(\psi_2))^* = \hat{\psi}_{1p}(\hat{\psi}_{2p})^*.$$

(3.6) results by symmetry.

Further, (3.1) together with Theorem 2.3 imply

$$(<[\psi, \psi]_p, f>, f) = (<\psi_p\bar{\psi}_x, f>, f) = (<\bar{\psi}_x, f>, <\bar{\psi}_x, f>) =$$
$$= \| <\bar{\psi}_x, f> \|^2 \geqslant 0 \text{ and (3.7) holds.}$$

For the same reason

$$[\psi_1, \psi_2]_p^* = (\psi_{1p}\bar{\psi}_{2x})^* = (\bar{\psi}_{2x})^*(\psi_{1p})^* = \psi_{2p}\bar{\psi}_{1x} = [\psi_2, \psi_1]_p,$$

that is (3.8) is true. (3.9) is an immediate consequence of the Definition 3.1.

Remark 3.2. Since Φ_c is embedded in $\mathcal{L}_c(L^2, L^2)$, Theorem 3.1 shows that the Fourier product may be considered something like an inner product on a space of linear continuous operators:

$$[A, B] = AB^*, \ A, B \in \mathcal{L}_c(H, H).$$

Theorem 3.2. *The space Φ_c can be normed so that it becomes isomorphic and isometric to $T_p(\Phi_c)$ or to $T_x(\Phi_c) \subset \mathcal{L}_c(L^2, L^2)$.*

Proof. We have already proved in Theorem 2.2 that Φ_c and $T_p(\Phi_c)$ are algebraically isomorphic. Consider the presumptive norms

(3.10) $$\|\psi\|_p = \|[\psi, \psi]_p\|^{1/2}_{\mathcal{L}_c(S,S)} .$$

(3.11) $$\|\psi\|_x = \|[\psi, \psi]_x\|^{1/2}_{\mathcal{L}_c(S,S)} \quad \text{on } \Phi_c .$$

Clearly, from (3.10) it follows that $\|\psi\|_p \geq 0$, $\forall \psi \in \Phi_c$; moreover $\|\psi\|_p^2 = \|\psi_p \bar{\psi}_x\| = \|\hat{\psi}_p (\hat{\psi}_p)^*/_S\| = \|T_p(\psi)(T_p(\psi))^*\|$ because the extension of ψ_p to $\hat{\psi}_p = T_p(\psi_p)$, from S to L^2, preserves the norm. Then

$$\|\psi\|_p = \|T_p(\psi)(T_p(\psi))^*\|^{1/2}_{\mathcal{L}_c(L^2,L^2)} = \|T_p(\psi)\|$$

and so $\|\psi\|_p = \|T_p(\psi)\|$; it turns out that, as presumed $\|\ \|_p$ is a norm on Φ. The last equality shows also that the embedding operator T_p is isometrical.

Let us also note that

$$\|\psi\|_p = \|(\hat{\bar{\psi}}_x)^* \hat{\bar{\psi}}_x\|^{1/2} = \|T_x(\bar{\psi})\| = \|\bar{\psi}\|_x . \tag{3.12}$$

It is to be observed that the above equality $\|\psi\|_p = \|\bar{\psi}\|_x$ implies that $\|\psi\|_x$ is an equivalent norm on Φ_c. This completes the proof of the theorem.

§ 4. The generalization of the Fourier transform. Definition 4.1. *The mappings ψ_x and ψ_p ($\hat{\psi}_x$ and $\hat{\psi}_p$), generated by a function $\psi : R^n \times R^n \to C$, $\psi \in \Phi$ (according to (2.4), (2.9), respectively (2.5), (2.10)) are said to be generalized Fourier transform operators if one of the following assertions is true :*

(4.1) *there exists $a > 0$, such that $[\psi, \psi]_p = \psi_p \psi_p^* = a^n I$;*

(4.2) *there exists $a > 0$, such that $[\psi, \psi]_x = \psi_x \psi_x^* = a^n I$;*

(4.3) *there exists $a > 0$, such that $[\psi, \psi]_p = [\psi, \psi]_x = a^n I$;*

($I \in \mathcal{L}(S, S)$ denotes the identity operator).

Remark 4.1. As already mentioned, the mappings φ_x and φ_p, where $\varphi(x, p)$ is given by (2.8), are the usual Fourier transform operators and

$$< \varphi_x \varphi_x^*, f > = < \varphi_x \bar{\varphi}_p, f > = < (2\pi)^n \varphi_x \varphi_x^{-1}, f > = (2\pi)^n f,$$

which is equivalent to the statement $\varphi_x \varphi_x^* = (2\pi)^n I$, i.e. (4.2) holds for φ. By symmetry of φ, (4.1) is also valid for φ.

Theorem 4.1. *On Φ_c, the conditions (4.1), (4.2) are equivalent respectively to*

(4.1)′ $\exists a > 0$, *such that* $\hat{\psi}_p (\hat{\psi}_p)^* = (\hat{\psi}_x)^* \hat{\psi}_x = a^n I \in \mathcal{L}_c(L^2, L^2)$;

(4.2)′ $\exists a > 0$, *such that* $\hat{\psi}_x (\hat{\psi}_x)^* = (\hat{\psi}_p)^* \hat{\psi}_p = a^n I \in \mathcal{L}_c(L^2, L^2)$.

Proof. First, as there is a unique possibility to extend ψ_p to L^2 preserving continuity, we have

$$< \hat{\psi}_p (\hat{\psi}_p)^*, f > = a^n f, \forall f \in L^2 \Leftrightarrow < \psi_p \psi_p^*, f > = a^n f, \forall f \in S.$$

Further this is equivalent to

$$< \overline{\psi_p \bar{\psi}_x}, f > = a^n \bar{f}, \forall f \in S \Leftrightarrow < \bar{\psi}_p \bar{\bar{\psi}}_x, \bar{f} > = a^n \bar{f}, \forall f \in S \Leftrightarrow$$

$$< (\psi_x)^* \psi_x, \bar{f} > = a^n \bar{f},\ f \in S \Leftrightarrow < (\hat{\psi}_x)^* \hat{\psi}_x \bar{f} > = a^n \bar{f},\ \forall f \in L^2 ;$$

replacing the arbitrary $f \in L^2$ by $\bar{f}$, we obtain

$$< (\hat{\psi}_x)^* \hat{\psi}_x, f > = a^n f, \forall f \in L^2, \text{ q.e.d.}$$

An exactly analogous argument enables us to infer (4.2)′.

Theorem 4.2. (4.1) *(or (4.2)) holds for $\psi \in \Phi_c$ if and only if it holds for $\bar{\psi}$. $A = \hat{\psi}_p$, $B = \hat{\psi}_x$ are generalized Fourier operators if and only if their adjoints $A^* = \hat{\bar{\psi}}_x$, $B^* = \hat{\bar{\psi}}_p$ are also generalized Fourier operators, and in this case we have respectively*

(4.1)″ $f = a^{-n} < AA^*, f > = a^{-n} < B^* B, f >, \forall f \in L^2$.

(4.2)″ $f = a^{-n} < BB^*, f > = a^{-n} < A^* A, f >, \forall f \in L^2$.

(4.3) $(Af, Ag) = a^n (f, g)$; $(Bf, Bg) = a^n (f, g), \forall f, g \in L^2$.

(4.4) $\|A\| = \|B\| = a^{n/2}$.

Proof. The proof follows immediately from (3.5), (3.6), Definition 4.1, Theorem 4.1 and usual calculus.

Theorem 4.3. *Consider $f, g \in S$ and their convolution $(f * g)(x) = \int f(y) g(x - y)\, dy$. If $\psi \in \Phi_c$ verifies (4.3) and there exist $\psi_1, \psi_2 \in \Phi_c$ which also verify (4.3), such that*

(4.6) $\psi(x + y, p) = \psi_1(x, p) \psi_2(y, p), \forall x, y, p \in R^n$,

then

(4.7) $< \hat{\psi}_p, f * g > = < \hat{\psi}_{2p}, f > < \hat{\psi}_{1p}, g > = < \hat{\psi}_{1p}, f > < \hat{\psi}_{2p}, g >$,

(4.8) $< \hat{\bar{\psi}}_{2x}, f > * < \hat{\bar{\psi}}_{1x}, g > = a^n < \hat{\bar{\psi}}_x, fg >, \forall f, g \in S$.

(Similar properties can be formulated changing x with p).

Proof. Applying Fubini's theorem to change the order of integration, for $f, g \in S$ and $\psi \in \Phi_c$ we have

$$< \hat{\psi}_p, f * g > = \int \psi(x, p) \left\{ \int f(z) g(x - z)\, dz \right\} dx =$$

$$= \int f(z) \left\{ \int \psi(x, p) g(x - z)\, dx \right\} dz = \int f(z) \left\{ \int \psi(z + y, p) g(y)\, dy \right\} dz =$$

$$= \int f(z) \left\{ \int \psi_1(z, p) \psi_2(y, p) g(y)\, dy \right\} dz = < \hat{\psi}_{1p}, f > < \hat{\psi}_{2p}, g >.$$

By commutativity

$$< \hat{\psi}_p, f * g > = < \hat{\psi}_p, g * f > = < \hat{\psi}_{1p}, g > < \hat{\psi}_{2p}, f >, \forall f, g \in S.$$

To prove (4.8) we shall use of (4.7) and (4.1)″ :

$$< \hat{\psi}_p, < \hat{\bar{\psi}}_{2x}, f > * < \hat{\bar{\psi}}_{1x}, g > > = < \hat{\psi}_{2p} \hat{\bar{\psi}}_{2x}, f > < \hat{\psi}_{1p} \hat{\bar{\psi}}_{1x}, g > = a^{2n} fg ;$$

further, by (4.2)′

$a^n(<\hat{\psi}_{2r}, f> * <\hat{\psi}_{1r}, g>) = <(\hat{\psi}_p)^* \hat{\psi}_p, <\hat{\psi}_{2r}, f> * <\hat{\psi}_{1r}, g>> =$
$= <(\hat{\psi}_p)^*. a^{2n} fg> = a^{2n} <\hat{\psi}_r, fg>$, i.e. (4.8) holds.

§5. Linear operators on Φ. One of the questions of interest is that to know when D_r^α, D_p^α (Remark 2.4) may be considered as linear operators on Φ.

Definition 5.1. *For $\alpha \in N_0^n$, $x \in R^n$ define D_r^α on $\mathcal{M}$ by*

(5.1) $$<D_r^\alpha \psi, f> = (-1)^{|\alpha|} <\psi_p, D_r^\alpha f>, \quad \psi \in \mathcal{M}.$$

Theorem 5.1. *The partial differential operator defined by (5.1) maps Φ into $\mathcal{L}(S, S)$. Let $\psi \in \Phi$ have continuous partial derivatives; then $D_r^\alpha \psi \in \Phi$, i.e. $D_r^\alpha \psi = (D_r^\alpha \psi)_p \in T_p(\Phi)$.*

Proof. The first assertion is self-evident since $D_r^\alpha \psi = \psi_p D_r^\alpha$, where $D_r^\alpha \in \mathcal{L}(S, S)$ and $\psi_p \in \mathcal{L}(S, S)$.

Next, if ψ is continuously partially differentiable with respect to x, then for fixed $p_0 \in R^n$

$$<D_r^\alpha \psi, f>(p_0) = (-1)^{|\alpha|} <\psi_p, D_r^\alpha f>(p_0) = <(D_r^\alpha \psi)_p, f>(p_0),$$

for all $f \in C_0^\infty$. But, for fixed p_0, $<D_r^\alpha \psi, \;.\;>(p_0) = \psi_{p_0}(D_r^\alpha(\;.\;))$ is a tempered distribution on S; since a distribution with compact support can be uniquely extended to a tempered distribution, it follows that $<D_r^\alpha \psi, .\;>(p_0)$ coincides with $<(D_r^\alpha \psi)_p, .\;>(p_0)$ on S.

This completes the proof, as p_0 is arbitrary.

Remark 5.1. For continuously partially differentiable $\psi \in \Phi_c$, we are not allowed to state that $D_r^\alpha \psi \in \Phi_c$; indeed in this case $D_r^\alpha \psi = \psi_p D_r^\alpha$ does not belong to $\mathcal{L}_c(S, S)$; $\psi_p \in \mathcal{L}_c(S, S)$, but $D_r^\alpha : S \to S$ is not continuous on S in the prehilbertian (1.5) topology (D_r^α is continuous on S only in the locally convex (1.3) topology).

Let us investigate now the adjointness with respect to the Fourier product.

Definition 5.2. *A linear operator $D^* : \Phi \to \Phi$ is called the adjoint of $D \in \mathcal{L}(\Phi, \Phi)$ with respect to the Fourier product $[\;,\;]$ if*

(5.2) $$[D\psi_1, \psi_2] = [\psi_1, D^* \psi_2], \forall \psi_1, \psi_2 \in \Phi.$$

Remark 5.2. If D^* is the adjoint of D, then $D^{**} = D$; (5.2) implies $[\psi_1, D^* \psi_2]^* = [D\psi_1, \psi_2]^*$, $\forall \psi_1, \psi_2$ and using (3.8) we obtain $[D^* \psi_2, \psi_1] = [\psi_2, D\psi_1]$, $\forall \psi_1, \psi_2$.

Theorem 5.2. *A linear operator $D : \Phi \to \Phi$ has an adjoint with respect to the Fourier product $[\;,\;]_p$, respectively $[\;,\;]_x$, if and only if*

(5.3) $$(D\psi)_p = (2\pi)^{-n} \psi_p \varphi_p^* (D\varphi)_p, \quad \forall \psi \in \Phi,$$

respectively

(5.3)′ $$(D\psi)_x = (2\pi)^{-n} \psi_x \varphi_x^* (D\varphi)_x, \quad \forall \psi \in \Phi,$$

where φ is given by (2.8).

Proof. Suppose that D^* is the adjoint of D with respect to $[\;,\;]_p$. Then by virtue of (5.2) and (3.5), $(D\psi_1)_p \psi_{2p}^* = \psi_{1p} (D^* \psi_2)_p^*$, so that for

$\psi_1 = \psi \in \Phi$, $\psi_2 = \varphi$ one has $(D\psi)_p \varphi_p^* = \psi_p (D^* \varphi)_p^*$. Further $(D\psi)_p \varphi_p^* \varphi_p = \psi_p (D^* \varphi)_p^* \varphi_p$ and therefore

(5.5) $$(D\psi)_p = (2\pi)^{-n} \psi_p (D^* \varphi)_p^* \varphi_p.$$

Analogously, $(D^* \psi)_p = (2\pi)^{-n} \psi_p (D\varphi)_p^* \varphi_p$ and so $(D^* \psi)_p^* = (2\pi)^{-n} \cdot \varphi_p^* (D\varphi)_p \psi_p^*$; take $\psi = \varphi$ and thus $(D^* \varphi)_p^* = (2\pi)^{-n} \varphi_p^* (D\varphi)_p \varphi_p^*$. Then (5.5) can be written $(D\psi)_p = (2\pi)^{-n} \psi_p (2\pi)^{-n} \varphi_p^* (D\varphi)_p \varphi_p^* \varphi_p \Rightarrow (D\varphi)_p = (2\pi)^{-n} \cdot \psi_p \varphi_p^* (D\varphi)_p$.

Conversely, suppose that (5.3) holds for $D \in \mathcal{L}(\Phi, \Phi)$. Define D^* as follows:

(5.6) $$(D^* \psi)_p = (2\pi)^{-n} \psi_p (D\varphi)_p^* \varphi_p, \; \forall \varphi \in \Phi.$$

Then

$$[\psi_1, D^* \psi_2]_p = \psi_{1p} (D^* \psi_2)_p^* = (2\pi)^{-n} \psi_{1p} \varphi_p^* (D\varphi)_p \psi_{2p}^*$$

and by (5.3)

$$[D\psi_1, \psi_2] = (D\psi_1)_p \psi_{2p}^* = (2\pi)^{-n} \psi_{1p} \varphi_p^* (D\varphi)_p \psi_{2p}^*.$$

Hence D^* is really the adjoint of D.

Theorem 5.3. *If the operator $D \in \mathcal{L}(\Phi, \Phi)$ admits an adjoint D^* with respect to the Fourier product, then this adjoint is unique.*

Proof. Suppose that (5.2) (in the case of $[\;.\;]_p$) is verified both by D_1^* and D_2^*. These two conditions combine to give

$$\psi_{1p} (D_1^* \psi_2)_p^* = \psi_{1p} (D_2^* \psi_2)_p^*, \text{ for all } \psi_1, \psi_2 \in \Phi.$$

Select $\psi_1 = \varphi$ (2.8) and thus by applying φ_p^* we have

$$\varphi_p^* \varphi_p (D_1^* \psi_2)_p^* = \varphi_p^* \varphi_p (D_2^* \psi_2)_p^*, \; \forall \psi_2 \in \Phi.$$

Because of (4.1) and Remark 5.2 it follos that $(D_1^* \psi_2)_p = (D_2^* \psi_2)_p$; whence, by the injectivity of the embedding operator in Theorem 2.2, it results

$$D_1^* \psi_2 = D_2^* \psi_2, \; \forall \psi_2 \in \Phi, \text{ q.e.d.}$$

A similar argument applies to the case of the adjoint with respect to $[\;.\;]_x$.

Theorem 5.4. *An operator $D \in \mathcal{L}(\Phi, \Phi)$ is selfadjoint with respect to the Fourier product $[\;,\;]_p$ (respectively $[\;,\;]_x$) if and only if (5.3) (respectively (5.3)′) holds and*

(5.7) $$D\varphi = D^* \varphi$$

where φ is given by (2.8).

Proof. If D is selfadjoint then conditions (5.3) and (5.7) are clearly satisfied. Conversely, if (5.3) holds then D has a unique adjoint defined by means of (5.6), (according to Theorem 5.2 and Theorem 5.3). Further, if (5.7) also holds, then we have $[\varphi, D\varphi]_p = [D\varphi, \varphi]_p$, i.e. $\varphi_p (D\varphi)_p^* = (D\varphi)_p \varphi_p^*$.

Consequently $\varphi_p (D\varphi)_p^* \varphi_p = (D\varphi)_p \varphi_p^* \varphi_p$ and thus $(D\varphi)_p = (2\pi)^{-n} \varphi_p (D\varphi)_p^* \varphi_p$; now, let us substitute this for $(D\varphi)_p$ in (5.3) and write

$$(D\psi)_p = (2\pi)^{-n}\,\psi_p\,\varphi_p^*\,(2\pi)^{-n}\,\varphi_p\,(D\varphi)_p^*\,\varphi_p = (2\pi)^{-n}\,\psi_p\,(D\varphi)_p^*\,\varphi_p\,.$$

Comparing this with (5.6) we may conclude that $(D\psi)_p = (D^*\psi)_p$, $\forall \psi \in \Phi$, which implies $D = D^*$.

Theorem 5.5. *Let $a: R^n \to C$ be a C^∞-tempered function such that $af \in S$, $\forall f \in S$. The operators $A\psi(x, p) = a(x)\,\psi(x, p)$ and $B\psi(x, p) = a(p)\,\psi(x, p)$ belong to $\mathcal{L}(\Phi, \Phi)$ and their adjoints are respectively $A^*\psi(x, p) = \bar{a}(x)\,\psi(x, p)$, $B^*\psi(x, p) = \bar{a}(p)\,\psi(x, p)$.*

Remark 5.3. We can choose $a(x) = x^\alpha$ or $a(x) =$ constant; if $a(x) \in R$, $\forall x$, then A and B are selfadjoints.

Proof. Since a is tempered, (2.1), (2.2) and (2.3) are verified for $a\psi$, $\forall \psi \in \Phi$; moreover, since $af \in S$, $\forall f \in S$, (2.6) is also satisfied for $a\psi$, $\forall \psi \in \Phi$. It follows that $a\psi \in \Phi$, $\forall \psi \in \Phi$. Obviously A and B are linear operators. Next, it is easily seen that

$$< (A\psi)_p, f > \; = \; < \psi_p\,, af > \; = (2\pi)^{-n} < \psi_p\,\varphi_p^*\,\varphi_p\,, af > \; =$$
$$= (2\pi)^{-n} < \psi_p\,\varphi_p^*\,(A\varphi)_p\,, f >, \; \forall f \in S,\; \forall \psi \in \Phi,$$

and so (5.3) stands for A. Then, by Theorem 5.2, A has the adjoint defined by (5.6): $(A^*\psi)_p = (2\pi)^{-n}\,\psi_p\,(A\varphi)_p^*\,\varphi_p$; but one can check that $< (A\varphi)_p^*, f > (x) = \bar{a}(x) < \varphi_p^*, f > (x)$, $\forall f \in S$, $x \in R^n$ so that $(A^*\psi)_p = (2\pi)^{-n}\,\psi_p\,(\bar{a}\varphi_p^*)\,\varphi_p = (\bar{a}\psi)_p$. The last equality means $A^*\psi(x, p) = \bar{a}(x)\,\psi(x, p)$.

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Received 9. VII. 1977

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Analele științifice ale Universității „Al. I. Cuza" din Iași
Tomul XXIII, S. I a, f. 2, 1977

SOLVING THE NETWORK FLOW MODEL ASSOCIATED TO A COOPERATIVE n-PERSON GAME

BY

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In two preceding papers, [1], [2], some concepts of solution for a cooperative n-person game have been introduced. The basic idea was that each player can belong to several coalitions, in consequence the outcomes of the game are described by a matrix. Concepts like bimputation, bicore, Shapley bivalue, similar to those commonly considered in the theory of n-person cooperative games, have been defined and their properties have been further studied. The aim of the present paper is that of solving a network flow model associated to a cooperative n-person game. The optimal solution of the model, which can be considered as a solution of the game, is found in two cases, though the model itself have been already defined in [1]. More precisely, we investigated the cases of linear and quadratic objective functions; explicit optimal solutions in both cases are given.

1. Two-dimensional concepts of solution for cooperative n-person games. Let $G = (N, v)$ be a cooperative n-person game with the set of players N, $|N| = n$, and the characteristic function $v(S)$ defined on the set $\mathcal{P}(N)$, $|\mathcal{P}(N) - \varnothing| = m = 2^n - 1$. We assume $v(\varnothing) = 0$ and $v(S) \geq 0$ for all $S \in \mathcal{P}(N)$.

In [1] we called bimputation a $n \times m$ matrix X such that

$$\text{(1)} \qquad \sum_{k | i \in S_k} x_{ik} \geqslant v(\{i\}), \qquad (i = 1, \ldots, n).$$

$$\text{(2)} \qquad \sum_{i \in S_k} x_{ik} \leqslant v(S_k), \qquad (k = 1, \ldots, m),$$

$$\text{(3)} \qquad x_{ik} = 0,\ (i \notin S_k),\ x_{ik} \geqslant 0,\ (i \in S_k),\ (i = 1, \ldots, n;\ k = 1, \ldots, m).$$

These conditions have been thought of as natural conditions imposed to the variables, taking account on the fact that x_{ik} has to be the gain of player i in the coalition S_k.

If we replace in (2) the inequalities by the corresponding equations, we define a subset of the set of bimputations. This subset was called in [1] the bicore of the game. It was also proved that the bicore is in some sense the set of undominated bimputations. In [2] the Shapley bivalue have been defined by a group of axioms; the superadditivity of the characte-