REFERENCES

1. Bauer F. L. — *An elementary proof of the Hopf inequality for positive operators*, Numerische Math., 7 (1965) 331—337.
2. Bronsted A. — *On a lemma of Bishop and Phelps*, Pacific Journal of Math., 55 (1974), 335—341.
3. Caristi J. and Kirk W. A. — *Geometric fixed point theory and inwardness conditions*, Lect. Notes in Math., vol. 490, Springer Verlag, New York, 1975, 74—83.
4. Bushell P. J. — *On a class of Volterra and Fredholm nonlinear integral equations*, Math. Proc. Cambr. Phil. Soc., 79 (1976), 329—335.
5. Ghermănescu M. — *Ecuații funcționale*, Ed. Acad. R.S.R., București, 1960, cap. IV.
6. Turinici M. — *Nonlinear contractions and applications to Volterra functional equations* An. șt. Univ. „Al. I. Cuza" Iași, t. 23 (1977), 43—50.


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# ALMOST TANGENT STRUCTURES OF ORDER $k$ ON SPHERES

BY

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**§1. Introduction.** The almost tangent structures of order $k$ are polynomial structures of the type $J^{k+1} = 0$, which appear as a generalization of the usual ones of first order (Clark-Bruckheimer [1]). Those structures were introduced by Eliopoulos [4].

We approach the problem of existence of the almost tangent structures of order $k$ on sphere $S^n$. Concretely : in §2 we give the strictly necessary definitions, used in this paper. In §3 we obtain in a constructive manner a structure of order 6 (type $J^7=0$) on sphere $S^7$ by using the distributions of vector fields on this sphere. There is only a similar case of existence in the bibliography ; concretely, for $S^3$ (Clark-Goel [2]).

Finally, in §4 we give results concerning obstruction to the existence of such structures on $S^n$. We finish with a theorem that gives a negative answer to the question, except in the cases $n = 3, 7, 15$. Thus it is an open problem the existence of structures of the type $J^3 = 0$, $J^5 = 0$ on $S^{15}$.

Thus we solve the question, proposed by Clark-Goel [2], for spheres $S^{2n}$, $3n \not\equiv 1$ (mod 4), concerning the existence of almost tangent structures of second order.

**§2. Definitions.** Let $M$ be a $(k+1)n$-dimensional differentiable manifold. A $G$-structure with group $G$ of matrices of the form

$$\begin{pmatrix} A_1 & 0 & \cdots & 0 \\ A_2 & A_1 & \cdots & \vdots \\ \vdots & & & 0 \\ A_{k+1} & \cdots & \cdots & A_1 \end{pmatrix} ; A_1 \in Gl(n, \mathbf{R})$$

will be said an almost tangent structure of order $k$.

We will say $M$ is an *almost tangent manifold of order* $k$ if such a structure is defined on $M$.

As it is known, a necessary and sufficient condition for a manifold $M$ the dimension of which is $(k+1)\,n$, to admit an almost tangent structure of order $k$, is the existence of a (1, 1) tensor field with constant rank $kn$, and such that $J^{k+1} = 0$.

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**§3. An almost tangent structure of order 6 ($J^7 = 0$), on $S^7$.** Let $S^7$ be the sphere in $\mathbf{R}^8$; and let $N$ be the unitary normal vector field on $S^7$. *If* $x = (\bar{x}, x^0, x^1, \dots, x^6)$ is a global chart, $N$ can be expressed by $N(x) = (\bar{x}, x^0, x^1, \dots, x^6)$.

$S^7$ is parallelizable, then it admits a global basis of vector fields. For instance, a globally orthogonal one is given by:

$$
\begin{aligned}
E_0(x) &= (-x^0,\ \bar{x},\ x^2,\ -x^1,\ x^4,\ -x^3,\ -x^6,\ x^5),\\
E_1(x) &= (-x^1,\ -x^2,\ \bar{x},\ x^0,\ x^5,\ x^6,\ -x^3,\ -x^4),\\
E_2(x) &= (-x^2,\ x^1,\ -x^0,\ \bar{x},\ x^6,\ -x^5,\ x^4,\ -x^3),\\
E_3(x) &= (-x^3,\ -x^4,\ -x^5,\ -x^6,\ \bar{x},\ x^0,\ x^1,\ x^2),\\
E_4(x) &= (-x^4,\ x^3,\ -x^6,\ x^5,\ -x^0,\ \bar{x},\ -x^2,\ x^1),\\
E_5(x) &= (-x^5,\ x^6,\ x^3,\ -x^4,\ -x^1,\ x^2,\ \bar{x},\ -x^0),\\
E_6(x) &= (-x^6,\ -x^5,\ x^4,\ x^3,\ -x^2,\ -x^1,\ x^0,\ \bar{x}).
\end{aligned}
$$

**Proposition 3.1.** *$S^7$ admits an almost tangent structure of order six.* In fact, the structure is defined in the following way:

$$JE_i = E_{i+1},\ i = 0, 1, \dots, 5;\ JE_6 = 0.$$

*Remark 1.* Clark-Goel [2] define a second order structure on the sphere $S^3$.

**§4. Obstruction to the existence of almost tangent structures on spheres.**

**Lemma 4.1.** *A manifold $M$ of dimension $2n$, such that $X(TM) \neq 0$, $H^n(M;\mathbf{Z}) = 0$, does not admit almost tangent structures of first order.*

*Proof*: The fact that an almost tangent structure of the first order is an extension of an orthogonal one, is known. (Clark-Bruckheimer [1]).

If a manifold $M$, with dimension $2n$, admits a structure of type $J^2 = 0$, the tangent bundle $TM$ splits in Whitney sum of two bundles $T_i$, $i = 1, 2$, with dimension $n$. ($TM = T_1 \oplus T_2$). Then, by Whitney's theorem for the Euler class, we have:

$$X(TM) = XT_1 \cup XT_2.$$

In addition to this, $XT_1$ and $XT_2$ are elements of $H^n(M:\mathbf{Z})$, and therefore, null.

**Corollary 4.2.** *No connected orientable manifold of dimension $2n$, and verifying $H^n(M;Z) = 0$, admits almost tangent structures of first order, if its Euler characteristic $\chi(M)$ is not null.*

*Proof*: In fact, in these conditions (see Deheuvels [3]) we have $X = \chi \cdot \mu$, $\mu$ being the fundamental cohomology class of the manifold, $\mu \in H^{2n}(M;\mathbf{Z})$.

**Lemma 4.3.** *The existence of a structure of the type $J^k = 0$, on a manifold $M$, of dimension $kn$, cocarries a chain of distributions in the tangent bundle with respective dimensions $n, 2n, \dots, (k-1)n$.*

*Proof*: As it is well known, the endomorphism $J_x : T_xM \to T_xM$, in a point $x \in M$, is linear, has rank $(k-1)n$, and verifies $J^k = 0$. Therefore the chain $(\operatorname{Ker} J)_x \subset (\operatorname{Ker} J^2)_x \subset \dots \subset (\operatorname{Ker} J^k)_x$ has respective dimensions $n$ and $kn$ in its extrems. Moreover in each step it cannot increase its dimension in more than $n$. Consequently we hawe:

$$\dim(\operatorname{Ker} J^i)_x = i \cdot n;\quad i = 1, \dots, k-1.$$

Q.E.D.



The functions $\operatorname{Ker} J^i : x \to (\operatorname{Ker} J^i)_x$ are $i \cdot n$-dimensional distributions on $M$, if the vector fields $X_1 \dots X_{k \cdot n}$ are local basis for $\operatorname{Ker} J^i$.

Therefore, it defines a structure on $M$, with group $G_i$ consisting of the non singular matrices

$$
\begin{pmatrix}
A_1 & B_1' & C_1' & \cdots & M_k' \\
B_1 & A_2 & B_2' & & \vdots \\
C_1 & B_2 & A_3 & & \vdots \\
\vdots & & & & B_{k-1}' \\
M_k & \cdots & C_{k-2} & B_{k-1} & A_k
\end{pmatrix},
$$

where the boxes in the right superior rectangle of $(k-i)$-lines of sub matrices, and $i$-columns are zero.

These $G_i$-structures has as its subordinated structure, one with group $G$, consisting of the matrices.

$$
\begin{pmatrix}
A_1 & 0 & 0 & \cdots & 0 \\
B_1 & A_2 & 0 & \cdots & \vdots \\
C_1 & B_2 & A_3 & \cdots & \vdots \\
\vdots & & & & \vdots \\
M_k & \cdots & & B_{k-1} & A_k
\end{pmatrix};\quad A_1, \dots, A_k \in Gl(n, \mathbf{R}).
$$

Then this structure has as its subordinated structure the almost tangent one.

*Remark 2.* It is easily understood that the only manifolds that can admit a structure of the type $J^k = 0$ are those of dimension $k \cdot n$. Then, the condition given in lemma 4.3 is not restrictive at all.

**Proposition 4.4.** *$S^m$ does not admit almost tangent structures of odd order.*

*Proof*: In fact, $S^{2n}$ is connected and oriented. We have $\chi(S^{2n}) = 2$, and $H^n(S^{2n}) = 0$. In the case where $m$ is odd, $S^m$ does not admit almost tangent structures of first order, by reasons of dimension.

It can clearly be seen that it does not admit any structures of odd order, because the equation $J^{2r} = (J^r)^2$, together with lemma 4.3 denotes that the existence of a structure of the type $J^{2r} = 0$, would cocarry one of the first order, which is not true.

**Proposition 4.5.** *The sphere $S^{kn}$, $kn$ even, does not admit almost tangent structures.*

*Proof*: Keeping in mind *remark* 2, it is sufficient to prove that $S^{kn}$ does not admit structures of the type $J^k = 0$ (order $k-1$).

$S^{kn}$ is paracompact and therefore if it would admit a structure $J^k = 0$, its tangent bundle could be split as Whitney sum of two bundles with dimensions $n$ and $(k-1)n$ respectively. ($TS^{kn} = T_n \oplus T_{(k-1)n}$: $T_n$ being the first of distributions in the chain, according to lemma 4.3).

On the other hand $\chi(S^{kn}) = 2$, and by using the property of Euler class with respect to Whitney sum, we have:

$$X(TS^{kn}) = X(T_n \oplus T_{(k-1)n}) = XT_n \cup XT_{(k-1)n},$$

where $\cup$ denotes the cup product.

By the consideration made in 4.2 it results:

$$X(TS^{kn}) = \chi \cdot \mu = 2 \neq 0.$$

However $XT_n$ and $XT_{(k-1)n}$ are null because they are elements of $H^n(S^{kn})$ and $H^{(k-1)n}(S^{kn})$ respectively. This fact contradicts the consideration.

**Proposition 4.6.** *$S^n$ does not admit almost tangent structures if $n$ is prime, different of 3 and 7.*

*Proof*: In fact, it could only admit almost tangent structures of type $J^n = 0$; and this would be equivalent to the existence of $n$-distributions of tangent 1-planes; that is: $S^n$ would be parallelizable; but it is known that this is true only for $S^1$ (where we cannot speak about $J^1 = 0$), and for $S^3$ and $S^7$. The existence of almost tangent structures on these spheres has been proved in a constructive manner.

**Lemma 4.7.** (Steenrod [ 7 ]). *On $S^n$, with $n = 2^m(2s+1) - 1$; $m, s > 0$ a distribution of tangent $q$-planes for $q$ verifying $2^m \leqslant q \leqslant n - 2^m$, does not exist.*

**Proposition 4.8.** *The sphere $S^n$, with $n = 2^m(2s+1) - 1$, does not admit almost tangent structures of any order.*

*Proof*: Suppose $S^n$ admits a structure of the type $J^k = 0$. By lemma 4.3 there exists a chain, the elements of which are distributions of $\lambda n/k$-planes: $\lambda = 1, \ldots, k-1$. (Note that $n/k$ is an integer).

We try to prove the existence of an integer $\lambda$, such that $q = \lambda \cdot n/k$ is in the conditions of the last lemma. We will thus prove the non existence of such a chain of distributions and, therefore, the impossibility of the existence of a structure verifying $J^k = 0$ on $S^n$. In fact,

$$2^m \leqslant \lambda \cdot \frac{n}{k} \leqslant 2^m(2s+1) - 1 - 2^m = 2^m \cdot 2s - 1;$$

$$\text{that is: } \frac{k \cdot 2^m}{2^m(2s+1)-1} \leqslant \lambda \leqslant \frac{2^m \cdot 2s \cdot k - k}{2^m(2s+1)-1}$$

Since $\lambda$ must be integer, its integer least bound is

$$\left[\frac{k \cdot 2^m}{2^m(2s+1)-1} + 1\right]; \text{ where } „[\;\;]" \text{ denotes the „integer part".}$$



The proposition will be proved if we can show that $\dfrac{k \cdot 2^m}{2^m(2s+1)-1} + 1 \leqslant \dfrac{2^m \cdot 2s \cdot k - k}{2^m(2s+1)-1}$, or equivalently:

$$((^*)) \qquad 2^m \cdot 2s(k-1) \geqslant k \cdot 2^m + 2^m - 1 + k.$$

By using the first proposition in a partial form, we can suppose $k \geqslant 3$; thus the expresion $((^*))$ is verified, except for $s = 1$, $k = 3$: which is eliminated because $2^m \cdot 3 - 1$ is not a multiple of 3.

Therefore at least $q = n/k$ is in the conditions of lemma 4.7 for $\lambda = [\lambda 2^m/(2^m(2s+1)-1) + 1]$.

Finally we verify that $1 < \lambda < k - 1$, as we desired. So, it is sufficient to verify that

$$\frac{k \cdot 2^m}{2^m(2s+1)-1} + 1 < k.$$

But this is equivalent to $2^m \cdot 2s(k-1) > 2^m - 1 + k$ which is true for $k \geqslant 3$.

With this previous study we have proved the following:

**Theorem 4.9.** *The sphere $S^n$, except for $n = 3$ or $7$, or $n$ non-prime with $n = 2^m - 1$ does not admit almost tangent structures of order $k - 1$ (Type $J^k = 0$)*

So, we must consider the case $n = 2^m - 1$, with $n$ non prime.

*Remark 3.* If $S^{2m-1}$, admits a structure of the type $J^k = 0$, it will exist a chain of distributions of tangent planes with respective dimensions, $\dfrac{2^m-1}{k}$, $2 \cdot \dfrac{2^m-1}{k}, \ldots, i \cdot \dfrac{2^m-1}{k}, \ldots, 2^m - 1$.

**Lemma 4.10** (Steenrod). *If $2p \leqslant n$, then $S^n$ admits a distribution of tangent $p$-planes, if and only if it admits $p$ independent vector fields.*

*Remark 4.* In accordance with the last lemma, there are $i(2^m-1)/k$ vector fields, when

$$2 \cdot i \cdot \frac{2^m-1}{k} \leqslant 2^m - 1; \text{ that is: } 2i \leqslant k.$$

**Theorem 4.11.** (Hurwitz-Radon-Adams). *Let $n = (2a(n)+1)\,2\exp b(n)$ and $b(n) = c(n) + 4d(n)$; $0 \leqslant c(n) \leqslant 3$. Then the maximum number of linearly independent vector fields on the sphere $S^{n-1}$ is $M(n-1)$; where: $M(n-1) = \rho(n) - 1$, and $\rho(n) = 2^{c(n)} + 8d(n)$.*

*Remark 5.* In our case, $n - 1 = 2^m - 1$; then $n = 2^m$.

So, $2^m = 2^b(2a+1)$. We obtain $b = m$, $a = 0$.

In this way: $m = c + 4d$; $(0 \leqslant c \leqslant 3$, $d = [m/4]$.

Then, we have: $M(n-1) = 2^c + 8[m/4] - 1$.

*Remark 6.* When $i(2^m-1)/k > M(n-1)$, for some $i$, such that $2i \leqslant k$, it results a contradiction between the remarks 4 and 5. (That is: the existence of a structure of type $J^k = 0$ implies the one of certain number of vector fields. If the number of these vector fields is greater than the one carried by $S^n$, then such a structure does not exist.

The inequality is :

$$2^c + 8\left[\frac{m}{4}\right] - 1 < i \cdot \frac{2^m - 1}{k}.$$

In particular, the most convenient case is $i = [k/2]$ ; which is (by reasons of dimension) $i = (k-1)/2$.

The last expression can be reduced to :

$$2^c + 8\left[\frac{m}{4}\right] - 1 < \frac{k-1}{k} \cdot \frac{2^m - 1}{2}.$$

By using the argument made in 4.6, and again considering reasons of dimension, we can suppose $k \geqslant 3$. So, it is verified $2/3 \leqslant (k-1)/k$. Then, we only must prove :

$$\frac{2 \cdot 2^c + 16\left[\frac{m}{4}\right] - 2}{2^m - 1} < \frac{2}{3}.$$

In fact, keeping in mind $[m/4] = (m-c)/4$, the various values of $c$, and several simplifications ; the last expression is equivalent to :
$6(m-c) + h < 2^m - 1$ ; where $h = 0, 3, 9, 9$, when $c$ is respectively $c = 0, 1, 2, 3$. The last expression is true for $m \geqslant 5$.

We conclude with the following

**Theorem 4.12.** a) *The sphere $S^n$ does not admit almost tangent structures of any order, except $n = 3, 7$ or $15$.*

b) *$S^3$ admits an almost tangent structure of order two ($J^3 = 0$).*

c) *admits an almost tangent structure of order six ($J^7 = 0$).*

$S^{15}$ could only admit structures of the type $J^3 = 0$, $J^5 = 0$.

BIBLIOGRAPHY


1. Clark, R. S. and Bruckheimer, M. R. — *Sur les structures presque-tangentes.* C.R. Acad. Sci. Paris 251 (1960) 327—349.
2. Clark, R. S. and Goel, D. — *Almost tangent manifolds of second order.* Toh. Math. J. 2 (1972) 79—92.
3. Dehcuvels, R. — *Clases caracteristicas.* Publ. Dpto. Geom. and Top. Univ. Santiago de Compostela n° 10 (1970).
4. Eliopoulos, H. — *Structures r-tangentes sur les variétés différentiables.* C. R. Acad. Sci. Paris Ser A—B 263 (1966), A 413—416.
5. Gadea, P. M. and Rosendo J. L. — *On almost tangent structures of order k.* An. şt. Univ. Iaşi 22 (1976) 213—220.
6. Schwartz, J. I. — *Differential Geometry and Topology.* Gordon and Breach (1968).
7. Steenrod, N. E. — *The topology of fibre bundles,* Princeton Univ. Press.
8. Steenrod, N. E. — and Epstein, D. — *Cohomology operations.* Princeton Univ. Press.
9. Yanamoto, H. — *On 7-dimensional riemannian manifold with vector crossproduct.* Nagaoka Tech. Coll. 3 (1972) 123—134.


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# ALMOST QUATERNAL STRUCTURES

BY

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1. **Introduction.** The *almost quaternal* structures are $G$-structures for which the structural group is $GL(m, \mathbf{H}) \cdot Sp(1)$ where $GL(m, \mathbf{H})$ is the linear group of the $m \times m$ nondegenerate matrices with entries quaterni-ons and $Sp(1) \cong SO(3)$ is the (multiplicative) group of unit quaternions. If the structural group is only $GL(m, \mathbf{H})$, we obtain the almost quater-nion structures. The almost quaternion and almost quaternal structures were intensively studied in the last years, especially in the case where there exist Kaehlerian metrics for them [1, 3, 4], i.e. the structural group is further reduced to $Sp(m)$ or to $Sp(m) \cdot Sp(1)$ and it is satisfied a sup-plementary condition. In this paper we intend to begin a study of the al-most quaternal structures from the point of view of the general theory of $G$-structures. The first problem touched is that of integrability. Looking for the almost quaternal connections, we obtain the obstructions to inte-grability given by the structure tensors of this structure. Let us recall that the first structure tensor is the Chern-Bernard invariant.

It is known that the integrable almost quaternion structures are affi-nely flat, i.e. the manifold which carries an integrable almost quaternion structure has an atlas defining its structure of differentiable manifold by affine coordinate transformations.

The typical example of integrable almost quaternal structure is that of the quaternion projective spaces.

2. **Definitions and algebra.** Let $\mathbf{H}^m$ be the $m$-dimensional right mo-dule over the quaternions $\mathbf{H}$. Its elements are columns of $m$ quaternions. If we think $\mathbf{H}^m$ as a vector space over the reals $\mathbf{R} \subset \mathbf{H}$ then it can be re-presented as a $4m$-dimensional real vector space $\mathbf{R}^{4m}$ as follows. Let ${}^tq = (q^1, \ldots, q^m) \in \mathbf{H}^m$ where $q^h = u^h + iv^h + jx^h + ky^h$ ; $h = 1, \ldots, m$ ; $i, j, k$ — the standard units of $\mathbf{H}$. If ${}^tU = (u^1, \ldots, u^m)$, ${}^tV = (v^1, \ldots, v^m)$, ${}^tX = (x^1, \ldots, x^m)$, ${}^tY = (y^1, \ldots, y^m)$, where $t$ denotes the transposition, then ${}^tQ = {}^tU + i\,{}^tV + j\,{}^tX + k\,{}^tY$ and the corresponding vector of $\mathbf{R}^{4m}$ is $(u^1, \ldots, u^m, v^1, \ldots, v^m, x^1, \ldots, x^m, y^1, \ldots, y^m) = ({}^tU, {}^tV, {}^tX, {}^tY)$, considered as a column. The linear group $GL(m, \mathbf{H})$ acts on $\mathbf{H}^m$ from the left by matrix multiplication while $\mathbf{H}$ acts on $\mathbf{H}^m$ from the right by the usual vector space multiplication. The resulting action of $GL(m, \mathbf{H})$ and $\mathbf{H}$ on $\mathbf{H}^m$ gives a