$$(D\psi)_p = (2\pi)^{-n} \psi_p \varphi_p^* (2\pi)^{-n} \varphi_p (D\varphi)_p^* \varphi_p = (2\pi)^{-n} \psi_p (D\varphi)_p^* \varphi_p .$$

Comparing this with (5.6) we may conclude that $(D\psi)_p = (D^* \psi)_p$, $\forall \psi \in \Phi$, which implies $D = D^*$.

Theorem 5.5. *Let $a : R^n \to C$ be a C^∞-tempered function such that $af \in S$, $\forall f \in S$. The operators $A\psi(x, p) = a(x)\psi(x, p)$ and $B\psi(x, p) = a(p)\psi(x, p)$ belong to $\mathcal{L}(\Phi, \Phi)$ and their adjoints are respectively $A^*\psi(x, p) = \overline{a}(x)\psi(x, p)$, $B^*\psi(x, p) = \overline{a}(p)\psi(x, p)$.*

Remark 5.3. We can choose $a(x) = x^2$ or $a(x) =$ constant; if $a(x) \in R$, $\forall x$, then A and B are selfadjoints.

Proof. Since a is tempered, (2.1), (2.2) and (2.3) are verified for $a\psi$, $\forall \psi \in \Phi$; moreover, since $af \in S$, $\forall f \in S$, (2.6) is also satisfied for $a\psi$, $\forall \psi \in \Phi$. It follows that $a\psi \in \Phi$, $\forall \psi \in \Phi$. Obviously A and B are linear operators. Next, it is easily seen that

$$< (A\psi)_p, f > = < \psi_p, af > = (2\pi)^{-n} < \psi_p \varphi_p^* \varphi_p, af > =$$
$$= (2\pi)^{-n} < \psi_p \varphi_p^* (A\varphi)_p, f >, \ \forall f \in S, \ \forall \psi \in \Phi,$$

and so (5.3) stands for A. Then, by Theorem 5.2, A has the adjoint defined by (5.6): $(A^*\psi)_p = (2\pi)^{-n} \psi_p (A\varphi)_p^* \varphi_p$; but one can check that $< (A\varphi)_p^*, f > (x) = \overline{a}(x) < \varphi_p^*, f > (x)$, $\forall f \in S$, $x \in R^n$ so that $(A^*\psi)_p = (2\pi)^{-n} \psi_p (\overline{a}\varphi_p^*) \varphi_p = (\overline{a}\psi)_p$. The last equality means $A^*\psi(x, p) = \overline{a}(x)\psi(x, p)$.

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SOLVING THE NETWORK FLOW MODEL ASSOCIATED TO A COOPERATIVE *n*-PERSON GAME

BY

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In two preceding papers, [1], [2], some concepts of solution for a cooperative n-person game have been introduced. The basic idea was that each player can belong to several coalitions, in consequence the outcomes of the game are described by a matrix. Concepts like bimputation, bicore, Shapley bivalue, similar to those commonly considered in the theory of n-person cooperative games, have been defined and their properties have been further studied. The aim of the present paper is that of solving a network flow model associated to a cooperative n-person game. The optimal solution of the model, which can be considered as a solution of the game, is found in two cases, though the model itself have been already defined in [1]. More precisely, we investigated the cases of linear and quadratic objective functions; explicit optimal solutions in both cases are given.

1. Two-dimensional concepts of solution for cooperative *n*-person games. Let $G = (N, v)$ be a cooperative n-person game with the set of players N, $|N| = n$, and the characteristic function $v(S)$ defined on the set $\mathcal{P}(N)$, $|\mathcal{P}(N) - \emptyset| = m = 2^n - 1$. We assume $v(\emptyset) = 0$ and $v(S) \geqslant 0$ for all $S \in \mathcal{P}(N)$.

In [1] we called bimputation a $n \times m$ matrix X such that

$$(1) \qquad \sum_{k/i \in S_k} x_{ik} \geqslant v(\{i\}), \qquad (i = 1, \ldots, n),$$

$$(2) \qquad \sum_{i \in S_k} x_{ik} \leqslant v(S_k), \qquad (k = 1, \ldots, m),$$

$$(3) \qquad x_{ik} = 0, \ (i \notin S_k), \ x_{ik} \geqslant 0, \ (i \in S_k), \ (i = 1, \ldots, n;\ k = 1, \ldots, m).$$

These conditions have been thought of as natural conditions imposed to the variables, taking account on the fact that x_{ik} has to be the gain of player i in the coalition S_k.

If we replace in (2) the inequalities by the corresponding equations, we define a subset of the set of bimputations. This subset was called in [1] the bicore of the game. It was also proved that the bicore is in some sense the set of undominated bimputations. In [2] the Shapley bivalue have been defined by a group of axioms; the superadditivity of the characteristic function was also assumed and an explicit formula for the Shapley bivalue, which is a bimputation belonging to the bicore, have been given.

Let us define now another bimputation belonging to the bicore, which can also represent a solution of the game. This bimputation is the optimal solution of the mathematical programming problem (P) : minimize

$$C(X) = \sum_{k=1}^{m} \sum_{i \in S_k} c_{ik} x_{ik} \tag{4}$$

subject to

$$\sum_{i \in S_k} x_{ik} = v(S_k), \quad (k = 1, \dots, m), \tag{5}$$

$$x_{ik} = 0, \ (i \notin S_k); \ x_{ik} \geq 0, \ (i \in S_k), \quad (i = 1, \dots, n ; k = 1, \dots, m).$$

where $c_{ik} = c_{ik}(X)$ are given real functions for $k = 1, \dots, m$, $i \in S_k$. Of course, the inequalities (1) are superfluous when (2) are replaced by equations. The problem (P) has been also defined in [1], but there it was given only a small example. Here we shall solve the problem in the cases of linear and quadratic objective functions, assuming in the second case a certain form of the cost functions $c_{ik}(X)$.

2. Solving the linear case. Let us remark that the restrictions (5) are separable. In the linear case, i.e. the c_{ik}'s are real nonnegative numbers for all $k = 1, \dots, m$, $i \in S_k$, the linear objective function $C(X)$ is also separable. Therefore, the problem (P) can be solved by solving m subproblems, namely (P_k) : minimize

$$C_k(X^k) = \sum_{i \in S_k} c_{ik} x_{ik} \tag{6}$$

subject to

$$\sum_{i \in S_k} x_{ik} = v(S_k), \ x_{ik} \geq 0, \ (i \in S_k). \tag{7}$$

Obviously, if the index i_k is determined by the formula

$$c_{i_k k} = \min_{i \in S_k} c_{ik}, \tag{8}$$

an optimal solution of (P_k) is

$$x_{ik} = 0, \ (i \in S_k ; \ i \neq i_k), \qquad x^k_{i_k} = v(S_k). \tag{9}$$

Thus we proved the following result :

Theorem 1. *If the cost function $C(X)$ of the problem (P) is linear and for each k an index i_k is determined by (8), then the problem (P) has the optimal solution*

$$x_{ik} = \begin{cases} v(S_k) & \text{if } i = i_k, \\ 0 & \text{if } i \neq i_k, \end{cases} \quad \begin{pmatrix} i = 1, \dots, n \\ k = 1, \dots, m \end{pmatrix} \tag{10}$$

and the optimal value is

$$C(X) = \sum_{k=1}^{m} c_{i_k k} v(S_k). \tag{11}$$

The linear case seems to be trivial because a game in which each coalition has only one winner is unrealistic.

3. Solving the quadratic case. A reasonable hypothesis is that any cost function $c_{ik}(X)$ depends only on the gains x_{ik}, $(i \in S_k)$, of the members of coalition S_k, i.e. $c_{ik} = c_{ik}(X^k)$, $(k = 1, \dots, m ; \ i \in S_k)$. We shall investigate the linear case

$$c_{ik}(X^k) = p_{ik} + \sum_{j \in S_k} q^i_{jk} x_{jk}, \quad (k = 1, \dots, m ; i \in S_k). \tag{12}$$

Of course, some new hypotheses have to be imposed to the coefficients in (12), taking account on the fact that c_{ik} is the cost of an unitar gain of player i in S_k. Thus, if $x_{jk} = v(S_k)$ for some $j \neq i$ and consequently $x_{hk} = 0$ for all $h \in S_k$, $h \neq j$, the cost c_{ik} has to equal zero. Therefore

$$p_{ik} + q^i_{jk} v(S_k) = 0, \ \forall j \in S_k, \ j \neq i. \tag{13}$$

Further, if $x_{ik} = v(S_k)$ and consequently $x_{hk} = 0$ for all $h \in S_k$, $h \neq i$, the cost c_{ik} has to be a maximum, let us denote it by m_{ik} ; so

$$m_{ik} = p_{ik} + q^i_{ik} v(S_k), \quad (i \in S_k). \tag{14}$$

From (12) multiplied by $v(S_k)$ and (13), (14) we get

$$c_{ik} v(S_k) = m_{ik} x_{ik}, \quad (k = 1, \dots, m ; i \in S_k). \tag{15}$$

If $c_{ik} = c_{ik}(X^k)$, $(k = 1, \dots, m ; i \in S_k)$, the problem (P) is also separable in m subproblems (P_k) : minimize

$$C_k(X^k) = \sum_{i \in S_k} c_{ik}(X^k) x_{ik} \tag{16}$$

subject to

$$\sum_{i \in S_k} x_{ik} = v(S_k), \quad x_{ik} \geq 0, \quad (i \in S_k). \tag{17}$$

A problem (P_k) in which $v(S_k) = 0$ has the unique solution $x_{ik} = 0$, $\forall i \in S_k$; therefore we shall solve only the problems (P_k) in which $v(S_k) > 0$. According (15), the objective function can be written as

$$C_k(X^k) = \frac{1}{v(S_k)} \cdot \sum_{i \in S_k} m_{ik} x^2_{ik}. \tag{18}$$

Now, for solving a problem (P_k) we shall put

$$x_{ik} = y_{ik} v(S_k), \quad (i \in S_k), \tag{19}$$

and in the transformed problem we shall multiply the objective function by $1/2\, v(S_k)$. Thus, we get the problem : maximize

$$f_k(y) = -\frac{1}{2} \sum_{i \in S_k} m_{ik} y_{ik}^2 \tag{20}$$

subject to

$$\sum_{i \in S_k} y_{ik} = 1, \quad y_{ik} \geqslant 0, \quad (i \in S_k). \tag{21}$$

For solving such a problem we omit for a moment the index k and we assume $S = \{1,\dots, p\}$, so that we shall consider the problem : maximize

$$f(y) = -\frac{1}{2} \sum_{h=1}^{p} m_h y_h^2 , \tag{22}$$

subject to

$$\sum_{h=1}^{p} y_h = 1, \quad y_h \geqslant 0, \quad (h = 1,\dots, p). \tag{23}$$

Let us remark that if $m_r = 0$, an optimal solution is $y_r = 1$, $y_h = 0$, $(h = 1,\dots, p\,;\, h \neq r)$. Therefore, we have to solve only a problem in which $m_h \neq 0$ for all $h = 1,\dots, p$.

Now, the Kuhn-Tucker conditions for the problem above will be written, by using the rules given in [3], p. 42. We denote

$$\begin{gathered} f(y) = -\frac{1}{2} \cdot \sum_{h=1}^{p} m_h y_h^2 . \\ g_h(y) = y_h \geqslant 0, \ (h = 1,\dots, p), \ g_0(y) = 1 - \sum_{h=1}^{p} y_h = 0. \end{gathered} \tag{24}$$

We get

$$y_h \geqslant 0, \ (h = 1,\dots, p), \quad 1 - \sum_{h=1}^{p} y_h = 0. \tag{25}$$

$$\lambda_h y_h = 0, \quad \lambda_h \geqslant 0, \quad (h = 1,\dots, p). \tag{26}$$

$$-m_h y_h + \lambda_h - \lambda_0 = 0, \quad (h = 1,\dots, p). \tag{27}$$

and we shall solve this system.

First, we multiply every equation in (27) by y_h and we shall add the results ; we have

$$\lambda_0 = -\sum_{h=1}^{p} m_h y_h^2 \tag{28}$$

according to (25), (26). Thus, in any solution of the system, $\lambda_0 < 0$. Therefore, $y_h \neq 0$ for all $h = 1,\dots, p$, otherwise $y_h = 0$ would imply $\lambda_h = \lambda_0$, i.e. $\lambda_h < 0$. In consequence, the quadratic equations (26) give $\lambda_h = 0$ for all $h = 1,\dots, p$. Then, from (27) we have

$$y_h = -\frac{\lambda_0}{m_h} > 0, \quad (h = 1,\dots, p), \tag{29}$$

and the last condition (25) gives

$$\lambda_0 = -\frac{1}{M}, \quad M = \sum_{h=1}^{p} \frac{1}{m_h} . \tag{30}$$

The system (25), (26), (27) has the unique solution

$$y_h = \frac{1}{M m_h}, \quad \lambda_0 = -\frac{1}{M}, \quad \lambda_h = 0, \quad (h = 1,\dots, p). \tag{31}$$

The functions $f(y)$, $g_0(y)$, $g_h(y)$, $(h = 1,\dots, p)$, are differentiable and concave functions, thus according theorem 2.15 in [3] we proved the following result :

Theorem 2. *If $m_h > 0$ for all $h = 1,\dots, p$, the problem*

$$\min\left\{\frac{1}{2} \sum_{h=1}^{p} m_h y_h^2 \,\middle|\, \sum_{h=1}^{p} y_h = 1, \ y_h \geqslant 0, \ (h = 1,\dots, p)\right\} \tag{32}$$

has the optimal solution

$$y_h = \frac{1}{M m_h}, \quad M = \sum_{h=1}^{p} \frac{1}{m_h}, \quad (h = 1,\dots, p), \tag{33}$$

and the optimal value is $1/2\, M$.

If $m_h = 0$ for some $h \in H \subset \{1,\dots, p\}$, the set of optimal solutions is the convex hull of the set of vectors

$$y_r = 1, \quad y_h = 0, \quad (h = 1,\dots, p : h \neq r). \tag{34}$$

and the optimal value equals zero.

Now, if we return to the initial variables and use the theorem 2, we get the result :

Theorem 3. *If $v(S_k) > 0$ and $m_{ik} \neq 0$ for all $i \in S_k$, the problem (P_k) has the optimal solution*

$$x_{ik} = \frac{v(S_k)}{M_k m_{ik}}, \quad (i \in S_k), \quad M_k = \sum_{i \in S_k} \frac{1}{m_{ik}}, \tag{35}$$

and the optimal value is

$$C_k(X^k) = \frac{v(S_k)}{M_k} . \tag{36}$$

If $v(S_k) > 0$ and $m_{ik} = 0$ for $i \in H_k \subset S_k$, $H_k \neq S_k$, the set of optimal solutions is the convex hull of the set of vectors

$$x_{ik} = v(S_k), \ x_{hk} = 0, \ (h \in S_k, \ h \neq i), \ (i \in H_k), \tag{37}$$

and the optimal value equals zero.

If $v(S_k) = 0$ the optimal solution is $x_{ik} = 0$, $\forall i \in S_k$, and the optimal value equals zero.

According theorem 3 we can construct an optimal solution of the problem (P). by constructing successively the columns of the optimal matrix.

Let us remark that in the case $m_{ik} = m_k$, $\forall i \in S_k$, we have $M_k = |S_k| / m_k$ and consequently in (35). (36) we get

$$x_{ik} = \frac{v(S_k)}{|S_k|}, \quad (i \in S_k). \quad C_k(X^k) = \frac{m_k \, v(S_k)}{|S_k|}. \tag{38}$$

Of course, to find an optimal solution of the problem (P) in more general cases is an open question.

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GENERATION POLYNOMIAL RANGES BY ITERATIVE ONE-DIMENSIONAL ARRAYS

BY

ADRIAN SOIL

In this Note we present a way to construct an iterative one-dimensional array of finite-state automata which generates the power k of all positive integers. for a given positive integer k. Using this array, we construct another one which generates the range of a polynomial with given non-negative integers as coeficients for positive integer values of the argument.

1. Preliminaries. Let $N = \{1, 2, 3, \ldots, n, n+1 \ldots\}$ and $N^* = N \cup \{0\}$. We will denote an automaton (see [1]) by $A = (I, S, O, f, g)$. For $n \in N^*$. the set $\{j^n \mid j \in N\}$ will be called the set of perfect powers n.

1.1. Definition ([2]). *An infinite iterative one-dimensional array (IIAI) is a set of finite-state automata* $\{A_n \; n \in N^*\}$ *with the following properties* :

i) for every n, $n \in N$, the automata are interconnected in such a way that the state of the automaton A_n at time t depends only on the states of the automata A_{n-1}, A_n, A_{n+1} at time $t-1$;

ii) all the automata of the array are identical with the exception of A_0 and, at time $t = 0$, they are all in the same initial state;

iii) the inputs and the outputs of the array are connected to A_0 and the state of A_0 at time t depends only on the input and on the states of A_0 and A_1 at time $t-1$.

1.2. Let $n \in N$; in section 2 we shall construct, with the finite-state automata A_0^n, A_1^n, ... an *IIAI* which will have the opuput 1 at time t, if t is a perfect power n and 0, for others t.

1.3. A_0^n may easily be transformed, so that it recognizes the perfect powers n, in the following way; the output at time t is 1 if the input at time $t-1$ was 1 and t is a perfect power n and zero otherwise. Excepting this transformation the array works like the *IIAI* that generates the perfect powers n.

1.4. We can interconnect the automata of the array so that we realise 1.1 *i*) and *iii*):

— each A_k, $k > 0$, has an input line with two terminals and an output line with two terminals;