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PROJECTIVE METRICS AND NONLINEAR PROJECTIVE CONTRACTIONS

BY

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1. Throughout this note, the following general notations will be used

i) $\Pi(X) = \{X_1 \subseteq X ; X_1 \neq \emptyset\}$, ($X$ an arbitrary nonempty set),

ii) $F(X, Y) = \{f ; f : X \to Y\}$, ($X$ and Y arbitrary nonempty sets),

iii) $N = \{0, 1, \dots\}$, $N' = \{1, 2, \dots\}$,

iv) T^n (respectively $T^{[n]}$) = the n-th iterate of $T \in F(X, X)$, (X an arbitrary nonempty set), $\forall n \in N'$,

v) $X^n = X \times X \times \dots \times X$, ($n$ times). (X an arbitrary nonempty set), $\forall n \in N'$.

vi) $C(X, Y) = \{f \in F(X, Y) ; f$ is continuous$\}$, (X and Y arbitrary topological spaces),

vii) $R =]-\infty, +\infty[$, $R_- =]-\infty, 0]$, $R_+ = [0, +\infty[$, $R_+^0 =]0, +\infty[$, $\bar{R}_- = [-\infty, 0]$, $\bar{R}_+ = [0, +\infty]$, $D = \{(t, s) ; 0 < t \leqslant s < +\infty\}$, $E = \{(t, s) ; 0 \leqslant s \leqslant t < +\infty\}$.

viii) $K = \{f \in F(R_+, R_+), f(0) = 0, f(t) < t, \ \forall t \in R_+^0\}$,

ix) $P = \{f \in K ; \lim_{n \to \infty} f^{[n]}(t) = 0, \ \forall t \in R_+\}$.

x) $(R^n, \|\cdot\|)$ = the euclidean n-dimensional space with the euclidean norm $\|\cdot\|$, $\forall n \in N'$.

2. In this paragraph, a fundamental notion will be introduced: the so-called Hilbert's projective metric [1], [4]. Firstly, let us denote

i) $X = C(R_+, R)$, $X_+ = C(R_+, R_+)$, $Y = \{f \in X ; f(R_+^0) \subseteq R_+^0\}$.

It is evident that

$$X^n = C(R_+, R^n), \quad X_+^n = C(R_+, (R_+)^n), \quad \forall n \in N', \tag{2.1}$$

$$Y^n = \{f \in X^n ; f(R_+^0) \subseteq (R_+^0)^n\}, \quad \forall n \in N'. \tag{2.2}$$

Moreover, X_+^n is a cone in X^n, $\forall n \in N'$. The order relation induced in X^n by X_+^n will be indicated by $\leqslant$.

Let us define $m \in F(X_+^n \times Y^n, R_+)$ and $M \in F(X_+^n \times Y^n, \bar{R}_+)$ by

ii) $m(x, y) = \sup\{\mu \in R_+ ;\ \mu y \leqslant x\}$

iii) $M(x, y) = \inf\{\lambda \in R_+ ;\ x \leqslant \lambda y\}$, if $\{\lambda \in R_+ ;\ x \leqslant \lambda y\} \neq \emptyset$,
$M(x, y) = +\infty$, if $\{\lambda \in R_+ ;\ x \leqslant \lambda y\} = \emptyset$

for every $(x, y) \in X_+^n \times Y^n$. The following properties hold [1]

(2.3) $0 \leqslant m(x, y) \leqslant M(x, y) \leqslant +\infty,\ \forall (x, y) \in X_+^n \times Y^n,$

(2.4) $m(x, y)\, y \leqslant x \leqslant M(x, y)\, y,\ \forall (x, y) \in X_+^n \times Y^n,$

(2.5) $M(x, y) = 1/m(y, x),\ \forall x, y \in Y^n$, (where $1/0 = +\infty$),

(2.6) $M(x, z) \leqslant M(x, y) \cdot M(y, z),\ \forall x \in X_+^n,\ \forall y, z \in Y^n,$

(2.7) $M(x + y, z) \leqslant M(x, z) + M(y, z),\ \forall x, y \in X_+^n,\ \forall z \in Y^n,$

(2.8) $M(x, z) \leqslant M(y, z),\ \forall x, y \in X_+^n,\ \forall z \in Y^n$, (where $x \leqslant y$).

(2.9) $M(\lambda x, y) = \lambda M(x, y),\ \forall (x, y) \in X_+^n \times Y^n,\ \forall \lambda \in R_+.$

By definition, Hilbert's projective metric is the mapping $d \in F((Y^n)^2, \overline{R}_+)$, expressed by

iv) $d(x, y) = \ln[M(x, y)/m(x, y)]$, if $0 < m(x, y) \leqslant M(x, y) < +\infty$
$d(x, y) = +\infty$, if $m(x, y) = 0$ or $M(x, y) = +\infty$

for every $(x, y) \in (Y^n)^2$. The following properties are well known [4]

(2.10) $d(x, x) = 0,\ \forall x \in Y^n$

(2.11) $d(x, y) < +\infty \Rightarrow d(y, x) = d(x, y)$

(2.12) $d(x, y) < +\infty,\ d(y, z) < +\infty \Rightarrow d(x, z) \leqslant d(x, y) + d(y, z)$

(2.13) $d(x, y) = d(\lambda x, \mu y),\ \forall x, y \in Y^n,\ \forall \lambda, \mu \in R_+^0$

(2.14) $d(x, y) = 0 \Rightarrow$ there exists a unique $\lambda \in R_+^0$ such that $x = \lambda y$.

Definition 2.1. *Let X be a given nonempty set. A mapping $d \in F(X^2, \overline{R}_+)$ is said to be a generalized pseudometric on X if it satisfies (2.10) – (2.12); In this case, the couple (X, d) is called a generalized pseudometric space (abbreviated g.p.m.s.).*

Remark 2.1. *From (2.10) – (2.12) and the above definition it follows that (Y^n, d) is a g.p.m.s.*

Now, for every $x \in X^n$ define $|x| \in X_+^n$ *by*

v) $|x| = (|x_1|, \ldots, |x_n|)$, if $x = (x_1, \ldots, x_n)$

and, if $g \in Y^n$ is a fixed element, denote

vi) $\|x\|_g = M(|x|, g),\ \forall x \in X^n$

Theorem 2.1. *$(X^n, \|\cdot\|_g)$ is a generalized Banach space.*

Proof. From (2.7) – (2.9) it follows immediately that $\|\cdot\|_g$ is a generalized norm on X^n. Now, for any $p \in N'$ let us denote

vii) $X_p^n = C([0, p], R^n)$

viii) $|||x|||_p = \sup\{\|x(t)\| ;\ t \in [0, p]\},\ \forall x \in X_p^n$

It is well known that $(X_p^n, |||\cdot|||_p)$ is a Banach space, $\forall p \in N'$. Furthermore, from (2.4) and (vi) we get

(2.15) $|x| \leqslant \|x\|_g \cdot g,\ \forall x \in X^n,\ \|x\|_g < +\infty.$

Taking into account the following properties of the euclidean norm

(2.16) $x, y \in X_+^n,\ x \leqslant y \Rightarrow \|x(t)\| \leqslant \|y(t)\|,\ \forall t \in R_+$

(2.17) $\||x|(t)\| = \|x(t)\|,\ \forall t \in R_+,\ \forall x \in X^n,$

we obtain at once

(2.18) $|||x|||_p \leqslant \|x\|_g |||g|||_p,\ \forall x \in X^n,\ \|x\|_g < +\infty,\ \forall p \in N'.$

Let $\{x_i ;\ i \in N\} \subset X^n$ be a $\|\cdot\|_g$ - Cauchy sequence, i.e., $\forall \varepsilon > 0,\ \exists n(\varepsilon) \in N$ such that $i, j \geqslant n(\varepsilon) \Rightarrow \|x_i - x_j\|_g < \varepsilon$. From (2.15), this will imply $|x_i(t) - x_j(t)| = |x_i - x_j|(t) \leqslant \|x_i - x_j\|_g \cdot g(t) < \varepsilon g(t),\ \forall t \in R_+$. On the other hand, from (2.18), $\{x_i ;\ i \in N\} \subset X^n$ is a $|||\cdot|||_p$ - Cauchy sequence, $\forall p \in N'$, and therefore, $x_i \xrightarrow[|||\cdot|||_p]{} x,\ \forall p \in N'$, for some $x \in X^n$. Performing the limit as $j \to +\infty$ in the above inequality we get $|x_i(t) - x(t)| \leqslant \varepsilon g(t),\ \forall i \geqslant n(\varepsilon),\ \forall t \in R_+$, i.e., $\|x_i - x\|_g \leqslant \varepsilon,\ \forall i \geqslant n(\varepsilon)$, which shows that $x_j \xrightarrow[\|\cdot\|_g]{} x$.

Now, let us denote

ix) $\Gamma = \{(x, y) \in (Y^n)^2 :\ 0 < m(x, y) \leqslant 1 \leqslant M(x, y) < +\infty\},$

x) $\Pi_\Gamma(Y^n) = \{Z \in \Pi(Y^n) :\ Z^2 \subseteq \Gamma\},$

xi) $X_g^n = \{x \in X^n :\ \|x\|_g < +\infty\},\ \tilde{X}_g^n = \{x \in X^n ;\ \|x\|_g = 1\},$
$\forall g \in Y^n,$

xii) $Y_g^n = \{x \in Y^n ;\ 0 < m(x, g) \leqslant M(x, g) < +\infty\},\ \tilde{Y}_g^n = Y_g^n \cap \tilde{X}_g^n,$
$\forall g \in Y^n,$

xiii) $Y_g^n(\alpha) = \{x \in Y^n ;\ m(x, g) \geqslant \alpha\},\ \tilde{Y}_g^n(\alpha) = Y_g^n(\alpha) \cap \tilde{X}_g^n,$
$\forall \alpha \in R_+^0,\ \forall g \in Y^n,$

xiv) $Y_g^n(\alpha, \beta) = \{x \in Y^n ;\ \alpha \leqslant m(x, g) \leqslant M(x, g) \leqslant \beta\},\ \tilde{Y}_g^n(\alpha, \beta) =$
$= Y_g^n(\alpha, \beta) \cap \tilde{X}_g^n,\ \forall (\alpha, \beta) \in D,\ \forall g \in Y^n.$

Remark 2.2. *It is easy to verify that $X_g^n, \tilde{X}_g^n, Y_g^n(\alpha)$, $(\forall \alpha \in R_+^0)$, $Y_g^n(\alpha, \beta)$, $(\forall (\alpha, \beta) \in D)$ are closed subsets in $(X^n, \|\cdot\|_g)$.*

A very important result concerning the connection between $\|\cdot\|_g$ $(g \in Y^n)$ and „d" can be stated as follows

Theorem 2.2. *Let $g \in Y^n$ be fixed. Then,*

(2.19) $\| x - y \|_g \leqslant [e^{d(x,y)} - 1] \inf \{\| x \|_g , \| y \|_g\}, \ \forall (x, y) \in \Gamma \cap (Y^n_g)^2$

(2.20) $\| x - y \|_g \geqslant \frac{1}{2} [1 - e^{-d(x,y)}] \sup \{m (x, g), m (y, g)\}, \ \forall (x, y) \in (Y^n_g)^2$

Proof. Let $(x, y) \in \Gamma \cap (Y^n_g)^2$ be given. From (2.4) and (ix) we get $x - y \leqslant [M (x, y) - m (x, y)] y = [M (x, y)/m (x, y) - 1] m (x, y) y \leqslant [M (x, y)/m (x, y) - 1] x = [e^{d(x,y)} - 1] x$, and analogously, $x - y \leqslant [M (x, y) - m (x, y)] y = [M (x, y)/m (x, y) - 1] m (x, y) y \leqslant [M (x, y)/m (x, y) - 1] y = [e^{d(x,y)} - 1] y$. Similar inequalities are to be obtained if we interchange x and y. Taking into account the following simple property

(2.21) $$x, y \in X^n, \ x \leqslant y, \ -x \leqslant y \Rightarrow | x | \leqslant y$$

and the properties (2.8) + (2.9) we get (2.19). Furthermore, let us remark that $x, y \in Y^n_g , \ \| x - y \|_g < + \infty \Rightarrow x - y \leqslant | x - y | \leqslant \| x - y \|_g . g \leqslant [\| x - y \|_g/m (y, g)] y$, and analogously, $y - x \leqslant | y - x | \leqslant \| x - y \|_g . g \leqslant [\| x - y \|_g/m (y, g)] y$, which gives $x \leqslant [1 + \| x - y \|_g/m (y, g)] y$ and respectively $x \geqslant [1 - \| x - y \|_g/m (y, g)] y$. Therefore, $M (x, y) \leqslant 1 + \| x - y \|_g/m (y, g)$ and $m (x, y) \geqslant 1 - \| x - y \|_g/m (y, g)$. A simple substraction yields $M (x, y) - m (x, y) \leqslant 2 \| x - y \|_g/m (y, g)$, and this implies $\| x - y \|_g \geqslant \frac{1}{2} m (y, g) [M (x, y) - m (x, y)] \geqslant \frac{1}{2} m (y, x) m (x, g) . [M (x, y) - m (x, y)] = \frac{1}{2} m (x, g) [1 - m (x, y)/M (x, y)] = \frac{1}{2} m (x, g) [1 - e^{-d(x,y)}]$. A similar inequality may be obtained by interchanging x and y, and this completes the proof. Q.E.D.

Theorem 2.3. *Let $g \in Y^n$ be given. Then, $(\tilde{Y}^n_g , d)$ is a complete metric space.*

Proof. Firstly, let us observe that $\tilde{Y}^n_g \in \tilde{\Pi} (Y^n), \ \forall g \in Y^n$. Let $\{x_p ; p \in N\} \subset \tilde{Y}^n_g$ be a d-Cauchy sequence. From (2.19), $\{x_p ; p \in N\}$ will be also a $\| . \|_g$ - Cauchy sequence, and therefore, (see theorem 2.1) $x_p \xrightarrow[\| \|_g]{} x$ for some $x \in X$. Since a Cauchy sequence is bounded, there exists $\mu \in R^0_+$, such that $d (x_p , g) \leqslant \mu, \ \forall p \in N$, i.e., $M (x_p , g)/m (x_p , g) \leqslant e^{\mu}, \ \forall p \in N$, and this implies (as $M (x_p , g) = \| x_p \|_g = 1, \ \forall p \in N$) $m (x_p , g) \geqslant e^{-\mu}$, $\forall p \in N$. Therefore, $\{x_p ; p \in N\} \subset \tilde{Y}^n_g (e^{-\mu}, 1)$; as this set is $\| . \|_g$ — closed (see Remark 2.2) we get $x \in \tilde{Y}^n_g (e^{-\mu}, 1)$. Now, from (2.20) we obtain $(e^{-\mu}/2) [1 - e^{-d(x_p, x)}] \leqslant \| x_p - x \|_g , \ \forall p \in N$, and this gives $x_p \xrightarrow[d]{} x$, completing the proof. Q.E.D.

Theorem 2.4. *(Y^n, d) is a complete g.p.m.s.*

Proof. Let $\{x_p ; p \in N\} \subset Y^n$ be a d-Cauchy sequence. Without loss of generality we may assume that $d (x_0 , x_p) \leqslant \mu, \forall p \in N$, for some $\mu \in R^0_+$. Let us denote $g = x_0 , \ y_p = x_p/\| x_p \|_g , \ \forall p \in N$. Clearly, $\{y_p ; p \in N\} \subset \tilde{Y}^n_g$ is a d-Cauchy sequence (from hypothesis and (2.13)) and therefore (from theorem 2.3) $y_p \xrightarrow[d]{} y$ for some $y \in \tilde{Y}^n_g$, which implies (from (2.13)) $x_p \xrightarrow[d]{} y$. Q.E.D.

3. Definition 3.1. *A mapping $T \in F (Y^n, Y^n)$ is said to have property (Q), if*

(3.1) $$d (x, y) = 0 \Rightarrow d (Tx, Ty) = 0.$$

A motivation for the study of such a class of mappings will be given in the next paragraph. Now, let us denote

(i) $F_0 (Y^n, Y^n) = \{T \in F (Y^n, Y^n) ; T$ has the property $(Q)\}$

(ii) $Z = F (R^0_+ , R^0_+)$.

A partial characterization of the class $F_0 (Y^n, Y^n)$ may be given as follows

Theorem 3.1. *For every $T \in F_0 (Y^n, Y^n)$ there exists a map*

(3.2) $$x \to T^* (x) \in Z, \ \forall x \in Y^n,$$

such that

(3.3) $$T (\lambda x) = [T^* (x) (\lambda)] Tx, \ \forall x \in Y^n, \ \forall \lambda \in R^0_+ .$$

(3.4) $$T^* (x) (\lambda \mu) = T^* (\mu x) (\lambda) . T^* (x) (\mu), \ \forall x \in Y^n, \ \forall \lambda, \mu \in R^0_+ .$$

Proof. Let $x \in Y^n$, $\lambda \in R^0_+$ be fixed. Since $d (\lambda x, x) = 0$, we must have (by hypothesis) $d (T (\lambda x), Tx) = 0$. By (2.14), there exists a unique $\mu \in R^0_+$, such that $T (\lambda x) = \mu Tx$. Let us put $T^* (x) (\lambda) = \mu$. Clearly, (3.3) holds. Furthermore (from (3.3)) $T (\lambda \mu x) = T (\lambda (\mu x)) = [T^* (\mu x) . (\lambda)] T (\mu x) = [T^* (\mu x) (\lambda)] [T^* (x) (\mu)] Tx$, and on the other hand (again from (3.3)) $T (\lambda \mu x) = T^* (x) (\lambda \mu) Tx$. From these two equalities, (3.4) follows at once. Q.E.D.

Another important class of mappings is the following

Definition 3.2. *A mapping $T \in F (Y^n, Y^n)$ is said to be monotone increasing (resp. decreasing) if*

(3.5) $$x, y \in Y^n, \ x \leqslant y \Rightarrow Tx \leqslant Ty \ (\text{resp. } Tx \geqslant Ty).$$

Lemma 3.1. *Let $T \in F_0 (Y^n, Y^n)$ be monotone increasing (resp. decreasing). Then, $T^* (x) \in Z$ is also monotone increasing (resp. decreasing) for every $x \in Y^n$.*

Proof. Evident.

Let us indicate by $F_1 (Y^n, Y^n)$ the class of all $T \in F_0 (Y^n, Y^n)$, such that $\{T^* (x) ; x \in Y^n\} = \{T^*\}$, i.e., the family $\{T^* (x) ; x \in Y^n\}$ is independent of $x \in Y^n$. Therefore, for every $T \in F_1 (Y^n, Y^n)$ we have

a) $T (\lambda x) = T^* (\lambda) Tx, \ \forall x \in Y^n, \ \forall \lambda \in R^0_+$; b) $T^* (\lambda \mu) = T^* (\lambda) T^* (\mu)$, $\forall \lambda, \mu \in R^0_+$.

Lemma 3.2. [5]. *Let $f \in Z$ be such that a) $f (\lambda \mu) = f (\lambda) f (\mu), \ \forall \lambda, \mu \in R^0_+$; b) f is monotone increasing (resp. decreasing). Then, there exists a $p \in R_+$ (resp. $p \in R_-$), such that $f (t) = t^p, \ \forall t \in R^0_+$.*

Proof. Let us put $p = \ln [f(e)]$, i.e., $f(e) = e^p$. It is simply to verify that $f(e^r) = e^{pr}$, for every rational number $r \in Q \cap R^0_+$. Let $\alpha \in R^0_+$ be an irrational number. Then, there exist $\{r_m ; m \in N\} \subset Q \cap R^0_+$, $\{s_m ; m \in N\} \subset Q \cap R^0_+$, such that $r_0 < r_1 < \ldots < r_m < \ldots < \alpha < \ldots < s_m < \ldots < s_1 < s_0$, and $\lim_{m \to \infty} r_m = \lim_{m \to \infty} s_m = \alpha$. By hypothesis, this implies $e^{pr_m} \leqslant f(e^\alpha) \leqslant e^{ps_m}$, $\forall m \in N$ (resp. $e^{pr_m} \geqslant f(e^\alpha) \geqslant e^{ps_m}$, $\forall m \in N$). Performing the limit as $m \to +\infty$, we get $f(e^\alpha) = e^{p\alpha}$. Therefore,

$$f(t) = f(e^{\ln t}) = e^{p \ln t} = t^p, \quad \forall t \in R^0_+. \qquad \text{Q.E.D.}$$

Definition 3.3. *A mapping $T \in F(Y^n, Y^n)$ is said to be p-homogeneous ($p \in R$) if*

$$(3.6) \qquad T(\lambda x) = \lambda^p \, Tx, \quad \forall x \in Y^n, \ \forall \lambda \in R^0_+.$$

Theorem 3.2. *Let $T \in F_1(Y^n, Y^n)$ be monotone increasing (resp. decreasing). Then, T is p-homogeneous for some $p \in R_+$ (resp. $p \in R_-$).*

Proof. Is a simple consequence of the above results.

4. Recall a fixed point result established in [6].

Theorem 4.1. *Let (X, d) be a complete metric space and let $T \in F(X, X)$ $f \in P$, be such that*

$$(4.1) \qquad d(x, y) \leqslant t \Rightarrow d(Tx, Ty) \leqslant f(t)$$

Then there exist $z \in X$ and $\rho \in F(X, R_+)$, such that

$$(4.2) \qquad \{x \in X ;\ x = Tx\} = \{z\}$$

$$(4.3) \qquad d(T^n x, z) \leqslant f^{[n]}(t), \quad \forall t \geqslant \rho(x), \ \forall n \in N', \ \forall x \in X.$$

Definition 4.1. *A mapping $T \in F(Y^n, Y^n)$ for wich (4.1) holds is called a nonlinear projective contraction (abbreviated n.p.c.)*

Remark 4. 1. If T is a n.p.c., then $T \in F_0(Y^n, Y^n)$.

As an application of this result, we have

Theorem 4.2. *Let $T \in F_0(Y^n, Y^n)$, $f \in P$, be such that*

$$(4.4) \qquad T \text{ is monotone increasing}$$

$$(4.5) \qquad T^*(\lambda x)(e^t) \leqslant e^{f(t)}, \quad \forall x \in Y^n, \ \forall \lambda, t \in R^0_+$$

$$(4.6) \qquad \text{the set } Y^n(T) = \{g \in Y^n ;\ \alpha g \leqslant Tg \leqslant \beta g, \text{ for some } (\alpha, \beta) \in D\}$$

is not empty ;

Then, there exists a map

$$(4.7) \qquad g \to \bar{g} = U(g) \in \tilde{Y}^n_g, \qquad \forall g \in Y^n(T)$$

and a map $\rho \in F(Y^n(T), R_+)$, such that

$$(4.8) \qquad T(\lambda \bar{g}) = T^*(\bar{g})(\lambda) \, \| T(\bar{g}) \|_g \, \bar{g}, \quad \forall \lambda \in R^0_+, \ \forall g \in Y^n(T)$$

$$(4.9) \qquad m \in N', \ t \geqslant \rho(g) \Rightarrow \lambda_m T^m(g) \leqslant \bar{g} \leqslant e^{f^{[m]}(t)} \lambda_m T^m(g)$$

(for some $\lambda_m \in R^0_+$).

Proof. It is simply to verify that from (4.4), (4.5) and (3.4) we get (4.1), i.e., T is a n.p.c. Now, let $g \in Y^n(T)$ be fixed. As (4.1) holds, we get $T \in F(Y^n_g, Y^n_g)$. Define $V_g \in F(Y^n_g, Y^n_g)$ by $V_g(x) = Tx / \| Tx \|_g$ $\forall x \in Y^n_g$. Clearly, V_g satisfies (4.1) too ; so, theorem 4.1 is applicable. Let us put $\bar{g} = U(g) =$ the (unique) fixed point of V_g, $\rho(g) =$ the same as in theorem 4.1, and let us remark that $T \in F_*(Y^n, Y^n) \Rightarrow V^{[m]}_g(g) = \mu_m T^m(g)$, (for some $\mu_m \in R^0_+$), $\forall m \in N'$. Combining this with (4.2) and (4.3) the conclusion follows. Q.E.D.

Remark 4.1. *Suppose that both (4.4) and (4.5) are replaced by*

$$(4.4)' \qquad T \text{ is monotone decreasing}$$

$$(4.5)' \qquad T^*(\lambda x)(e^t) \geqslant e^{-f(t)}, \quad \forall x \in Y^n, \ \forall \lambda, t \in R^0_+$$

Then, the above conclusions (4.8) and (4.9) are still valid.

Corollary 4.1. *Suppose $\lambda_0 \in R^0_+$ is a fixed point of the map $\lambda \to \| T(\bar{g}) \|_g . T^*(\bar{g})(\lambda)$. Then, $\lambda_0 \bar{g} \in Y^n_g$* is a fixed point of T i.e., $T(\lambda_0 \bar{g}) = \lambda_0 \bar{g}$

Proof. Evident (from (4.8)).

Theorem 4.3. *Let $T \in F_1(Y^n, Y^n)$ $p \in [0, 1[$ (resp. $p \in]-1, 0]$) be such that* a) *T is monotone increasing (resp. decreasing) ;* b) *T is p-homogeneous ;* c) *(4.6) holds. Then, there exist a map*

$$(4.10) \qquad g \to g^* = V(g) \in Y^g_n, \quad \forall g \in Y^n(T)$$

and a map $\rho \in F(Y^n(T), R_+)$, such that

$$(4.11) \qquad T(g^*) = g^*, \quad \forall g \in Y^n(T)$$

$$(4.12) \qquad \begin{gathered} m \in N', \ t \geqslant \rho(g) \Rightarrow \lambda_m T^m(g) \leqslant g^* \leqslant e^{f^{[m]}(t)} \lambda_m T^m(g) \\ \textit{(for some } \lambda_m \in R^0_+). \end{gathered}$$

Proof. Define $f \in P$ by $f(t) = |p| \cdot t$, $\forall t \in R_+$. Clearly, theorem 4.2 is applicable. On the other hand, the mapping $\lambda \to [\| T(\bar{g}) \|_g] \lambda^p$ has a (unique) fixed point $\lambda_0 = \lambda_0(g) = [\| T(\bar{g}) \|_g]^{1/(1-p)}$, $\forall g \in Y^n(T)$, i.e. corollary 4.1 is also applicable. Q.E.D.

5. In what follows, a distinct class of mappings will be investigated.

Definition 5.1. *Let (X, d) be a metric space. A mapping $f \in F(X, R)$ is said to be lower semi-continuous (abbreviated l.s.c.) on X if $\{x \in X ;\ f(x) \leqslant \lambda\}$ is d-closed, $\forall \lambda \in R$.*

A fundamental result used here is

Theorem 5.1. [2], [3]. *Let (X, d) be a complete metric space and let $T \in F(X, X)$, $f \in F(X, R)$ be such that*

$$(5.1) \qquad d(Tx, x) \leqslant f(x) - f(Tx), \quad \forall x \in X$$

$$(5.2) \qquad f \text{ is l.s.c. on } X$$

$$(5.3) \qquad \inf \{f(x) ;\ x \in X\} > -\infty$$

Then, $\{x \in X;\ x = Tx\} \neq \emptyset$.

Definition 5.1. *A mapping* $T \in F(Y^n, Y^n)$ *is said to be an I-mapping with respect to* $\psi \in F(Y^n, R^0_+)$, *if*

(5.4) $$\psi(Tx)\, x \leqslant Tx \leqslant \psi(x)\, x, \quad \forall x \in Y^n$$

Definition 5.2. *A mapping* $\varphi \in F(Y^n, R)$ *is said to be* g-*l.s.c.* ($g \in Y^n$) *if* $\{x \in \tilde{Y}^n_g;\ \varphi(x) \leqslant \lambda\}$ *is d-closed*, $\forall \lambda \in R$.

As an application of theorem 5.1 we have

Theorem 5.2. *Let* $T \in F(Y^n, Y^n)$, $\psi \in F(Y^n, R^0_+)$ *be such that*

(5.5) T *is an I-mapping with respect to* ψ

(5.6) *the set* $Y^n(\psi)$ *of all* $g \in Y^n$ *such that* a) ψ *is 0-homogeneous on* Y^n_g, *i.e.*, $\psi(\lambda x) = \psi(x)$, $\forall x \in Y^n_g$, $\forall \lambda \in R^0_+$; b) ψ *is* g-*l.s.c.*; c) $\inf\{\psi(x);\ x \in \tilde{Y}^n_g\} > 0$, *is not empty*.

Then, there exists a map

(5.7) $$g \to U(g) \in \Pi(\tilde{Y}^n_g), \quad \forall g \in Y^n(\psi)$$

such that

(5.8) $$Tz = [\|Tz\|_g] \,.\, z, \quad \forall z \in U(g), \quad \forall g \in Y^n(\psi)$$

Proof. Let $g \in Y^n(\psi)$ be fixed. From (5.5) we get $T \in F(Y^n_g, Y^n_g)$. Let $V_g \in F(\tilde{Y}^n_g, \tilde{Y}^n_g)$ be defined by $V_g(x) = Tx/\|Tx\|_g$, $\forall x \in \tilde{Y}^n_g$, and let $\varphi \in F(\tilde{Y}^n_g, R)$ be given by $\varphi(x) = \ln[\psi(x)]$, $\forall x \in \tilde{Y}^n_g$. Clearly, (5.5) and (5.6) (a) imply $d(V_g(x), x) = d(Tx, x) \leqslant \varphi(x) - \varphi(Tx) = \varphi(x) - \varphi(V_g(x))$, $\forall x \in \tilde{Y}^n_g$, i.e., V_g satisfies (5.1). On the other hand, from (5.6) (b) + (c) it follows that φ satisfies (5.2) + (5.3). So, theorem 5.1 is applicable. Define $U(g) = \{x \in Y^n_g;\ x = V_g(x)\}$, $\forall g \in Y^n(\psi)$; obviously, (5.8) is satisfied. Q.E.D.

6. In this paragraph, as an application of the above theorems, an existence and uniqueness result concerning a class of Volterra functional equations will be presented. Firstly, let us denote

i) $J = \{\alpha \in \Pi(R_+);\ \alpha = \text{interval}\}$,

ii) $M_n = C(E, (R_+)^{n^2})$, $M^0_n = C(E, (R^0_+)^{n^2})$ (the set of all positive (resp. strictly positive) continuous $n \times n$ matrix functions),

For every $x = (x_1, \ldots, x_n) \in (R_+)^n$, and $p \in R^0_+$ denote

iii) $x^p = (x_1^p, \ldots, x_n^p)$.

Let $\varphi \in F(R_+, J)$ be a given mapping. For the sake of simplicity we shall denote henceforward

iv) $\varphi(t) = \tilde{t}$, $\forall t \in R_+$; $\varphi(R_+) = \tilde{R}_+$

Let us indicate by H the class of all (h, p), $h \in F(Y^n \times \tilde{R}_+, R^n)$, $p \in F(Y^n, R^0_+)$, such that

(6.1) $\forall x \in Y^n$, the map $t \to h(x, \tilde{t})$ is in Y^n,

(6.2) $$x, y \in Y^n,\ x \leqslant y,\ t \in R_+ \Rightarrow h(x, \tilde{t})^{p(x)} \leqslant h(y, \tilde{t})^{p(y)},$$

(6.3) $$x \in Y^n,\ t \in R_+,\ \lambda \in R^0_+ \Rightarrow h(\lambda x, \tilde{t}) = \lambda h(x, \tilde{t}).$$

Now, let $A \in M^0_n$, $(h, p) \in H$, be fixed mappings. Consider the following Volterra functional equation

(6.4) $$x(t) = \int_0^t A(t, s)\, [h(x, \tilde{s})]^{p(x)}\, ds, \quad \forall t \in R_+$$

Theorem 6.1. *Suppose that*

(6.5) $$\sup\{p(x);\ x \in Y^n\} = q \in\,]0, 1[$$

(6.6) *there exist* $B \in M^0_n$, $g \in Y^n$, $\lambda, \mu \in R^0_+$ *such that*

a) $$\lambda B \leqslant A \leqslant \mu B.$$

b) $$g(t) = \int_0^t B(t, s)\, [h(g, \tilde{s})]^{p(g)}\, ds, \quad \forall t \in R_+.$$

Then, there exist $\bar{g} \in Y^n_g$ *and* $\rho \in R_+$, *such that*

(6.7) $\bar{g}$ *is a solution of* (6.4).

(6.8) *the sequence of succesive approximations* $\{x_m;\ m \in N\} \subset Y^n_g$,

$$x_0(t) = g(t), \quad \forall t \in R_+, \quad x_{m+1}(t) = \int_0^t A(t, s)\, [h(x_m, \tilde{s})]^{p(x_m)}\, ds,$$

$\forall t \in R_+$, *has the property* $\lambda_m x_m \leqslant \bar{g} \leqslant e^{q^m \cdot \sigma} \lambda_m x_m$, (*for some* $\lambda_m \in R^0_+$), $\forall m \in N'$, $\forall \sigma \geqslant \rho$.

Proof. Let us define $T \in F(Y^n, Y^n)$ by the right hand of (6.4) and $f \in P$ by $f(t) = q \,.\, t$, $\forall t \in R_+$. It is evident that T is monotone increasing and that $T \in F_0(Y^n, Y^n)$, with $T^*(x)(\lambda) = \lambda^{p(x)}$, $\forall x \in Y^n$, $\forall \lambda \in R^0_+$. On the other hand, the map $\lambda \to \mu \,.\, \lambda^{p(x)}$ has a unique fixed point, $\forall \mu \in R^0_+$, $\forall x \in Y^n$. Therefore, theorem 4.1 as well as corollary 4.1 are applicable, and this completes the proof. Q.E.D.

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ALMOST TANGENT STRUCTURES OF ORDER k ON SPHERES

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§1. Introduction. The almost tangent structures of order k are polynomial structures of the type $J^{k+1} = 0$, which appear as a generalization of the usual ones of first order (Clark-Bruckheimer [1]). Those structures were introduced by Eliopoulos [4].

We approach the problem of existence of the almost tangent structures of order k on sphere S^n. Concretely : in §2 we give the strictly necessary definitions, used in this paper. In §3 we obtain in a constructive manner a structure of order 6 (type $J^7=0$) on sphere S^7 by using the distributions of vector fields on this sphere. There is only a similar case of existence in the bibliography ; concretely, for S^3 (Clark-Goel [2]).

Finally, in §4 we give results concerning obstruction to the existence of such structures on S^n. We finish with a theorem that gives a negative answer to the question, except in the cases $n = 3, 7, 15$. Thus it is an open problem the existence of structures of the type $J^3 = 0$, $J^5 = 0$ on S^{15}.

Thus we solve the question, proposed by Clark-Goel [2], for spheres S^{3n}, $3n \not\equiv 1 \pmod 4$, concerning the existence of almost tangent structures of second order.

§2. Definitions. Let M be a $(k+1)n$-dimensional differentiable manifold. A G-structure with group G of matrices of the form

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & A_1 & \cdots & \vdots \\ \vdots & & & 0 \\ A_{k+1} & \cdots & \cdots & A_1 \end{pmatrix} ; \; A_1 \in Gl\,(n, \mathbf{R})$$

will be said an almost tangent structure of order k.

We will say M is an *almost tangent manifold of order* k if such a structure is defined on M.

As it is known, a necessary and sufficient condition for a manifold M the dimension of which is $(k+1)\,n$, to admit an almost tangent structure of order k, is the existence of a (1, 1) tensor field with constant rank kn, and such that $J^{k+1} = 0$.

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