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SUMAR

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ON SOME PROPERTIES OF GENERALIZED SEMI IDEALS OF A SEMIGROUP

BY

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A nonempty subset A of a semigroup S is called generalized left semi (abbreviation g.l.s.) ideal iff $x^2A \subset A$ for every $x \in S$.

In [2] the first author has shown that every left ideal [$SA \subset A$] is always a g.l.s. ideal but the converse is not true [2]. Also with the help of this notion, in [1], [2] the first author has characterized regularity of a semigroups and formulated a number of criteria for determining when a semigroup is a group or when a semigroup is a semilattice of groups.

The purpose of this paper is to study some new properties of g.l.s. ideal, to define generalized simple semigroup and to obtain criteria when a commutative semigroup will be generalized simple and when a group will be generalized simple group.

1. Let A be a nonempty subset of a commutative semigroup S. Then the collection of all g.s. ideals of S containing A is nonempty since S itself is a g.s. ideal of S containing A. Now the intersection of all g.s. ideals of S containing A is also a g.s. ideal of S containing A. It is called the g.s. ideal generated by A.

Proposition 1. If A is a nonempty subset of a commutative semigroup S then $\bar{S}A \cup A$ is the generalized semi ideal of S generated by A, where $\bar{S} = \{x^2 \mid x \in S\}$.

Proof. Obviously $\bar{S}A \cup A$ is a generalized semi ideal of S. Also let R be any other generalized semi ideal of S containing A. Since $A \subset R$ so $\bar{S}A \subset R$. Thus $\bar{S}A \cup A \subset R$. So $\bar{S}A \cup A$ is the smallest generalized semi ideal of S containing A. Hence the proposition.

In Example 3 [2] we found that a group may contain generalized semi ideal. Regarding groups we can prove the following proposition.

Proposition 2. Let T be the collection of all principal generalized semi ideals of a commutative group G; then T is a partition of G.

Proof. Let
$$T = \{\bar{G}a \mid a \in G\}.$$
Obviously any element x of G belongs to $\bar{G}x$ i.e. to some member of T. Also any two members of T are disjoint. On the contrary, if possible let $\bar{G}a \cap \bar{G}b \neq \varnothing$ Let $x \in \bar{G}a \cap \bar{G}b$. Then there exist g_1, g_2 in G such that $x = g_1^2 a = g_2^2 b$. So

$$a = g_1^{-2} g_2^2 b = (g_1^{-1} g_2)^2 b \in \bar{G}b.$$

Thus, $\bar{G}a \subseteq \bar{G}b$. Similarly $\bar{G}b \subseteq \bar{G}a$. Hence $\bar{G}a = \bar{G}b$. Thus any two members of T are either equal or disjoint. Hence T forms a partition of G. Let A, B be two subsets of a semigroup S such that A is a generalized semi ideal of B and B is a generalized semi ideal of S. Then A may not be a generalized semi ideal of S. We now prove the following.

Proposition 3. Let A, B be two subsets of a semigroup S such that $A^2 = A$ and A is a left ideal of B and B is a generalized left semi ideal of S. Then A is a generalized left semi ideal of S.

Proof. Let $a \in A$. Now since $A^2 = A$ so $a = a_1 a_2$ where a_1, $a_2 \in A$. Let $x \in S$. Then $x^2 a_1 \in B$ since $a_1 \in A \subset B$ and B is a g.l.s. ideal of S. Again since $a_2 \in A$ and A is a left ideal of B so:

$$x^2 a = x^2 (a_1 a_2) = (x^2 a_1)\, a_2 \in A.$$

Thus A is a g.l.s. ideal of S.

Let A, B be two subsets of a group G such that A is a g.l.s. ideal of B and B is a g.l.s. ideal of G, then in general A may not be a g.l.s. ideal of G. This fact follows from the following example.

Example 1. Let G be the group of all integers w.r.t. addition. Then the subset B of all even integers forms a generalized semi ideal of G. Now the subset $A = \{4K \mid K \text{ is an integer}\}$ of B forms a generalized semi ideal of B. But A is not a generalized semi ideal of G, since $4 \in A$ and $1 \in G$ but $(1+1)+4 = 6 \notin A$.

Proposition 4. *If A is a g.l.s. ideal of B and B is a g.l.s. ideal of the group G then A is g.l.s. ideal of G iff $B - A$ (the complement of A in B) is a g.l.s. ideal of G.*

Proof. It is given that A is a g.l.s. ideal of B and B is a g.l.s. ideal of G. Assume that $B - A$ is a g.l.s. ideal of G. It is known [1], that in a group, the difference of two g.l.s. ideals is also a g.l.s. ideal. Now $A = B - (B - A)$. Hence, from the above result, we find that A is a g.l.s. ideal of G. The converse follows from the above known result.

2. A semigroup S will be called a generalized left simple iff S is the only generalized left semi ideal of S. A generalized right simple semigroup is defined analogously.

A semigroup which is both generalized right simple and generalized left simple is called simply generalized simple.

Example 2. The semigroup $S = \{1, w, w^2\}$ where $w^3 = 1$ does not contain any generalized semi-ideal other than S itself. So S is a generalized simple semigroup.

Example 3. Let R be the set of all nonzero rational numbers. Then R is a multiplicative group. Since R is a group it is simple but R is not generalized simple as $\bar{R}$ is a g.s. ideal of R.

In example 1 we find that S is both simple and generalized simple semigroup but in the second example R is simple but not generalized simple semigroup.

Proposition 5. *A commutative semigroup S will be generalized simple iff for every a in S we have $\bar{S}a = S$.*

Proof. Let us first suppose that S is generalized simple and $a \in S$. Let $x^2 a \in \bar{S}a$. Then for any $y \in S$, $y^2 x^2 a = (yx)^2 a$ (Since S is commutative) $\in \bar{S}a$. So $\bar{S}a$ is a generalized semi-ideal of S. Hence $\bar{S}a = S$ since S is generalized simple.

Conversely let $\bar{S}a = S$ for every a in S. Let A be any generalized semi-ideal of S and $a \in A$. Then $S = \bar{S}a \subseteq A \subseteq S$. So $S = A$. Thus S contains no proper generalized semi-ideal. Hence the proposition.

Proposition 6. *A semigroup will be a group if it is generalized simple.*

Proof. Let S be a generalized simple semigroup. So S is also simple. But it is known ([3], p. 142) that every simple semigroup is a group. Hence S is a group.

The converse of the above proposition does not hold in general. i.e. a group may not be generalized simple, and this follows from the following example.

Example 4. Let R be the set of all non-zero rational numbers. Then R is a multiplicative group. This group R is not generalized simple, since the set $S = \{2r^2 \mid r \in R\}$ is a proper generalized semi-ideal of R.

Proposition 7. *A commutative group S will be generalized simple iff $\bar{S} = S$.*

Proof. Let S be a generalized simple group. Obviously $\bar{S}$ is a generalized semi ideal of S. So $\bar{S} = S$. Next let $\bar{S} = S$. Let R be any proper generalized semi-ideal of S. Then either $1 \in R$ or $1 \in S - R$ where 1 is the identity element of $\bar{S}$. So either $\bar{S} \subset R$ or $\bar{S} \subset S - R$ since R and $S - R$ are both generalized semi ideals of S. Then either $S \subset R$ or $S \subset S - R$ both of which are impossible since R is a proper subset of S. Thus S contains no proper generalized semi-ideal. Hence the proposition.

Proposition 8. A finite cyclic group S will be generalized simple iff its order is an odd integer.

Proof. Let S be a cyclic group whose order is an odd integer n. We shall show $S = \bar{S}$.

Let $S = \{1, a, \ldots, a^{n-1}\}$, $a^n = 1$ is the identity element of S. Suppose $a^i \neq a^j$ $(i \neq j)$ $1 \leqslant i, j \leqslant n-1$, we shall show $a^{2i} \neq a^{2j}$. If $2i, 2j < n$ then obviously $a^{2i} \neq a^{2j}$ since $i \neq j$. If $2i < n$ but $2j > n$. Then $2i \neq 2j - n$ since $2i$ is even but $2j - n$ is odd. So $a^{2i} \neq a^{2j-n}$. If both $2i, 2j > n$. Then $2i - n \neq 2j - n$ since $i \neq j$, $a^{2i-n} \neq a^{2j-n}$. Thus $\bar{S}$ contains n distinct elements of S and S also contains n elements. So $\bar{S} = S$. Hence by proposition 3, S is generalized simple group.

Conversely, we assume that $S = \langle a \rangle$ is a finite cyclic group which is also generalized simple. So by proposition 3, we have $\bar{S} = S$. If possible, let the order of S be an even integer $2r$. Then $a^{2r} = 1$. Now $a^r \in S$ and $a^{2r} = 1$. Also $1^2 = 1$. So $a^{2r} = 1^2$. So the number of elements in $\bar{S}$ is less than or equal to $2r - 1$.

Hence $\bar{S} \subset S$, which contradicts $\bar{S} = S$. So the order of S is an odd integer.

From the above proposition we find that a cyclic group of even order is not generalized simple. Regarding the cyclic group of even order we have the following proposition.

Proposition 9. *Every cyclic group of even order has two generalized semi ideals.*

Proof. Let S be a cyclic group of even order n generated by g. Then $S = \{1, g, \ldots, g^{n-1}\}$. Now $S_1 = \{1, g^2, g^4, \ldots, g^{n-2}\}$ and $S_2 = \{g, g^3, \ldots, g^{n-1}\}$ are two generalized semi-ideals of S. Thus S contains two generalized semi-ideals. Next we shall show that these are the only generalized semi-ideals of S. If possible let A be any other generalized semi-ideal of S. Then either $1 \in A$ or $1 \notin A$. If $1 \in A$, then A contains all the elements of S_1, and no element of S_2. Because if any one of the elements of S_2 belongs to A then A coincides with S which contradicts that A is a proper generalized semi ideal of S. So if $1 \in A$ then $A = S_1$. If $1 \in A$ then A does contain any element of S_1 and contains all the elements of S_2. So $A = S_2$, Hence the proposition.

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REFERENCES

1. S e n, M. K. — *On Generalized semiideals of semigroups.* Mathematica Balkanica 3 (1973), pp. 453—457.
2. S e n, M. K. — *On the Generalized semiideals of a semigroup.* Glasnik Matematicki, Vol. 10 (30) 1975, pp. 43—46.
3. L j a p i n E. S. — *Semigroups*, Vol. 3. Translation of Math. Monographs A.M.S. 1963.

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ON WARD QUASIGROUPS

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The purpose of the present note is to solve a problem suggested in [2] which is concerned with the structures of quasigroups and groups in a same set.

Definition 1. *A subtractive groupoid is a groupoid $(G, 0)$ which satisfies the axioms:*

$$(S\,1)\quad a \circ (b \circ c) = c \circ (b \circ a), \text{ for every } a, b, c, \text{ in } G,$$

$$(S\,2)\quad a \circ (b \circ b) = a, \text{ for every } a, b, \text{ in } G.$$

Examples. The set Z of integers with $x \circ y = x - y$. The set Q^* of rationals different of zero with $x \circ y = x/y$.

Theorem 1. *If x and y are elements of a subtractive groupoid $(G, \circ)$ then*

$$x \circ x = y \circ y.$$

Proof. $x \circ x = x \circ (x \circ (y \circ y)) = (y \circ y) \circ (x \circ x) = y \circ y.$

Theorem 2. *If $(G, \circ)$ is a subtractive groupoid, the element $i = x \circ x$, for every $x \in G$, is a right identity in G.*

Proof. By $(S\,2)$, $a \circ (x \circ x) = a$.

Theorem 3. *If a, b, c, d are elements of the subtractive groupoid $(G, \circ)$, then*

$$(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d).$$

Proof.

$$(a \circ b) \circ (c \circ d) = d \circ (c \circ (a \circ b)) = d \circ (b \circ (a \circ c)) =$$
$$= (a \circ c) \circ (b \circ d).$$

Theorem 4. *If i is the right identity of the subtractive groupoid $(G, \circ)$ then*

(i) $i \circ (a \circ b) = b \circ a$, *for every a, b in G,*
(ii) $(a \circ b) \circ c = (a \circ c) \circ b$, *for every a, b, c in G,*
(iii) $(a \circ b) \circ c = a \circ (c \circ (i \circ b))$, *for every a, b, c in G.*

Proof.

(i) $i \circ (a \circ b) = b \circ (a \circ i) = b \circ a,$
(ii) $(a \circ b) \circ c = i \circ (c \circ (a \circ b)) = i \circ (b \circ (a \circ c)) = (a \circ c) \circ b,$
(iii) $a \circ (c \circ (i \circ b)) = a \circ (b \circ (i \circ c)) = (i \circ c) \circ (b \circ a) =$
$= (i \circ (b \circ a)) \circ c = (a \circ b) \circ c.$

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