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A UNIFIED APPROACH OF THE DESCENT METHODS FOR MINIMIZATION OF CONVEX FUNCTIONALS

Part I. GENERAL THEORY

BY

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The usual assumptions for the descent methods of minimization are the convexity and the uniform differentiability of the functional. These hypotheses can be weakened considerably by introducing certain generalized concepts of convexity (see for example [3]), but so far, the descent methods are essentially defined for differentiable functionals.

The present paper is an attempt to introduce descent iterative methods for minimization of convex, not everywhere differentiable, functionals. The algorithms are natural extensions of many known methods and they are presented in a unifying manner, which is a direct extension of the J. C é a's [1] approach. Primarily we shall be concerned with the unconstrained problem — that is, minimization over all the space of definition — but we shall also consider methods for the constrained problem, when they are natural extensions of earlier methods.

1. Brief survey of convex analysis. Let us begin with a revue of the main results of convex analysis which we need later. Let E be a real Banach space, E^* the dual space with the duality denoted by $<.,.>$ and $J : E \to \overline{R}$ a proper convex functional with the effective domain $D(J)$. (The notations and the usual concepts are essentially those of R o c k a f e l l a r's monograph [4]). For any $x \in D(J)$ and $h \in E$, the lateral directional differentials are denoted by $J'_+(x, h)$ and $J'_-(x, h)$. For $x \in \text{Int } D(J)$ there exist $J'_+(x, h)$, $J'_-(x, h)$ convex, respectively concave with respect to h and

$$J'_+(x, h) \geq J'_-(x, h). \tag{1.1}$$

If J is continuous at x, there exists $x^* \in E^*$, such that

$$J(x+h) - J(x) \geq J'_+(x, h) \geq <h, x^*> \geq J'_-(x, h); \tag{1.2}$$

x^* is a subgradient of J at x, the set of all subgradients at x, the subdifferential $\partial J(x)$, is a bounded, closed set in the norm of E^* and

$$J'_+(x, h) = \sup\{<h, x^*>,\ x^* \in \partial J(x)\}, \tag{1.3}$$

$$J'_-(x, h) = \inf\{<h, x^*>,\ x^* \in \partial J(x)\}. \tag{1.4}$$

When J is Gâteaux derivable, grad $J(x)$ is the unique subgradient at x.

A theorem which we need here is the following [4] :

Theorem 1.1. *Let $J : E \to \bar{R}$ be a convex propre functional, $x \in \operatorname{Int} D(J)$ and $h \in E$. Then*

$$J'_+(x - \lambda h, h) \leqslant J'_-(x, h) \leqslant J'_+(x, h) \leqslant J'_-(x + \lambda h, h), \qquad \forall \lambda \geqslant 0, \tag{1.5}$$

and

$$\lim_{\lambda \to 0+} J'_+(x + \lambda h, h) = \lim_{\lambda \to 0+} J'_-(x + \lambda h, h) = J'_+(x, h), \tag{1.6}$$

$$\lim_{\lambda \to 0-} J'_+(x + \lambda h, h) = \lim_{\lambda \to 0-} J'_-(x + \lambda h, h) = J'_-(x, h). \tag{1.7}$$

Also, when J is continuous on $\operatorname{Int} D(J)$, if $x_n \in \operatorname{Int} D(J)$, such that $\lim_{n\to\infty} \| x_n - x \| = 0$, and $h_n \in E$ with $\lim_{n\to\infty} \| h_n - h \| = 0$, then :

$$\lim \sup_{n \to \infty} J'_+(x_n, h_n) \leqslant J'_+(x, h). \tag{1.8}$$

A survey of properties of the lower semicontinuous functionals is given in the next theorem :

Theorem 1.2. *A functional $J : E \to \bar{R}$ is (weakly) lower semicontinuous, iff any level set $\{ x \in E, J(x) \leqslant \alpha \}$, $\alpha \in R$, is (weakly) closed in E. A convex lower semicontinuous functional is weakly lower semicontinuous and also it is continuous in the interior of the effective domain.*

2. Unconstrained minimization problem and iteratived escent methods. Let E be a reflexive space, $J : E \to \bar{R}$ a convex proper lower semicontinuous functional and assume that :

$$\exists\, u_0 \in D(J), \quad W(u_0) = \{ x \in E, \ J(x) \leqslant J(u_0) \} \ \text{ is bounded.} \tag{2.1}$$

The hypothesis (2.1) is verified, for example, when a condition of coercivity

$$\lim_{\| x \| \to \infty} J(x) = \infty \tag{2.2}$$

is fulfilled. Under these conditions the problem of finding a minimum-point $u \in E$, such that

$$\min_{v \in E} J(v) = J(u), \tag{2.3}$$

has at least a solution, characterized by the equation

$$0 \in \partial J(u). \tag{2.4}$$

We shall consider a large class of iterative methods for approximating the minimum points u of J, which, at each iteration u_n, provide a direction $h_n \in E$ along which we move to the next point,

$$u_{n+1} = u_n + \lambda_n h_n, \tag{2.5}$$

with steplength $\lambda_n \in R$, such that $J(u_{n+1}) < J(u_n)$.

We call *descent methods* these procedures. The essential question for them is how to choose the directions $h_n \in E$ and the steplength factors $\lambda_n \in R$ at each iteration, so that any limit-point of the sequence $\{ u_n \}$, to be a minimum-point of J. We have no loss of generality if we take λ_n positive and $h_n \neq 0$, or even $\| h_n \| = 1$, for each n. The assumptions on h_n and λ_n are contained in the conditions of the next definitions.

Definition 2.1. *A sequence $\{ h_n \} \subset E$ with $\| h_n \| = 1$ is called a set of directions of convergence for the iterative methods (2.5), if the following conditions are verified :*

$$J'_+(u_n, h_n) < 0, \quad n = 0, 1, \ldots, \tag{2.6}$$

$$\lim_{n\to\infty} J'_+(u_n, h_n) = 0 \Rightarrow \exists\, u_n^* \in \partial J(u_n), \ \lim_{n\to\infty} \| u_n^* \|_* = 0. \tag{2.7}$$

The inequality (2.6) is a condition of decreasing along the direction h_n starting from u_n. The implication (2.7) is an essential hypothesis for the theorem of convergence. It is easy to prove that, for a bounded sequence $\{ h_n \}$, $h_n \neq 0$, $n = 0, 1, \ldots$, satisfying (2.6) and, (2.7), $\{ h_n / \| h_n \| \}$ is a set of directions of convergence.

Definition 2.2. *If $\{ h_n \}$ is a set of directions of convergence for (2.5), a sequence of positive scalars $\{ \lambda_n \}$ is called a set of factors of convergence associated to $\{ h_n \}$ if they satisfy :*

$$J(u_n + \lambda_n h_n) < J(u_n), \quad n = 0, 1, \ldots, \tag{2.8}$$

and the implication

$$\lim_{n \to \infty} \{ J(u_n) - J(u_n + \lambda_n h_n) \} = 0 \Rightarrow \lim_{n \to \infty} J'_+(u_n, h_n) = 0. \tag{2.9}$$

We can prove now the main theorem of convergence :

Theorem 2.1. *If $\{ h_n \}$ is a set of directions of convergence and $\{ \lambda_n \}$ an associated set of factors of convergence for the iterative method (2.5), then the sequence $\{ u_n \}$ has weak limit points and any one of them is a minimum-point of J. Also $\{ J(u_n) \}$ is convergent and*

$$\lim_{n \to \infty} J(u_n) = \min_{v \in E} J(v). \tag{2.10}$$

Proof. $\{ J(u_n) \}$ is non-increasing and lower bounded, hence it is convergent and

$$\lim_{n \to \infty} \{ J(u_n) - J(u_{n+1}) \} = 0.$$

As a consequence of (2.9), we get $\lim_{n\to\infty} J'_+(u_n, h_n) = 0$ and, by (2.7), there exists $u_n^* \in \partial J(u_n)$ such as

$$\lim_{n \to \infty} \| u_n^* \|_* = 0. \tag{2.11}$$

Because of (2.8) and (2.1), $\{ u_n \} \subset W(u_0)$ and it is bounded ; then, since the space is reflexive, there exists at least a subsequence $\{ u_{n'} \}$ weakly convergent : $u_{n'} \to \bar{u}$, $n' \to \infty$. According to Theorem 1.2, J is weakly lower semicontinuous and $W(u_0)$ is weakly closed. Hence $\bar{u} \in W(u_0)$ and $\lim_{n' \to \infty} J(u_{n'}) \geqslant J(\bar{u})$. By the definition of the subgradient, we have

$$J(v) \geq J(u_{n'}) + < v - u_{n'}, u^*_{n'} >, \ \forall v \in E,$$

where from, by using (2.11), we obtain

$$J(v) \geq \lim_{n' \to \infty} J(u_{n'}) \geq J(\bar{u}), \ \ \forall v \in E,$$

which means that $\bar{u}$ is minimum-point of J. For $v = \bar{u}$ we get also the equality (2.10).

Consequence 2.1. *Under the conditions of the previous theorem, if J is strictly convex, the sequence $\{u_n\}$ is weakly convergent to the unique minimum-point u of the functional.*

Let us now investigate under which stronger hypotheses, the sequence of iterations are norm-convergent.

Definition 2.3. *The functional $J : E \to (-\infty, +\infty]$ is called uniformly convex [3], if there exists a real-valued continuous, monotone-increasing function $\gamma(t)$, with the properties:*

(2.12) $$\gamma(t) > 0, \text{ for } t > 0; \ \gamma(0) = 0, \ \lim_{t \to \infty} \gamma(t)/t = \infty,$$

and a real valued function $\alpha(\lambda)$, defined for $\lambda \in [0,1]$, with

(2.13) $$\alpha(\lambda) > 0, \text{ for } \lambda \in (0,1); \ \alpha(0) = \alpha(1) = 0, \ \lim_{\lambda \to 0} \alpha(\lambda)/\lambda = \beta \neq 0,$$

such that, for any $x, y \in D(J)$ and $\lambda \in [0,1]$,

(2.14) $$\lambda J(x) + (1 - \lambda) J(y) - J(\lambda x + (1 - \lambda) y) \geq \alpha(\lambda) \gamma(\| x - y \|).$$

Proposition 2.1. *Any uniformly convex functional $J : E \to (-\infty, +\infty]$ is strongly convex and satisfies:*

(2.15) $$J(v + h) - J(v) \geq J'_+(v, h) + \beta \gamma(\| h \|), \ \ \forall v \in D(J), \ \forall h \in E.$$

Proof. The properties of α and γ, together with (2.14) imply that J is strongly convex and

$$(J(y + \lambda(x - y)) - J(y))/\lambda \leq J(x) - J(y) - \alpha(\lambda) \gamma(\| x - y \|)/\lambda.$$

By taking $y = v \in D(J)$, $x - y = h \in E$ and by passing to the limit for $\lambda \to 0$, we obtain (2.15).

The inequality (2.15) is not generally sufficient for the uniform convexity of J.

Proposition 2.2. *Suppose that there exists a real-valued monotone-increasing, continuous function γ_1, with the property:*

(2.16) $$\gamma_1(ct) \geq c \gamma_1(t) > 0, \ \ \forall c > 0, \ \ \forall t > 0,$$

such that the functional $J : E \to (-\infty, +\infty]$ satisfies:

(2.17) $$J(v + h) - J(v) \geq J'_+(v, h) + \| h \| \gamma_1(\| h \|), \ \forall v \in D(J), \ \forall h \in E.$$

Then J is uniformly convex.

Proof. Let $x, y \in D(J)$ and $\lambda \in [0,1]$. Then:

$$J(x) - J(\lambda x + (1 - \lambda) y) \geq J'_+(\lambda x + (1 - \lambda) y, (1 - \lambda)(x - y)) +$$
$$+ (1 - \lambda)^2 \| x - y \| \gamma_1(\| x - y \|),$$

and

$$J(y) - J(\lambda x + (1 - \lambda) y) \geq J'_+(\lambda x + (1 - \lambda) y, \ \lambda(y - x)) +$$
$$+ \lambda^2 \| x - y \| \gamma_1(\| x - y \|).$$

By adding the first inequality multiplied by λ, with the second one, multiplied by $(1 - \lambda)$ and by using also (1.1) and (1.3), we obtain:

$$\lambda J(x) + (1 - \lambda) J(y) - J(\lambda x + (1 - \lambda) y) \geq \lambda(1 - \lambda) \| x - y \|$$
$$\times \gamma_1(\| x - y \|).$$

So (2.14) holds with $\alpha(\lambda) = \lambda(1 - \lambda)$, $\gamma(t) = t \gamma_1(t)$, which obviously satisfy the properties (2.12), (2.13) because $\gamma(t)/t = \gamma_1(t) \geq t \gamma_1(1)$.

Consequence 2.2. *Let J be a twice-Gâteaux-differentiable functional in the convex domain $D(J) \subseteq E$, whose second differential satisfies:*

(2.18) $$J''(x, h, h) \geq \| h \| \gamma_1(\| h \|), \ \ \forall x \in D(J), \ \ \forall h \in E,$$

where γ_1 is a real-valued function verifying the conditions of Proposition 2.2. Then J is uniformly convex.

Proof. By Taylor's formula, there exists $\theta \in (0,1)$, such that

$$J(v + h) - J(v) = J'(v, h) + J''(v + \theta h, h, h)/2 \geq J'(v, h) +$$
$$+ \| h \| \gamma_1(\| h \|)/2,$$

which implies the uniform convexity.

Now we are able to prove a theorem of strong convergence for the descent methods (2.5).

Theorem 2.2. *Let J be a lower semicontinuous, uniformly convex-propre functional, defined on the reflexive space E. Then J has a unique minimum-point u and the sequence of iterations $\{u_n\}$ constructed by the formula (2.5), where $\{h_n\}$ and $\{\lambda_n\}$ are sets of directions, respectively, factors of convergence, is norm-convergent to u, for any $u_0 \in D(J)$ and*

(2.19) $$\gamma(\| u_n - u \|) \leq \beta^{-1} \{ J(u_n) - J(u) \}, \ n = 0,1, \ldots$$

Proof. According to Proposition 2.1, J is strictly convex, hence J has no more than one minimum-point. Also, by (2.15), for $v \in \operatorname{Int} D(J)$ we have

$$J(v + h) - J(v) \geq < h, v^* > + \beta\gamma(\| h \|) \geq \| h \| \{ - \| v^* \|_* +$$
$$+ \beta \gamma(\| h \|)/\| h \| \},$$

for a subgradient $v^* \in \partial J(v)$. Therefore, the condition (2.2) is satisfied, so that there exists a unique minimum-point. According to Consequence 2.1 u_n is weakly convergent to u. Also, we have $(u_n + u)/2 \to u$, for $m \to \infty$. By the semicontinuity and the uniform convexity of J and by using (2.10), we can write:

$$0 \leqslant \liminf_{n\to\infty} \gamma(\| u_n - u \|)\, \alpha(1/2) \leqslant \limsup_{n\to\infty} \gamma(\| u_n - u \|)\, \alpha(1/2) \leqslant$$

$$\leqslant \frac{1}{2} J(u) + \frac{1}{2}\lim_{n\to\infty} J(u_n) - \liminf_{n\to\infty} J\left(\frac{u_n + u}{2}\right) \leqslant \frac{1}{2} J(u) + \frac{1}{2} J(u) - J(u) = 0,$$

where from $\lim_{n\to\infty} \gamma(\| u_n - u \|) = 0$. Since γ is monotone and continuous, it is inversable and the norm-convergence of u_n to u is proved. By using Proposition 2.1, we obtain also

$$J(u_n) - J(u) \geqslant J'_+(u, u_n - u) + \beta\gamma(\| u_n - u \|), \quad n = 0,1,\ldots$$

Because of (2.4) and (1.5), $J'_+(u, u_n - u) \geqslant 0$, which leads to the estimation (2.19).

Consequence 2.3. *If J is twice-differentiable and the second differential satisfies (2.18), then the iterations $\{u_n\}$, constructed with directions and factors of convergence, are norm-convergent to the unique minimum-point u of J.*

Particularly, when the hessian of J is positive definite, i.e.

$$J''(v, h, h) = < v, H(v) h > \geqslant m \| h \|^2 > 0, \ \forall v \in D(J), \ \forall h \neq 0,$$

the estimation (2.19) has the form:

$$\| u_n - u \|^2 \leqslant 2 \{ J(u_n) - J(u) \}/m. \tag{2.20}$$

3. Choice of the directions h_n of convergence. In the form (2.5) and (2.6), the conditions imposed on the directions h_n are unpractical. We shall prove now sufficient conditions under which $\{h_n\}$ are directions of convergence for (2.5).

Definition 3.1. *Let $u \in D(J)$ fixed. Any $h \in E$, $h \neq 0$ is called a descent direction at u of the functional, if*

$$J'_+(u, h) < 0. \tag{3.1}$$

For a descent direction h at u, the functional can be diminished along the line $\{u + \lambda h, \lambda \geqslant 0\}$, because, by the definition of $J'_+(u, h)$, for $0 < \varepsilon < -J'_+(u, h)$, there exists $\lambda(\varepsilon) > 0$ such that $(J(u + \lambda h) - J(u))/\lambda < J'_+(u, h) + \varepsilon < 0$, for $0 < \lambda < \lambda(\varepsilon)$. When $J'_+(u, h) \geqslant 0$, we have, by (1.2), $J(u + \lambda h) - J(u) \geqslant \lambda J'_+(u, h) \geqslant 0, \forall \lambda \geqslant 0$, so that the functional achieves its minimum along the direction h, at u.

Proposition 3.1. *$h \in E$ is a descent direction of J at $u \in D(J)$, iff there exists $\tau > 0$ such that*

$$- < h, u^* > \ > \tau \| u^* \|_*, \quad \forall u^* \in \partial J(u). \tag{3.2}$$

Proof. If h is a descent direction at u, by (1.3) we have

$$- < h, u^* > / \| u^* \|_* \geqslant - J'_+(u, h)/\alpha = \tau > 0,$$

where $\alpha = \sup \{ \| u^* \|_*, \ u^* \in \partial J(u) \}$. Conversely, the inequality (3.2) implies that $0 \notin \partial J(u)$ and then, since the subdifferential is a closed set in E^*, $\inf \{ \| u^* \|_*, \ u^* \in \partial J(u) \} = \beta > 0$. The hypothesis (3.2) leads to $< h, u^* > \ \leqslant -\tau\beta$, $\forall u^* \in \partial J(u)$, where from we obtain $J'_+(u, h) \leqslant -\tau\beta < 0$.

We state now a simple theorem concerning sufficient conditions for satisfying (2.6) and (2.7).

Theorem 3.1. *Let $h_n \in E$, $\| h_n \| = 1$, $n = 0,1,\ldots$ and suppose that there exists $\rho > 0$ such that*

$$- < h_n, u^*_n > \ \geqslant \rho \| u^*_n \|_*, \quad \forall u^*_n \in \partial J(u_n), \ n = 0,1,\ldots \tag{3.3}$$

Then $\{h_n\}$ is a set of directions of convergence for the iterative method (2.5).

Proof. According to Proposition 3.1, h_n is a descent direction of J at u_n, i.e. $J'_+(u_n, h_n) < 0$. Because of (1.3), for each n and for $0 < \varepsilon_n \leqslant -J'_+(u_n, h_n)$, there exists a subgradient $u^*_n \in \partial J(u_n)$ such that

$$-\rho \| u^*_n \|_* \geqslant \ < h_n, u^*_n > \ > J'_+(u_n, h_n) - \varepsilon_n \geqslant 2 J'_+(u_n, h_n).$$

Therefore, there exists $u^*_n \in \partial J(u_n)$ with

$$\| u^*_n \| \leqslant -2 J'_+(u_n, h_n)/\rho,$$

which leads to the implication (2.7).

Remark 3.1. Since, under the hypothesis (3.3), $- < h_n, u^*_n > \ = | < h_n, u^*_n > |$, we have:

$$0 < \rho \leqslant 1. \tag{3.4}$$

The condition (3.2) is quite restrictive, because it is required to be verified for any subgradient u^*_n at each iteration u_n. A less stronger assumption is given by the following theorem:

Theorem 3.2. *If no one of the iterations u_n is a minimum-point of J, any sequence $\{h_n\} \subset E$, $\| h_n \| = 1$, satisfying*

$$-J'_+(u_n, h_n) \geqslant \rho \inf \{ \| v^*_n \|_*, \ v^*_n \in \partial J(u_n) \}, \ n = 0,1,\ldots, \text{ where } \rho > 0 \tag{3.5}$$

is a constant, is a set of directions of convergence for (2.5).

Proof. $0 \in E^*$ is not a subgradient of J at u_n, so that

$$\inf \{ \| v^*_n \|_*, v^*_n \in \partial J(u_n) \} = \beta_n > 0, \quad n = 0,1,\ldots$$

Hence h_n is a descent direction at u_n and, for $\varepsilon > 0$, there exists $u^*_n \in \partial J(u_n)$, such that $\beta_n(1 + \varepsilon) > \| u^*_n \|_*$, $n = 0,1,\ldots$ Consequently, for each n, there exists $u^*_n \in \partial J(u_n)$, for which

$$-J'_+(u_n, h_n) > \frac{\rho}{1 + \varepsilon} \| u^*_n \|_*$$

and the implication (2.7) is following.

Remark 3.2. The hypothesis (3.3) of the previous theorem is stronger than (3.5) and the constant ρ of the latter satisfies also (3.4).

Proof. From (3.3) we obtain $- < h_n, u^*_n > \ \geqslant \rho\beta_n$, $\forall u^*_n \in \partial J(u_n)$ which implies (3.5), by using the property (1.3). Also because of (1.3) we have:

$$-J'_+(u_n, h_n) \leqslant - < h_n, u^*_n > \ \leqslant \| u^*_n \|_*, \quad \forall u^*_n \in \partial J(u_n),$$

where from $-J'_+(u_n, h_n) \leqslant \beta_n$, $n = 0,1,\ldots$, which leads to (3.4) when (3.5) is satisfied.

Let us discuss now some special cases.

1) When J is Gâteaux derivable, the conditions (3.3) and (3.5) are equivalent to:

(3.6) $\exists \rho > 0, \; -<h_n, \text{ grad } J(u_n)> \geqslant \rho \| \text{grad } J(u_n) \|_*$.

A particular choice for h_n is, then $h_n = -y_n$, where y_n is a subgradient of the norm of E^* at the point grad $J(u_n) \in E^*$, i.e.

$$\| y_n \| = 1, \quad <y_n, \text{ grad } J(u_n)> = \| \text{ grad } J(u_n) \|_*.$$

2). When E is a Hilbert space the condition (3.3) becomes

(3.7) $-\cos(\widehat{h_n, u_n^*}) \geqslant \rho > 0, \; \forall u_n^* \in \partial J(u_n), \; n = 0,1,\ldots$

The meaning of this relation is that h_n is any direction which does not tend to be orthogonal to any subgradient at u_n, when $n \to \infty$.

3) When J is Gâteaux derivable and $D_n : E^* \to E$, $n = 0,1,\ldots$, are linear bounded operators satisfying

(3.8) $<D_n v, v> \geqslant m \| v^2 \|_* > 0, \; \forall v \in E^*, \; v \neq 0, \; \| D_n \| \leqslant \delta$.

another possible choice is

$$h_n = -z_n / \| z_n \|, \; z_n = D_n \text{ grad } J(u_n).$$

This class contains especially the well-known quasi-Newton methods.

4) Let J be Gâteaux derivable and $\{ g_n \}$ a set of directions of convergence for (2.5). Define $\{ h_n \}$ by the recurrence

(3.9) $h_{n+1} = g_{n+1} + \mu_n h_n$,

where $\mu_n \in R$ is taken such that $\{ h_n \}$ is bounded and $\lambda_n \in R$ in such a way that

(3.10) $\mu_n <h_n, \text{ grad } J(u_n)> \leqslant 0$.

Then we have

$$J'(u_n, h_n) = <h_n, \text{ grad } J(u_n)> \leqslant <g_n, \text{ grad } J(u_n)> = J'(u_n, g_n),$$

which implies (2.6) and (2.7) for $\{ h_n \}$. The conjugate gradients methods and many other procedures of that type, belong to this class. The parameters μ_n can be used for improving the speed of convergence of these methods.

In the second part of this paper [2] we shall be concerned with the choice of λ_n satisfying the conditions of Definition 2.2.

Also some particular methods will be considered and the speed of convergence will be estimated.

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ON SOME NEW DISCRETE INEQUALITIES AND THEIR APPLICATIONS TO A CLASS OF SUM-DIFFERENCE EQUATIONS

BY

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1. Introduction. In recent years considerable attention has been given to the development of the qualitative theory of difference equations. The interest in the study of these equations in both mathematics and engineering stems from their usefulness in various applications. In addition, any system whose internal structure may be unknown but whose input output behavior can be obtained through experiment is a candidate for a difference equation model, and this accounts, in part, for the popularity of such models with economists, psychologists and statisticians (see [4]). Among numerous papers dealing with various problems in the theory of difference equations we refer the interested readers to [1], [2], [7]—[14], [16], and some of the references therein. In this paper we wish to study the boundedness, asymptotic behavior and the rate of growth of the solutions of a class of sum-difference equations allowing more general perturbations. The main tools in our analysis are the variation of constants formula recently developed by N. Luca and P. Talpalaru in [7] and the discrete inequalities which are new to the literature.

2. Discrete Inequalities. In this section we present some new discrete inequalities which can be used in the theory of finite difference equations as handy tools. Before giving the main results in this section, we first recollect a few of the basic notions and definitions from [9]—[11]. Let N be a set of points $n_0 + k (k = 0, 1, 2, \ldots)$ where $n_0 \geqslant 0$ is a given integer. The expression $\sum_{s=n_0}^{n-1} p(s)$ represents a solution of the linear difference equation $\Delta x(n) = p(n)$ for all $n \in N$ under the initial condition $x(n_0) = 0$, where Δ is the operator defined by $\Delta x(n) = x(n+1) - x(n)$. It is supposed that $\sum_{s=n_0}^{n_0-1} p(s) = 0$. The expression $\prod_{s=n_0}^{n-1} q(s)$ represents a solution of the linear difference equation $x(n+1) = q(n)\, x(n)$ for all $n \in N$ under the initial condition $x(n_0) = 1$. It is supposed that $\prod_{s=n_0}^{n_0-1} q(s) = 1$.