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HOLOMORPHIC SOLUTIONS OF LINEAR TOTAL DIFFERENTIAL SYSTEMS AT A SINGULAR POINT

BY

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§ 0. **Introduction.** The problem of holomorphic solutions of total differential equations at a singular point was first considered in two papers of T o k u i S a t o [16], [17], where this problem was investigated in the case of a scalar total differential equation of the form $udz = P(u, v, z)\,du + uQ(u, v, z)\,dv$, where P and Q are holomorphic at $u = v = z = 0$. But, to our knowledge the linear case, which is so rich in results for ordinary equations, was not yet investigated for total differential systems.

The purpose of this paper is to establish total differential analogues for some known results of the theory of linear ordinary systems. (For informations concerning ordinary systems we send to P. H a r t m a n [12], Chapter IV and to M. H u k u h a r a, T. K i m u r a and T. M a t u d a [13]).

The statements and proofs are closely related to those of the papers of H a r r i s, S i b u y a and W e i n b e r g ([11], see also [10]) and J. K. H a l e [9], with the modifications required by the treatment of total differential systems. In other words, the method of bifurcation (determining) equations is applied to prove the existence of holomorphic solutions at a singular point. (This method, which has proved to be successfull in the study of a lot of boundary value problems for ordinary and partial differential systems is usually called the C e s a r i - H a l e method. The reader interested to get a deeper knowledge of this method, as well as further references, should go to [2], [3], [7] and especially to [4], [5], [8], [9], where some variants of this method are given).

The analysis of the (bifurcation) determining equations is the main point in establishing further properties of the solutions (Theorems 2, 3, 4, 5 below). Let us remark that Theorem 2 is a total differential analogue to the well-known *Lettenmeyer-Perron* theorem [14].

Let us also mention that we have obtained similar results in the nonlinear case, They are total differential analogues of some results of R. W. B a s s [1] and W. A. H a r r i s Jr. [10] and, at the same time, they generalize some of T. Sato's results of [16], [17]. These results will appear in a separate paper. Let us finally mention that singular points of total differential systems were already studied in some papers, but from a quite different point of vue and with other intentions (see, for example, B. P i n i [15]).

Preliminaries. Let $A(u, v) = (a^{is}(u, v))$ be an n by n matrix-valued function and $h(u, v) = (h^1(u, v), \ldots, h^n(u, v))^T$ an n by 1 vector-valued function, holomorphic in the real (complex) variables u and v in the interval („circle") of convergence $|u| < \rho_1$, $|v| < \rho_2$. Denote

$$h^i(u, v) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} h^i_{jk} u^j v^k, \tag{0.1}$$

$$h(u, v) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} h_{jk} u^j v^k, \quad h_{jk} = (h^1_{jk}, \ldots, h^n_{jk})^T, \tag{0.1 bis}$$

$$a^{is}(u, v) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a^{is}_{jk} u^j v^k, \tag{0.2}$$

$$A(u, v) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A_{jk} u^j v^k, \quad A_{jk} = (a^{is}_{jk})_{i, s=1, \ldots, n}. \tag{0.2 bis}$$

Then, the above series converge absolutely (and uniformly) in

$$|u| \leqslant \delta, \quad |v| \leqslant \varepsilon, \tag{P}$$

for every fixed δ, ε, $0 < \delta < \rho_1$, $0 < \varepsilon < \rho_2$. Denote by $S_{\delta, \varepsilon}$ the Banach space made up by the matrix-valued functions (0.2 bis) whose power series expansions are absolutely convergent in (P) and whose norm is defined by

$$\| A \|_{\delta, \varepsilon} = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty}\sum_{i=1}^{n}\sum_{s=1}^{n} |a^{is}_{jk}| \delta^j \varepsilon^k = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} |A_{jk}| \delta^j \varepsilon^k. \tag{0.3}$$

(No confusion can arise by designating both the vector-valued and the matrix-valued functional space with the same symbol. In order to avoid complication in notations, we shall drop the subscripts δ, ε to the symbol norm whenever no confusion is possible).

Lemma 0.1. *For every n by n matrix $A \in S_{\delta, \varepsilon}$ and every n by 1 vector $h \in S_{\delta, \varepsilon}$ we have*

$$\| Ah \| \leqslant \| A \| . \| h \| . \tag{0.5}$$

Proof: Denoting by $[E]^{\alpha\beta\gamma}_{jk}$ the coefficient of $u^j v^k$ in the expansion of the expression $E^{\alpha\beta\gamma}\ldots$ (matrix, vector, scalar, no matter what), we have obviously

$$[Ah]^i_{jk}\,\delta^j \varepsilon^k = \sum_{p=1}^{n} \sum_{q+s=j} \sum_{r+t=k} [a]^{ip}_{qr} \delta^q \varepsilon^r . [h]^p_{st} \delta^s \varepsilon^t,$$

and the proof becomes trivial.

Corollary 0.2. *For every n by n matrix $A \in S_{\delta, \varepsilon}$ and every n by 1 vectors $h, k \in S_{\delta, \varepsilon}$, we have*

$$\| A(h - k) \| \leqslant \| A \| . \| h - k \|. \tag{0.6}$$

Consider now, for d_i $(i = 1, \ldots, n)$ given non negative integers,

$\sum_{i=1}^{n} d_i = d \leqslant n$, for any n by n matrix $A(u, v) \in S_{\delta, \varepsilon}$ (respectively for any n by 1 vector $h(u, v) \in S_{\delta, \varepsilon}$) and for a positive integer N, $(N > n)$ the operators $\mathfrak{A}_N : S_{\delta, \varepsilon} \to S_{\delta, 0} \subset S_{\delta, \varepsilon}$, $\mathfrak{L} : S_{\delta, \varepsilon} \to S_{\delta, \varepsilon}$ defined by

$$(\mathfrak{A}_N(A^{is}(u) \overset{\text{def}}{=} \sum_{j=N}^{\infty} a^{is}_{j0} \frac{u^{j-d_i+1}}{j - d_i + 1}; \qquad (i, s = 1, \ldots, n),$$

$$(\mathfrak{A}_N h)^i(u) \overset{\text{def}}{=} \sum_{j=N}^{\infty} h^i_{j0} \frac{u^{j-d_i+1}}{j - d_i + 1}, \qquad (i = 1, \ldots, n),$$

$$(\mathfrak{L}A)^{is}(u, v) \overset{\text{def}}{=} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} a^{is}_{jk} u^j \frac{v^{k+1}}{k+1}, \qquad (i, s = 1, \ldots, n),$$

$$(\mathfrak{L}h)^i(u, v) \overset{\text{def}}{=} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} h^i_{jk} u^j \frac{v^{k+1}}{k+1}, \qquad (i = 1, \ldots, n)$$

Then, by (0.1) — (0.5) we have

$$\| \mathfrak{A}_N h \|_{\delta\varepsilon,} = \sum_{j=N}^{\infty}\sum_{i=1}^{\infty} |h^i_{j0}| \frac{\delta^{j-d_i+1}}{j - d_i + 1} \leqslant \left(\sum_{i=1}^{\infty} \frac{\delta^{1-d_i}}{N - d_i + 1} \right) \|h\|_{\delta, 0} = \tag{0.7}$$
$$= \alpha_N \|h\|_{\delta, 0} \leqslant \alpha_N \|h\|_{\delta, \varepsilon} .$$

$$\| \mathfrak{L}h \|_{\delta, \varepsilon} \leqslant \varepsilon \| h \|_{\delta, \varepsilon} \tag{0.8}$$

with

$$\alpha_N = \sum_{i=1}^{n} \frac{\delta^{1-d_i}}{N - d_i + 1} \tag{0.9}$$

no matter how small for N sufficiently large. Now, by (0.7), (0.8), (0.5) we get

$$\| \mathfrak{A}_N(Ah) \| \leqslant \alpha_N \| Ah \| \leqslant \alpha_N \| A \| . \| h \| = \alpha'_N \| h \|, \tag{0.10}$$

$$\| \mathfrak{L}(Bh) \| \leqslant \varepsilon \| Bh \| \leqslant \varepsilon \| B \| . \| h \| = \varepsilon' \| h \|, \tag{0.11}$$

where $\alpha'_N = \alpha_N \| A \|$, $\varepsilon' = \varepsilon \| B \|$ are no matter how small for N sufficiently large and ε sufficiently small. Furthermore, we have from (0.6), (0.7) and (0.8).

$$\| \mathfrak{A}_N A(h - k) \| \leqslant \alpha'_N \| h - k \|, \quad \| \mathfrak{L}B(h - k) \| \leqslant \varepsilon' \| h - k \|. \tag{0.12}$$

Proposition 0.3. *Suppose $A(u, v)$, $B(u, v) \in S_{\delta, \varepsilon}$ are given matrix-valued functions*

$$A(u, v) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} A_{jk} u^j v^k, \quad B(u, v) = \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} B_{jk} u^j v^k ,$$

where N is a fixed, sufficiently large, positive integer, and ε is sufficiently small to have

$$\alpha'_N + \varepsilon' = \alpha_N \| A \| + \varepsilon \|B\| < 1. \tag{0.13}$$

Then for every fixed polynomial $\varphi(u) = (\varphi^1, \dots, \varphi^n)^T$,

$$\varphi^i(u) = \sum_{j=0}^{N-d_i} \varphi^i_{j0} u^j, \tag{0.14}$$

there exists a unique vector-valued solution $z_\varphi \in S_{\delta,\varepsilon}$ *of the equation*

$$z = \varphi + \mathfrak{A}_N(Az) + \mathfrak{L}(Bz). \tag{0.15}$$

Proof. Consider the operator $T: S_{\delta,\varepsilon} \to S_{\delta,\varepsilon}$ defined by

$$Tz = \varphi + \mathfrak{A}_N(Az) + \mathfrak{L}(Bz). \tag{0.16}$$

For any $\hat{z}, \bar{z} \in S_{\delta,\varepsilon}$ we have from (0.16), (0.13), (0.12)

$$\|T\hat{z} - T\bar{z}\| \leqslant (\alpha'_N + \varepsilon')\, \|\hat{z} - \bar{z}\|,$$

thus T is a contraction and Proposition 0.3 is proved.

§ 1. Solutions of total differential systems at a singular point. Consider now the total differential system

$$u^{d_i} dz^i = \sum_{s=1}^{n} a^{is}(u, v) z^s du + u^{d_i} \sum_{s=1}^{n} b^{is}(u, v) z^s dv, \tag{1.1}$$

$(i = 1, \dots, n)$ which can be also written as

$$u^D dz = A(u, v) z du + u^D B(u, v) z dv, \tag{1.1 bis}$$

with $D = \operatorname{diag}(d_1, d_2, \dots, d_n)$, $A = (a^{is})_{i,s=1\dots n}$ $z = (z^1, \dots, z^n)^T$, u, v real (complex) independent variables. The system (1.1) or (1.1 bis) can be rewritten in the form

$$u^{d_i} \frac{\partial z^i}{\partial u} = \sum_{s=1}^{n} a^{is}(u, v) z^s, \quad \frac{\partial z^i}{\partial v} = \sum_{s=1}^{n} b^{is}(u, v) z^s, \quad (i = 1, \dots, n) \text{ or} \tag{1.1.1}$$

$$u^D \frac{\partial z}{\partial u} = A(u, v) z, \quad \frac{\partial z}{\partial v} = B(u, v) z. \tag{1.1.1 bis}$$

Hypotheses H : *Suppose the matrix-valued functions*

$$A(u, v) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} A_{jk} u^j v^k, \qquad B(u, v) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} B_{jk} u^j v^k,$$

with $A_{jk} = (a^{is}_{jk})$, $B_{jk} = (b^{is}_{jk})$ *have absolutely convergent expansions in*

$$|u| \leqslant \delta, \quad |v| \leqslant \varepsilon. \tag{P}$$

(In other words, $A, B \in S_{\delta,\varepsilon}$*) and suppose* ε *is sufficiently small to have (0.13). Furthermore, the condition of complete integrability of (1.1), that is*

$$\frac{\partial a^{ir}}{\partial v} + \sum_{s=1}^{n} a^{is} b^{sr} = u^{d_i} \frac{\partial b^{ir}}{\partial u} + u^{d_i} \sum_{s=1}^{n} b^{is} u^{-d_s} a^{sr}, \tag{1.2}$$

holds for every $(u, v) \in P$.

Commentary : Inasmuch as the first three terms in (1.2) are holomorphic, so should be the last one. If $d_s > 0$, it is sufficient, for example, that $u^{-d_s} b^{is}$ be holomorphic functions for $i \neq s$.

Theorem 1. *Under hypotheses H, the necessary and sufficient condition for the (unique) solution of (0.15) to be a solution of system (1.1) is that the polynomial* $\varphi(u)$ *satisfies the bifurcation (determining) equations (1.8) below.*

Proof. Let $z(u, v)$ be the (unique) solution of (0.15). Then

$$z_\varphi(u, v) = \varphi(u) + \mathfrak{A}_N(Az_\varphi) + \mathfrak{L}(Bz_\varphi), \tag{1.3}$$

that is

$$z^i_\varphi(u, v) = \varphi^i(u) + \sum_{j=N}^{\infty} [Az_\varphi]^i_{j0} \frac{u^{j-d_i+1}}{j - d_i + 1} + \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} [Bz_\varphi]^i_{jk} u^j \frac{v^{k+1}}{k+1} \tag{1.3.i.}$$

with

$$\varphi^i(u) = \sum_{j=0}^{N-d_i} \varphi^i_{j0} u^j, \tag{1.4}$$

$(i = 1, \dots, n)$. From (1.3.i) we find

$$\frac{\partial z^i_\varphi(u, v)}{\partial v} = \sum_{s=1}^{n} b^{is}(u, v) z^s_\varphi(u, v), \tag{1.5}$$

that is

$$\frac{\partial z_\varphi(u, v)}{\partial v} = B(u, v) z_\varphi(u, v). \tag{1.6}$$

On the other hand, we obviously obtain from (1.3)

$$u^{d_i} \frac{\partial z^i_\varphi(u, 0)}{\partial u} = \sum_{s=1}^{n} a^{is}(u, 0) z^s_\varphi(u, 0) + u^{d_i} \frac{\partial \varphi^i(u)}{\partial u} - \sum_{j=0}^{N-1} [Az_\varphi]^i_{j0} u^j. \tag{1.7}$$

Thus the necessary and sufficient condition for $z_\varphi(u, v)$ to satisfy the first group of (1.1.1) for $v = 0$ (that is, anyway, *a necessary condition for* z_φ *to satisfy (1.1.1)*) is to have for $\varphi^i(u)$

$$Q^i(u; \varphi; N; A) \stackrel{\text{def}}{=} u^{d_i} \frac{\partial \varphi^i(u)}{\partial u} - \sum_{j=0}^{N-1} [Az_\varphi]^i_{j0} u^j = 0, \tag{1.8}$$

$(i = 1, \dots, n)$. We call (1.8) the *bifurcation* (determining) *equations*. Let us show that *the fulfilment of conditions (1.8) by* $\varphi(u)$ *is also sufficient* for z_φ to be a solution of the first group of equations (1.1.1). Remark first that if (1.8) holds, then (1.7) becomes

$$u^{d_i} \frac{\partial z^i_\varphi(u, 0)}{\partial u} = \sum_{s=1}^{n} a^{is}(u, 0) z^s_\varphi(u, 0), \tag{1.9}$$

$(i = 1, \dots, n)$ or

$$u^D \frac{\partial z_\varphi(u, 0)}{\partial u} = A(u, 0) z_\varphi(u, 0). \tag{1.10}$$

Now, if (1.8) hold, we obtain from (1.6), taking (1.9) into account :

$$u^{d_i}\frac{\partial z^i_\varphi(u,v)}{\partial u}=\sum_{s=1}^n a^{is}(u,0)z^s_\varphi(u,0)+u^{d_i}\int_0^v\sum_{s=1}^n b^{is}(u,v_1)\frac{\partial z^s_\varphi(u,v_1)}{\partial u}dv_1+ \tag{1.11}$$
$$+u^{d_i}\int_0^v\sum_{s=1}^n\frac{\partial b^{is}(u,v_1)}{\partial u}z^s_\varphi(u,v_1)dv_1,$$

$(i=1,\ldots,n)$. Now, replacing in the last integral $u^{d_i}(\partial b^{is}/\partial u)$ by its value given by the complete integrability condition (1.2) we find, after regrouping terms and integrating an obvious total differential :

$$u^{d_i}\int_0^v\sum_{s=1}^n\frac{\partial b^{is}(u,v_1)}{\partial u}z^s_\varphi(u,v_1)dv_1=\sum_{s=1}^n a^{is}(u,v)z^s_\varphi(u,v)- \tag{1.12}$$
$$-\sum_{s=1}^n a^{is}(u,0)z^s_\varphi(u,0)-u^{d_i}\int_0^v\sum_{s=1}^n\sum_{r=1}^n b^{ir}(u,v_1)u^{-d_r}a^{rs}(u,v_1)z^s_\varphi(u,v_1)dv_1,$$

$(i=1,\ldots,n)$. If we introduce now (1.12) into (1.11) and denote

$$E^i_\varphi(u,v)=u^{d_i}\frac{\partial z^i_\varphi(u,v)}{\partial u}-\sum_{s=1}^n a^{is}(u,v)z^s_\varphi(u,v), \tag{1.13}$$

$(i=1,\ldots,n)$ we obtain

$$E^i_\varphi(u,v)=u^{d_i}\int_0^v\sum_{r=1}^n u^{-d_r}b^{ir}(u,v_1)E^r_\varphi(u,v_1)\,dv_1, \tag{1.14}$$

$(i=1,\ldots,n)$. Hence, by virtue of (1.14), $E^i_\varphi(u,v)$ satisfies the ordinary, homogeneous, linear system

$$\frac{\partial E^i_\varphi(u,v)}{\partial u}=u^{d_i}\sum_{r=1}^n u^{-d_r}b^{ir}(u,v)E^r_\varphi(u,v), \tag{1.15}$$

(which is, by hypotheses H a system with holomorphic coefficients), and the initial conditions

$$E^i_\varphi(u,0)=0,\qquad (i=1,\ldots,n). \tag{1.16}$$

Hence from (1.15) and (1.16) we get $E^i_\varphi(u,v)\equiv 0$, thus, by (1.13) Theorem 1 is proved.

Remark 1. The problem of holomorphic at (0,0) solutions of (1.1) reduces (under hypotheses H) by theorem 1 to the problem of solvability of the bifurcation equations (1.8). If $\varphi(u,v)$ can be determined from (1.8), the solution $z_\varphi\in S_{\delta,\varepsilon}$ of (0.15) is a solution of (1.1). Let us only remark that for this solution z_φ of (1.1) we have further $u^D(z_\varphi-\varphi)=0(u^{D+1})$. Remark also that the coefficients of z_φ are linear and homogeneous in the coefficients of φ. Furthermore, by this same, the coefficients of $q(u;\varphi;N;A)=(q^1,\ldots,q^n)^T$ are linear homogeneous in the coefficients of φ also.

Now, we can state and prove a total differential analogue of Lettenmayer's theorem (in other words its generalization for total differential systems), namely :

Theorem 2. *Under hypotheses H, suppose $n-\sum_{i=1}^n d_i=n-d\geqslant 0$. Then the system (1.1) has at least $n-d$ linearly independent solutions, holomorphic at $u=v=0$.*

Proof. The determining equations (1.8) are at most $\sigma=\Sigma N=Nn$ linear, homogeneous equations in the $\rho=\Sigma(N-d_i+1)=Nn+n-d$ unknowns φ^i_{j0} $(i=1,\ldots,n)$. Hence there exist at least $\rho-\sigma=n-d$ independent coefficients φ^i_{j0}, the remaining ones depending linearly on them. The rest of the proof is a corollary to Theorem 1 and Remark 1.

It is now easy to see that equations (1.8) can be written in the form

$$j\varphi^i_{j0}=[Az_\varphi]^i_{j+d_i-1,0}\qquad(i=1,\ldots,n;\ j=1,2,\ldots,N-d_i). \tag{1.8.1}$$

We obtain at the same time from (1.8) the conditions

$$[Az_\varphi]^i_{00}=[Az_\varphi]^i_{10}=\ldots=[Az_\varphi]^i_{d_i-1,0}=0,\qquad(i=1,\ldots,n). \tag{1.17}$$

The case $d_1=\ldots=d_n=1$. Let us mention that the nonlinear case of the case $d_1=\ldots=d_n=1$ was first studied by T. S a t o [16], [17], but only in the case of an equation, that is $n=1$. Take everywhere $d_1=\ldots=d_n=1$. System (1.1) (or its equivalent form (1.1.1)) becomes

$$u\,dz=A(u,v)\,z\,du+uB(u,v)\,z\,dv, \tag{1.18}$$

that is

$$u\frac{\partial z}{\partial u}=A(u,v)z,\qquad\frac{\partial z}{\partial v}=B(u,v)z, \tag{1.18.1}$$

whilst the complete integrability condition of hypotheses H becomes

$$\frac{\partial A}{\partial v}+AB=u\frac{\partial B}{\partial u}+BA,\qquad\text{for any }(u,v)\in P. \tag{1.19}$$

Theorem 3. *Consider the system (1.18), where*

$$A(u,v)=\sum_{j=0}^\infty\sum_{k=0}^\infty A_{jk}u^jv^k,\qquad B(u,v)=\sum_{j=0}^\infty\sum_{k=0}^\infty B_{jk}u^jv^k$$

are matrix-valued functions, whose expansions are convergent in the interval („circle") of convergence (K): $|u|<\rho_1$, $|v|<\rho_2$. Assume (1.19) holds in (K) and suppose

$$z(u,v)=\sum_{j=0}^\infty\sum_{k=0}^\infty z_{jk}u^jv^k$$

is a (vector-valued) formal solution to system (1.8). Then $z(u,v)$ is convergent.

Proof. The determining equations (1.8.1) and (1.17) become for $d_1 = .. = d_n = 1$

$$(1.20)\qquad (jI - A_{00})\,\varphi_{j,0} = \sum_{q=1}^{j} A_{q,0}\,\varphi_{j-q,0},$$

$(j = 1, .., N-1)$, where I is the unit matrix,

$$(1.21)\qquad A_{00}\,\varphi_{00} = 0.$$

Now, (1.20), (1.21) are the first N conditions for $z(u, v)$ to be a formal solution of the first equation (1.18.1), for $v = 0$. Thus the determining equations are satisfied by assumption and thus every formal solution of system (1.18) is absolutely convergent in $|u| \leqslant \delta$, $|v| \leqslant \varepsilon$, for arbitrary δ, ε, $0 < \delta < \rho_1$, $0 < \varepsilon < \rho_2$, provided ε is sufficiently small.

Let us now discuss the existence of solutions of (1.18.1) of the form $z = u^{\lambda} y(u, v)$.

Theorem 4. *Let λ be a fixed complex constant and n_λ $(\geqslant 0)$ the number of linearly independent vectors y satisfying $A_{00}y = \lambda y$. Then the number N_λ $(\geqslant 0)$ of linearly independent solutions of system (1.18.1) of the form*

$$z = u^{\lambda} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} z_{jk}\, u^j v^k.$$

satisfies $\max\,(n_\lambda, n_{\lambda+1}, \ldots) \leqslant N_\lambda \leqslant n_\lambda + n_{\lambda+1} + \ldots$.

Proof. By introducing $z(u, v) = u^{\lambda}\, y(u, v)$ in (1.18.1) we obtain

$$(1.22)\qquad u\frac{\partial y}{\partial u} = (A(u, v) - \lambda I) y(u, v), \qquad \frac{\partial y}{\partial v} = B(u, v) y(u, v).$$

Let n_λ, N_λ be the above numbers. The determining equations corresponding to (1.20) — (1.21) are

$$(1.23)\qquad ((j + \lambda)I - A_{00})\, y_{j,0} = \sum_{q=0}^{j} A_{q,0}\, y_{j-q,0},$$

$(j = 0, .., N-1\,;\; y_{-1,0} = 0)$, where $y(u, v) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} y_{jk}\, u^j v^k$,

Now, from (1.23) it follows that $N_\lambda \leqslant n_\lambda + n_{\lambda+1} + \ldots$ and $N_\lambda \geqslant N_{\lambda+1}$ (see for example H a r t m a n [12], chapter IV, Lemma 11.4). Let us show that $N_\lambda \geqslant n_\lambda$. Without loss of generality, assume $\lambda = 0$ and A_{00} is in Jordan canonical form. Then each eigenvector of A_{00} corresponds to a zero row for A_{00}. Thus, the corresponding $d_i = 1$ can be reduced to nought. Hence, Theorem 2 implies the existence of at least $n - (n - n_0) = n_0$ solutions analytic at (0,0). This proves $N_\lambda \geqslant n_\lambda$, and in connection with $N_\lambda \geqslant N_{\lambda+1}$, the theorem is proved.

Corollary 1. (*The classical result*). *If $A_{00} = 0$, then, by* (1.19)

$$kA_{0k} + \sum_{t=0}^{k-1} A_{0t} B_{0,k-t-1} = \sum_{t=0}^{k-1} B_{0t} A_{0,k-t-1},$$

$(k = 1, 2, \ldots)$, hence we obtain step by step $A_{0k} = 0$. $(k = 0, 1, 2, \ldots)$. Thus A reduces to $A(u, v) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} A_{jk}\, u^j v^k$ and hence system (1.18) *has no singular point.* But $A_{00} = 0$ implies the existence of just $n_0 = n$ linearly independent solutions, analytic at (0.0) (that is a fundamental system of solutions holomorphic at (0,0)). We refind this way the classical result for total differential systems.

Corollary 2. *If no positive integer is eigenvalue of A_{00}, then $N_0 = n_0$, that is for every $z_{00} \in \ker A_{00}$, there exists a solution of (1.18.1) with $z(0,0) = z_{00}$.*

Remark 2. If neither nougth nor any positive integer are eigenvalues of A_{00}, then, by (1.20), (1.21) we obtain $\varphi = 0$, and therefore $z = 0$.

Finally, we state and prove.

Theorem 5. *If no two eigenvalues of A_{00} differ by a positive integer, then there exists, under hypothesis (1.19), a fundamental system $W(u, v)$ of (1.18.1), that is such that*

$$(1.24)\qquad u\frac{\partial W}{\partial u} = A(u, v) W(u, v), \qquad \frac{\partial W}{\partial v} = B(u, v) W(u, v),$$

of the form

$$(1.25)\qquad W(u, v) = P(u, v)\, u^{A_{00}},$$

where $P(u,v)$ is an n by n matrix-valued function analytic at (0,0) and $P(0,0) = I$.

Proof. A matrix $W(u, v)$ of the form (1.25) is a solution of (1.24) if and only if

$$(1.26)\qquad u\frac{\partial P}{\partial u} = A_{00}P - PA_{00} + (A(u, v) - A_{00})P,$$

$$(1.27)\qquad \frac{\partial P}{\partial v} = BP.$$

On the other hand, $P(u, v) = \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} P_{jk}\, u^j v^k$ satisfies formally (1.26), (1.27) if and only if

$$(1.28)\qquad A_{00} P_{00} - P_{00} A_{00} = 0,$$

$$(1.29)\qquad jP_{j0} = A_{00} P_{j0} - P_{j0} A_{00} + \sum_{q=0}^{j} A_{q,0} P_{j-q,0},$$

$$(1.30)\qquad kP_{jk} = \sum_{s=0}^{j} \sum_{t=0}^{k-1} B_{s,t}\, P_{j-s,k-t-1},$$

$(j = 0, 1, \ldots\,;\; k = 1, 2, \ldots)$, where (1.30) is a recurrence formula.

Now, instead of considering system (1.26), (1.27) as a system in the n by n matrix P, we can consider it as a system in the n^2 by 1 vector p, where p is built up by the elements of P. It is known from the algebra that (1.26) can be rearranged as

$$u\frac{\partial p}{\partial u} = C_{00}p + D(u, v)p, \tag{1.26.1}$$

with C_{00} a constant, n by n matrix and $D(u, v)$ a n^2 by n^2 matrix with $D(0,0) = 0$ (see Hartman [12], chapter IV, §10). Furthermore, if no two eigenvalues of A_{00} differ by a positive integer, then no eigenvalue of C_{00} is a positive integer. Thus, by the above Corollary 2, there are as many linearly independent solutions of (1.26.1) — (1.27.1) (where (1.27.1) is the corresponding „rearranged" system of (1.27)) as the dimension of ker C_{00}. Moreover, any $p_{00} \in$ ker C_{00} yields a solution p of (1.26.1) — (1.27.1) holomorphic at $p(0,0) = p_{00}$. From the definition of C_{00} and p, the p corresponding to $P = I$ is in ker C_{00}. Thus (1.26) — (1.27) has a solution $P(u, v)$ analytic at (0,0), and such that $P(0,0) = I$. Q.E.D.

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SUR LES SOLUTIONS PERIODIQUES ET PRESQUE-PERIODIQUES DES SYSTEMES D'EQUATIONS DIFFERENTIELLES NON-LINEAIRES DU PREMIER ORDRE

PAR

CONSTANTIN SIMIRAD

Nous allons considérer le système non-linéaire :

$$\dot{x}_i = f_i(t, x_1, \ldots, x_n), \quad i = 1, \ldots, n, \tag{1}$$

où les fonctions $f_i(t, x)$ sont définies sur $R \times \Omega$, Ω étant un ensemble convexe dans l'espace R^n. Dans la suite nous allons supposer que les fonctions $f_i(t, x)$ sont à valeurs réelles.

Théorème 1. *Si :*

1° *les fonctions $f_i(t, x)$ sont définies et continues sur $R \times \Omega$,*

2° $\frac{\partial f_i}{\partial x_j}(t, x)$ *sont définies et continues sur $R \times \Omega$ et de plus, supposons qu'il existe une matrice de fonctions intégrables $\Psi(t) = (\psi_{ij}(t))$, telle que*

$$\left|\frac{\partial f_i}{\partial x_j}(t, x)\right| \leqslant \psi_{ij}(t), \ \forall i \neq j \ \text{et} \ \frac{\partial f_i}{\partial x_i}(t, x) \leqslant \psi_{ii}(t), \ i, j = 1, 2, \ldots, n,$$

$$3°. \lim_{\theta \to -\infty} \inf \{ \exp \int_\theta^t [A(s) + a(s)]\, ds \} = 0, \text{ où}$$

$$A(s) = \max_j \sum_{i \neq j} \psi_{ij}(s), \ a(s) = \max_i \psi_{ii}(s),$$

alors le système (1) ne peut avoir qu'une seule solution bornée.

Démonstration. Supposons le contraire, c.a.d. que le système (1) a deux solutions bornées, disons $x(t)$ et $z(t)$. Posons $u_i(t) = x_i(t) - z_i(t)$, alors

$$\dot{u}_i(t) = \sum_{j=1}^n h_{ij}(t) u_j(t), \quad i = 1, \ldots, n, \tag{2}$$

où $h_{ij}(t) = \int_0^1 \frac{\partial f_i}{\partial \omega_j}(t, \omega(s))\, ds$, $\omega(s) = z(t) + s\,u(t)$, ce qui résulte du lemme d'Hadamard. Evidemment, $h_{ij}(t)$ sont des fonctions continues et on a $|h_{ij}(t)| \leqslant \psi_{ij}(t)$, $\forall i \neq j$ et $h_{ii}(t) \leqslant \psi_{ii}(t)$, $i, j = 1, \ldots, n$.

En considérant i fixé dans le système (2) nous obtenons une équation linéaire et sa solution sera :