Let us discuss now some special cases.

1) When J is Gâteaux derivable, the conditions (3.3) and (3.5) are equivalent to:

(3.6) $\exists\, \rho > 0,\ -<h_n\,,\ \mathrm{grad}\ J(u_n)> \geqslant \rho \,\|\, \mathrm{grad}\ J(u_n)\,\|_*$.

A particular choice for h_n is, then $h_n = -y_n$, where y_n is a subgradient of the norm of E^* at the point grad $J(u_n) \in E^*$, i.e.

$$\| y_n \| = 1,\quad <y_n,\ \mathrm{grad}\ J(u_n)> = \|\ \mathrm{grad}\ J(u_n)\|_*\,.$$

2). When E is a Hilbert space the condition (3.3) becomes

(3.7) $-\cos\,(\widehat{h_n\,,u_n^*}) \geqslant \rho > 0,\ \ \forall\, u_n^* \in \partial J(u_n),\ n = 0,1,\ldots$

The meaning of this relation is that h_n is any direction which does not tend to be orthogonal to any subgradient at u_n, when $n \to \infty$.

3) When J is Gâteaux derivable and $D_n : E^* \to E,\ n = 0,1,\ldots,$ are linear bounded operators satisfying

(3.8) $<D_n\, v, v> \geqslant m\, \| v^2 \|_* > 0,\ \forall\, v \in E^*,\ v \neq 0,\ \| D_n \| \leqslant \delta.$

another possible choice is

$$h_n = -z_n / \| z_n \|\ ,\ z_n = D_n\ \mathrm{grad}\ J(u_n).$$

This class contains especially the well-known quasi-Newton methods.

4) Let J be Gâteaux derivable and $\{g_n\}$ a set of directions of convergence for (2.5). Define $\{h_n\}$ by the recurrence

(3.9) $h_{n+1} = g_{n+1} + \mu_n\, h_n,$

where $\mu_n \in R$ is taken such that $\{h_n\}$ is bounded and $\lambda_n \in R$ in such a way that

(3.10) $\mu_n <h_n,\ \mathrm{grad}\ J(u_n)> \leqslant 0.$

Then we have

$J'(u_n\,, h_n) = <h_n,\ \mathrm{grad}\ J(u_n)> \leqslant\ <g_n,\ \mathrm{grad}\ J(u_n)> = J'(u_n\,, g_n),$

which implies (2.6) and (2.7) for $\{h_n\}$. The conjugate gradients methods and many other procedures of that type, belong to this class. The parameters μ_n can be used for improving the speed of convergence of these methods.

In the second part of this paper [2] we shall be concerned with the choice of λ_n satisfying the conditions of Definition 2.2.

Also some particular methods will be considered and the speed of convergence will be estimated.

REFERENCES

1. C é a J. — *Optimisation. Théorie et algorithmes*, Dunod, Paris, 1971.
2. I l i o i C. — *A unified approach of the descent methods for minimization of convex functionals. Part. II. Step-length algorithms*, to appear.
3. L e v i t i n E.S., P o l i a k B. I. — *Minimization Methods with Restrictions* (Russian). J. Vicisl. Mat. i Matem. Fiz. 11 (1967) 787—823.
4. R o c k a f e l l a r R. T. — *Convex analysis*, Princeton University Press, 1969.

Received 4.I.1978

Faculty of Mathematics
University of Iași
Iași, R. S. Romania

Analele științifice ale Universității „Al. I. Cuza" Iași
Tomul XXIV, f. 2, s. I a, 1978

ON SOME NEW DISCRETE INEQUALITIES AND THEIR APPLICATIONS TO A CLASS OF SUM-DIFFERENCE EQUATIONS

BY

B. G. PACHPATTE

1. Introduction. In recent years considerable attention has been given to the development of the qualitative theory of difference equations. The interest in the study of these equations in both mathematics and engineering stems from their usefulness in various applications. In addition, any system whose internal structure may be unknown but whose input output behavior can be obtained through experiment is a candidate for a difference equation model, and this accounts, in part, for the popularity of such models with economists, psychologists and statisticians (see [4]). Among numerous papers dealing with various problems in the theory of difference equations we refer the interested readers to [1], [2], [7]—[14], [16], and some of the references therein. In this paper we wish to study the boundedness, asymptotic behavior and the rate of growth of the solutions of a class of sum-difference equations allowing more general perturbations. The main tools in our analysis are the variation of constants formula recently developed by N. L u c a and P. T a l p a l a r u in [7] and the discrete inequalities which are new to the literature.

2. Discrete Inequalities. In this section we present some new discrete inequalities which can be used in the theory of finite difference equations as handy tools. Before giving the main results in this section, we first recollect a few of the basic notions and definitions from [9]—[11]. Let N be a set of points $n_0 + k (k = 0, 1, 2, \ldots)$ where $n_0 \geqslant 0$ is a given integer. The expression $\sum_{s=n_0}^{n-1} p(s)$ represents a solution of the linear difference equation $\Delta x(n) = p(n)$ for all $n \in N$ under the initial condition $x(n_0) = 0$, where Δ is the operator defined by $\Delta x(n) = x(n+1) - x(n)$. It is supposed that $\sum_{s=n_0}^{n_0-1} p(s) = 0$. The expression $\prod_{s=n_0}^{n-1} q(s)$ represents a solution of the linear difference equation $x(n+1) = q(n)\, x(n)$ for all $n \in N$ under the initial condition $x(n_0) = 1$. It is supposed that $\prod_{s=n_0}^{n_0-1} q(s) = 1$.

We now state and prove a useful discrete inequality in the following form.

Theorem 1. *Let $u(n)$, $a(n)$, $b(n)$, and $c(n)$ be real-valued nonnegative functions defined on N, for which the inequality*

$$u(n) \leqslant u_0 + \sum_{s=n_0}^{n-1} a(s)u(s) + \sum_{s=n_0}^{n-1} b(s)\left(u(s) + \sum_{t=n_0}^{s-1} c(t)u(t)\right), \tag{1}$$

holds for all $n \in N$, where u_0 is a nonnegative constant. Then

$$u(n) \leqslant u_0 \prod_{s=n_0}^{n-1} [1 + a(s)] \left[1 + \sum_{s=n_0}^{n-1} b(s) \prod_{t=n_0}^{s-1} [1 + a(t) + b(t) + \right.$$
$$\left. + c(t)] \prod_{\tau=n_0}^{s} [1 + a(\tau)]^{-1}\right], \tag{2}$$

for all $n \in N$.

Proof. Define a function $m(n)$ by the right hand member of (1). Then

$$\Delta m(n) = a(n)\,u(n) + b(n)\left(u(n) + \sum_{t=n_0}^{n-1} c(t)\,u(t)\right), \quad m(n_0) = u_0,$$

which in view of (1) implies

$$\Delta m(n) \leqslant a(n)\,m(n) + b(n)\left(m(n) + \sum_{t=n_0}^{n-1} c(t)\,m(t)\right). \tag{3}$$

If we put

$$v(n) = m(n) + \sum_{t=n_0}^{n-1} c(t)\,m(t), \quad v(n_0) = m(n_0) = u_0, \tag{4}$$

it follows from (4), (3) and the fact that $m(n) \leqslant v(n)$, that the inequality $v(n+1) \leqslant [1 + a(n) + b(n) + c(n)]v(n)$ is satisfied, this implies the estimation for $v(n)$ such that

$$v(n) \leqslant u_0 \prod_{s=n_0}^{n-1} [1 + a(s) + b(s) + c(s)].$$

Substituting this value of $v(n)$ in (3), we have

$$m(n+1) - [1 + a(n)]m(n) \leqslant u_0\, b(n) \prod_{s=n_0}^{n-1} [1 + a(s) + b(s) + c(s)].$$

Multiplying by $\prod_{s=n_0}^{n} [1 + a(s)]^{-1}$ t_0 both sides of the above inequality, setting $n = 3$, and then summing up both sides from n_0 to $n-1$, we obtain the estimate for $m(n)$:

$$m(n) \leqslant u_0 \prod_{s=n_0}^{n-1} [1 + a(s)] \left[1 + \sum_{s=n_0}^{n-1} b(s) \prod_{t=n_0}^{s-1} [1 + a(t) + b(t) \right.$$
$$\left. + c(t)] \cdot \prod_{\tau=n_0}^{s} [1 + a(\tau)]^{-1}\right].$$

Now, substituting the value of $m(n)$ in (1) the desired bound in (2) follows.

Another interesting and useful discrete inequality is embodied in the following theorem.

Theorem 2. *Let $u(n)$, $a(n)$, $b(n)$ and $c(n)$ be real-valued nonnegative functions defined on N, for which the inequality*

$$u(n) \leqslant u_0 + \sum_{s=n_0}^{n-1} a(s)u(s) + \sum_{s=n_0}^{n-1} b(s)u^{\alpha}(s)\left(u(s) + \sum_{t=n_0}^{s-1} c(t)u(t)\right), \tag{5}$$

holds for all $n \in N$, where u_0 is a positive constant and $0 < \alpha < 1$.

If $\alpha u_0^{\alpha} \sum_{t=n_0}^{n-1} b(t) \prod_{\tau=n_0}^{t} [1 + a(\tau) + c(\tau)]^{\alpha} < 1$ for all $n \in N$, then

$$u(n) \leqslant \left(\prod_{s=n_0}^{n-1} [1 + a(s)]\right)\left[u_0^{1-\alpha} + (1-\alpha) \sum_{s=n_0}^{n-1} b(s) \prod_{t=n_0}^{s} [1 + a(t)]\right]^{\alpha-1} u_0$$
$$\prod_{t=n_0}^{s-1} [1 + a(t) + c(t)] \Big/ \left[1 - \alpha u_0^{\alpha} \sum_{t=n_0}^{s-1} b(t) \prod_{\tau=n_0}^{t} [1 + a(\tau) + c(\tau)^{\alpha}]^{\frac{1}{\alpha}}\right] \tag{6}$$

for all $n \in N$.

Proof. Define a function $m(n)$ by the right hand member of (5).

Then $\Delta m(n) = a(n)u(n) + b(n)u^{\alpha}(n)\left(u(n) + \sum_{t=n_0}^{n-1} c(t)u(t)\right)$, $m(n_0) = u_0$,

which in view of (5) implies

$$\Delta m(n) \leqslant a(n)m(n) + b(n)m^{\alpha}(n)\left(m(n) + \sum_{t=n_0}^{n-1} c(t)m(t)\right). \tag{7}$$

If we put

$$v(n) = m(n) + \sum_{t=n_0}^{n-1} c(t)m(t), \; v(n_0) = m(n_0) = u_0, \tag{8}$$

then it follows from (7), (8) and the fact that $m(n) \leqslant v(n)$, that the inequality

$$v(n+1) - v(n) - [a(n) + c(n)]\, v(n) \leqslant b(n)v^{\alpha-1}(n), \tag{9}$$

is satisfied. Now, by following the similar argument as in the proof of theorem 4 given in [11] (see, also [14]) we obtain the estimation for $v(n)$ such that

$$v(n) \leqslant \frac{u_0 \prod_{s=n_0}^{n-1} [1 + a(s) + c(s)]}{\left[1 - \alpha u_0^{\alpha} \sum_{s=n_0}^{n-1} b(s) \prod_{t=n_0}^{s} [1 + b(t) + c(t)]^{\alpha}\right]^{\frac{1}{\alpha}}} = Q(n). \tag{10}$$

Substituting this value of $v(n)$ in (7) we obtain

$$m(n+1) - m(n) - a(n)m(n) \leqslant b(n)\, Q(n) m^{\alpha}(n).$$

Again by following the similar argument as mentioned above we obtain the estimation for $m(n)$ such that

$$m(n) \leqslant \left(\prod_{s=n_0}^{n-1} [1 + a(s)]\right)\Bigg[u_0^{1-\alpha} + (1-\alpha) \sum_{s=n_0}^{n-1} b(s) \prod_{t=n_0}^{s} [1 + a(t)]^{\alpha-1} Q(s)\Bigg]^{\frac{1}{1-\alpha}}. \tag{11}$$

Now, substituting the value of $Q(n)$ from (10) in (11) and then substituting this value of $m(n)$ in (5) we obtain the desired bound in (6).

We next state a slightly different version of Theorem 2 which can be used in some applications.

Theorem 3. *Let $u(n)$, $a(n)$, $b(n)$, and $c(n)$ be real-valued nonnegative functions defined on N, for which the inequality*

$$u(n) \leqslant u_0 + \sum_{s=n_0}^{n-1} a(s)u(s) + \sum_{s=n_0}^{n-1} b(s)u(s)\left(u(s) + \sum_{t=n_0}^{s-1} c(t)u(t)\right), \tag{12}$$

holds for all $n \in N$, where u_0 is a positive constant. If $u_0 \sum_{t=n_0}^{n-1} b(t) \prod_{\tau=n_0}^{t} [1 + a(\tau) + c(\tau)] < 1$ for all $n \in N$, then

$$u(n) \leqslant u_0 \prod_{s=n_0}^{n-1} \left[1 + a(s) + b(s) \frac{u_0 \prod_{t=n_0}^{s-1} [1 + a(t) + c(t)]}{1 - u_0 \sum_{t=n_0}^{s-1} b(t) \prod_{\tau=n_0}^{t} [1 + a(\tau) + c(\tau)]}\right], \tag{13}$$

for all $n \in N$.

The proof of this theorem follows by the similar argument as in the proof of Theorem 2 with suitable modifications. We omit the details.

We note that in the special case when $b(n) = 0$. Theorems 1–3 reduces to the discrete analogue of the Gronwall's inequality given in [10, p. 348]. In the special case when $a(n) = 0$ our results in Theorems 1–3 are new to the literature.

3. Applications to Sum-Difference Equations. In this section we present our results on the behavior of solutions of perturbed nonlinear systems of sum-difference equations of the form

$$(P)\quad x(n+1) = Ax(n) + \sum_{m=0}^{n-1} B(n-m-1)\, x(m) + f(n, x(n)) + H(n, x(.)),\ x(0) = x_0,$$

under some suitable conditions on f, H and on the solutions of the corresponding unperturbed linear system

$$(L)\quad y(n+1) = Ay(n) + \sum_{m=0}^{n-1} B(n-m-1)\, y(m),\ y(0) = x_0.$$

Here, x, y are the elements of R^p, the p-dimensional real Euclidean space with norm $|x| = \sum_{i=1}^{p} |x_i|$, $x = (x_1, x_2, \ldots, x_p)$; A and $B(n)$ are the elements of M^p, the space of all $p \times p$ matrices $C = (c_{ij})$ with norm $|.|$ given by $|C| = \sum_{i=1}^{p} \sum_{j=1}^{p} |c_{ij}|$; $f(n, x)$ is defined for $n \in N$ and $x \in R^p$; and $H(n, x(.))$ is an operator for $n \in N$. We assume in the following that $B(-n) = 0$, $n \in N$, $n \neq 0$. The identity matrix is denoted by E.

The problems of stability and asymptotic behavior of the solutions of (P) and various forms of (P) have been studied by Luca and Talpalaru [7], Bykov and Linenko [1], Coffman [2], Masolockaja [8], Pachpatte [9]–[11], and others by using different techniques. The allowable perturbation terms here are motivated by recent studies of the authors in [1], [7]. In particular the perturbations we have in mind are of the form

$$H(n, x(\,.\,)) = x(n) h\left(n, x(n), \sum_{m=0}^{n-1} k(n, m, x(m))\right), \tag{13}$$

$$H(n, x(\,.\,)) = x^{\alpha}(n) h\left((n, x(n), \sum_{m=0}^{n-1} k(n, m, x(m))\right), \tag{14}$$

$$H(n, x(\,.\,)) = h\left(n, x(n), \sum_{m=0}^{n-1} k(n, m, x(m))\right), \tag{15}$$

and so on. Our results in the present paper on the behavior of solutions of (P) (when the perturbation term $H(n, x(\,.\,))$ is of the form (13) or (14) or (15)) are of interest because of the week assumptions on the functions involved in (P) and the approach to the problem is different from those cited in the references [1], [7].

Preparatory to our results in this section, we require the following definition and Lemma 1 recently given by N. Luca and P. Talpalaru in [7].

Definition 1. *The difference resolvent $R(n)$ associated with the lineart system (L) is the solution of the matrix equation*

$$R(n+1) = AR(n) + \sum_{m=0}^{n-1} B(n-m-1) R(m),\ R(0) = E. \tag{16}$$

Lemma 1 (Luca and Talpalaru [7]). *If $R(n)$ is the difference resolvent associated with the linear system (L); then the solution of the initial value problem (P) satisfies the equation*

$$x(n) = R(n)\, x_0 + \sum_{m=0}^{n-1} R(n-m-1) [f(m, x(m)) + H(m, x(\,.\,))]. \tag{17}$$

We are now prepared to state and prove our results on the behavior of solutions of (P) as a perturbation of (L) with $H(n, x(\,.\,))$ in (P) as defined in (13). Our first theorem in this section establishes the boundedness of the solution of sum-difference equation of the form

$$(P^*) \qquad x(n+1) = Ax(n) + \sum_{m=0}^{n-1} B(n-m-1)\, x(m) + f(n, x(n)) + x(n)h\left(n, x(n), \sum_{m=0}^{n-1} k(n, m, x(m))\right),$$

with the initial condition $x(0) = x_0$, where $x_0 \neq 0$ is a constant, under some suitable conditions on the perturbation terms f, k, and h.

Theorem 4. *Let the resolvent $R(n)$ associated with the linear system (L) satisfies the condition*

$$(18) \qquad |R(n)| \leqslant M \text{ for all } n \in N,$$

where M is a positive constant. Suppose that the functions f, h, and k in (P^*) satisfy

$$(19) \qquad |f(n, x(n))| \leqslant a(n)|x(n)|, \qquad n \in N,$$

$$(20) \qquad |h(n, x(n), z)| \leqslant b(n)[|x(n)| + |z|], \qquad n \in N,$$

$$(21) \qquad |k(n, s, x(s))| \leqslant c(s)|x(s)|, \qquad n, s \in N,$$

where $a(n)$, $b(n)$, and $c(n)$ are real-valued nonnegative functions defined on N such that $M|x_0| \sum_{t=n_0}^{n-1} b(t) \prod_{\tau=n_0}^{t} [1 + a(\tau) + b(\tau)] < 1$, for all $n \in N$, and

$$(22) \qquad \prod_{s=0}^{n-1}\left[1 + a(s) + b(s)\frac{M|x_0| \prod_{t=0}^{m-1} [1 + Ma(t) + c(t)]}{1 - M|x_0| \sum_{t=0}^{m-1} Mb(t) \prod_{\tau=0}^{t} [1 + Ma(\tau) + c(\tau)]}\right] < \infty,$$

for all $n \in N$. Then all solutions of (P^*) are bounded on N.

Proof. Using the variation of constants formula developed by Luca and Talpalaru [7] the solution of (P^*) is represented by

$$(23) \qquad x(n) = R(n)x_0 + \sum_{m=0}^{n-1} R(n-m-1)\left[f(m, x(m)) + x(m)h\left(m, x(m), \sum_{t=0}^{m-1} k(m, t, x(t))\right)\right].$$

Using (18), (19), (20), and (21) in (23) we obtain

$$|x(n)| \leqslant M|x_0| + \sum_{m=0}^{n-1} Ma(m)|x(m)| + \sum_{m=0}^{n-1} Mb(m)|x(m)|\left[|x(m)| + \sum_{t=0}^{m-1} c(t)|x(t)|\right].$$

Now, an application of Theorem 3 yields

$$|x(n)| \leqslant M|x_0| \prod_{m=0}^{n-1}\left[1 + a(m) + b(m)\frac{M|x_0| \prod_{t=0}^{m-1} [1 + Ma(t) + c(t)]}{1 - M|x_0| \sum_{t=0}^{m-1} Mb(t) \prod_{\tau=0}^{t} [1 + Ma(\tau) + c(\tau)]}\right].$$

The above estimation in view of the assumption (22) implies the boundedness of all solutions of (P^*) on N.

It is important to note that Theorem 4 implies not only the boundedness, but also the stability of $x(n)$, if $M|x_0|$ is small enough. However, the above estimation does not prove the asymptotic stability.

Our next theorem shows that under some suitable conditions on the functions f, h, and k in (P^*) and on the resolvent $R(n)$ associated with the linear system (L) all the solutions of (P^*) approach zero as $n \to \infty$.

Theorem 5. *Let the resolvent $R(n)$ associated with the linear system (L) satisfies the condition*

$$(24) \qquad |R(n)| \leqslant Me^{-\alpha n}, \qquad n \in N,$$

where M and α are positive constants. Suppose that the functions f, h, and k in (P^*) satisfy

$$(25) \qquad |f(n, x(n))| \leqslant a(n)|x(n)|, \qquad n \in N,$$

$$(26) \qquad |h(n, x(n), z)| \leqslant b(n)[|x(n)| + |z|], \qquad n \in N,$$

$$(27) \qquad |k(n, m, x(m))| \leqslant e^{-\alpha(n-m-1)} c(m)|x(m)|, \qquad n, m \in N,$$

where $a(n)$, $b(n)$, and $c(n)$ are real-valued nonnegative functions defined on N such that

$$M|x_0| \sum_{t=0}^{n-1} Mb(t)e^{\alpha} \prod_{\tau=0}^{t} [1 + Ma(\tau)e^{\alpha} + c(\tau)e^{\alpha}] < 1 \text{ for all } n \in N \text{ and}$$

$$(28) \qquad \prod_{m=0}^{n-1}\left[1 + Ma(m)e^{\alpha} + Mb(m)e^{\alpha}. M|x_0| \prod_{t=0}^{m-1} [1 + Ma(t)e^{\alpha} + c(t)e^{\alpha}] \right. \left.\left[1 - \sum_{t=0}^{m-1} Mb(t)e^{-\alpha(t-1)} \prod_{\tau=0}^{t} [1 + Ma(\tau)e^{\alpha} + c(\tau)e^{\alpha}]\right]\right] < +\infty,$$

for all $n \in N$. Then all solutions of (P^*) approach zero as $n \to \infty$.

Proof. By using the variation of constants formula the solution of (P^*) is represented by the equation (23). Using (24), (25), (26), and (27) in (23) we obtain

$$|x(n)| \leqslant M|x_0|e^{-\alpha n} + \sum_{m=0}^{n-1} Me^{-\alpha(n-m-1)} a(m)|x(m)| + \sum_{m=0}^{n-1} Me^{-\alpha(n-m-1)} b(m)|x(m)|\left[|x(m)| + \sum_{t=0}^{m-1} e^{-\alpha(m-t-1)}|x(t)|\right].$$

The above inequality can be rewritten as

$$|x(n)|\,e^{\alpha n} \leqslant M|x_0| + \sum_{m=0}^{n-1} Ma(m)\,e^{\alpha}\,|x(m)|\,e^{\alpha m} +$$

$$\sum_{m=0}^{n-1} M\,b(m)\;e^{-\alpha(m-1)}\,|x(m)|\,e^{\alpha m}\left[\,|x(m)|\,e^{\alpha m} + \sum_{t=0}^{m-1} c(t)\,e^{\alpha}\,|x(t)\,e^{\alpha t}\right].$$

Now, applying Theorem 3 with $u(n) = |\,x(n)\,|\,e^{\alpha n}$, then multiplying by $e^{-\alpha n}$, we obtain

$$|x(n)| \leqslant M\,|x_0|\,e^{-\alpha n}\prod_{m=0}^{n-1}\Big[1 + Ma(m)\,e^{\alpha} + Mb(m)\,e^{\alpha}\,.\,M\,|x_0|\prod_{t=0}^{m-1}[1 +$$

$$+ Ma(t)\,e^{\alpha} + c(t)\,e^{\alpha}\Big]\Big/\Big[1 - M|\,x_0|\sum_{t=0}^{m-1} Mb(t)\,e^{-\alpha(t-1)}\prod_{\tau=0}^{t}[1 + Ma(\tau)e^{\alpha} + c(\tau)e^{\alpha}]\Big].$$

The above estimation in view of the assumption (28) yields the desired result as $n \to \infty$ if we choose $M\,|\,x_0\,|$ small enough, and the proof of the theorem is complete.

We say that a function $z(n)$ is *slowly growing* if and only if for every $\alpha > 0$ there exists a constant M, which may depend on α such that $|\,z(n)\,| \leqslant Me^{\alpha n}$, for all $n \in N$.

Our last theorem in this section proves that all the solutions of (P^*) grow more slowly than any positive exponential.

Theorem 6. *Let the resolvent $R(n)$ associated with the linear system (L) satisfies the condition*

(29) $$|R(n)| \leqslant Me^{\alpha n}, \qquad n \in N,$$

where M and α are positive constants. Suppose that the functions f, h, and k in (P^*) satisfy

(30) $$|\,f(n, x(n))\,| \leqslant a(n)\,|\,x(n)\,|, \qquad n \in N,$$

(31) $$|\,h(n, x(n), z)\,| \leqslant b(n)[\,|x(n)| + |\,z\,|], \qquad n \in N,$$

(32) $$|\,k(n, m, x(m))\,| \leqslant e^{\alpha(n-m-1)}c(m)\,|\,x(m)\,|, \qquad n, m \in N,$$

where $a(n)$, $b(n)$, and $c(n)$ are real-valued nonnegative functions defined on N such that

$$M\,|x_0|\sum_{t=0}^{n-1} Mb(t)\,e^{\alpha(t-1)}\prod_{\tau=0}^{t}[1 + Ma(\tau)e^{\alpha} + c(\tau)e^{\alpha}] < 1,$$

for all $n \in N$, and

(33) $$\prod_{m=0}^{n-1}\Big[1 + Ma(m)\,e^{-\alpha} + Mb(m)\,e^{-\alpha}\,.\,M\,|x_0|\prod_{t=0}^{m-1}[1 + Ma(t)e^{-\alpha} + c(t)e^{-\alpha}]\Big/\Big[1 - M\,|x_0|\sum_{t=0}^{m-1} Mb(t)e^{\alpha(t-1)}\prod_{\tau=0}^{t}[1 + Ma(\tau)e^{-\alpha} + c(\tau)\,e^{-\alpha}]\Big] < \infty.$$

for all $n \in N$. Then all solutions of (P^*) are slowly growing.

The proof of this theorem follows by the similar argument as in the proof of Theorem 5 with suitable modifications, and hence we omit the details.

We remark that our results in this section can be modified very easily to cover the case when the perturbation term H in (P) is defined by (14) or (15) by using Theorem 2 or Theorem 1 respectively, with suitable modifications and assumptions on perturbation terms.

In concluding this section, we remark that in [7], Luca and Talpalaru have studied the behavior of solutions of (P) when $f(n, x(n)) = 0$ by using a technique similar to that in the continous case [3, 5, 6]. Very interesting contributions concerning the boundedness and stability of the solutions of (P) when the perturbation term H in (P) is of the form $\sum_{m=0}^{n-1} k(n, m, x(m))$ have been made in a recent paper by B y k o v and L i n e n k o [1] by using a different technique.

4. Further Applications. In this section we are concerned with the relations between the solutions of the perturbed nonlinear systems of sum difference equations of the form

(S) $$x(n) = g(n) + \sum_{m=0}^{n-1} q(n-m-1)[x(m) + f(m, x(m)) + H(m, x(.))],$$

and the solutions of the corresponding unperturbed sum difference system

(U) $$y(n) = g(n) + \sum_{m=0}^{n-1} q(n-m-1)y(m),$$

where $q(n) \in M^p$, for any $n \in N$; x, y, g, and f are the elements of R^p and the functions f and H are as explained in section 3.

Following [7] we call the resolvent of (U) the matrix $r(n)$, that satisfies the equation

(34) $$r(n) = -q(n) + \sum_{m=0}^{n-1} q(n-m-1)r(m), \qquad n \in N.$$

It is known [7], if $r(n)$ is a solution of (34) then the solution $y(n)$ of (U) is given by the equation

(35) $$y(n) = g(n) - \sum_{m=0}^{n-1} r(n-m-1)g(m).$$

Rewrite the equation (S) in the form

(36) $$x(n) = F(n) + \sum_{m=0}^{n-1} q(n-m-1)x(m),$$

where $F(n) = g(n) + \sum_{m=0}^{n-1} q(n-m-1)\;[f(m, x(m)) + H(m, x(.))]$. On applying formula (35) to the equation (36) it follows that $x(n) = F(n) -$

$$-\sum_{m=0}^{n-1} r(n-m-1)F(m) = g(n) + \sum_{m=0}^{n-1} q(n-m-1)\,[f(m, x(m)) + H(m, x(.))] -$$

$$-\sum_{m=0}^{n-1} r(n-m-1)\Big[g(m) + \sum_{s=0}^{m-1} q(m-s-1)[f(s, x(s)) + H(s, x(.))]\Big],$$

since

$$\sum_{m=0}^{n-1} r(n-m-1)\left(\sum_{s=0}^{m-1} q(m-s-1)[f(s, x(s))+H(s, x(.))]\right)=$$

$$=\sum_{k=0}^{n-1} [f(k, x(k)) + H(k, x(.))] \sum_{s=0}^{n-k-2} r(n-k-s-2)q(s)$$

we have

$$x(n)=g(n)-\sum_{m=0}^{n-1} r(n-m-1)g(m)+\sum_{m=0}^{n-1} q(n-m-1)[f(m, x(m))+$$

$$+H(m, x(.))]-\sum_{k=0}^{n-1} [f(k, x(k)) + H(k, x(.))] \sum_{s=0}^{n-k-2} r(n-k-s-2)q(s)=$$

$$=y(n)+\sum_{m=0}^{n-1} [q(n-m-1)-\sum_{s=0}^{n-m-2} r(n-m-s-2)q(s)]\,.\,[f(m, x(m))+$$

$$+H(m, x(\cdot))].$$

Now by using (34) and the fact that

$$\sum_{s=0}^{n-m-2} q(n-m-s-2)r(s)=\sum_{s=0}^{n-m-2} r(n-m-s-2)q(s),$$

we obtain

$$x(n)=y(n)-\sum_{m=0}^{n-1} r(n-m-1)\,[f(m, x(m))+H(m, x(.))], \tag{37}$$

for all $n\in N$.

Before closing this section, we remark that theorems 4—6 though general, admit some further interesting extensions for sum-difference equations of the form (S) when the perturbation term H in (S) is of the form (13) or (14) or (15). These extensions, (when the perturbation term H in (S) is of the form (15)) formulated below as Theorems 7—9, are rather technical and follow the same lines of the technique used in section 3.

Before giving the main results in this section we require the following definitions which are the discrete versions of the definitions recently introduced by the present author in [15].

Definition 2. *The system (U) will be called globally uniformly stable relative to its resolvent $r(n)$, if for any solution $y(n)$ of (U), there exist constants $\delta>0$, $M>0$ such that $|y(n)|\leqslant\delta|x_0|$ and $|r(n)|\leqslant\delta$ for all $n\in N$ and $|x_0|<\infty$.*

Definition 3. *The system (U) will be called exponentially asymptotically stable relative to its resolvent $r(n)$, if for any solution $y(n)$ of (U), there exist constants $\delta>0$, $M>0$, $\alpha>0$ such that $|y(n)|\leqslant\delta|x_0|e^{-\alpha n}$, and $|r(n)|\leqslant Me^{-\alpha n}$ for all $n\in N$ and $|x_0|$ sufficiently small.*

Definition 4. *The system (U) will be called uniformly slowly growing relative to its resolvent $r(n)$, if and only if for every $\alpha>0$ there exist constants $\delta>0$, $M>0$ such that $|y(n)|\leqslant\delta|x_0|e^{\alpha n}$, and $|r(n)|\leqslant Me^{\alpha n}$ for all $n\in N$ and $|x_0|<\infty$.*

We are now prepared to establish our main results on the behavior of solutions of the summary difference equation of the form

$$x(n)=g(n)+\sum_{m=0}^{n-1} q(n-m-1)\Big[x(m)+f(m, x(m))+ \tag{S^*}$$

$$+h\Big(m, x(m), \sum_{t=0}^{m-1} k(m, t, x(t))\Big)\Big],$$

as a perturbation of the linear system (U), with the same initial values $x(0)=y(0)=f(0)=x_0$, where x_0 is a constant.

Theorem 7. *Let the solutions $y(n)$ of (U) be globally uniformly stable relative to its resolvent $r(n)$. Suppose that the functions f, h, and k in (S^*) satisfy the hypotheses (19), (20) and (21) of Theorem 4, where $a(n)$, $b(n)$ and $c(n)$ are real-valued nonnegative functions defined on N such that*

$$\prod_{m=0}^{n-1} [1+Ma(m)]\Big[1+\sum_{m=0}^{n-1} Mb(m)\prod_{t=0}^{m-1} [1+Ma(t)+Mb(t)+ \tag{38}$$

$$+c(t)]\,.\prod_{\tau=0}^{m} [1+Ma(\tau)]^{-1}\Big]<\infty,$$

for all $n\in N$. Then all solutions of (S^*) are bounded on N.

Theorem 8. *Let the solution $y(n)$ of (U) be exponentially asymptotically stable relative to its resolvent $r(n)$. Suppose that the functions f, h, and k in (S^*) satisfy the hypotheses (25), (26), and (27) of Theorem 5, where $a(n)$, $b(n)$, and $c(n)$ are real-valued nonnegative functions defined on N such that*

$$\prod_{m=0}^{n-1} [1+Ma(m)e^{\alpha}]\Big[1+\sum_{m=0}^{n-1} Mb(m)e^{\alpha}\,.\prod_{i=0}^{m-1} [1+Ma(m)e^{\alpha}+ \tag{39}$$

$$+Mb(m)e^{\alpha}+c(m)e^{\alpha}]\prod_{\tau=0}^{m} [1+Ma(\tau)e^{\alpha}]^{-1}\Big]<\infty,$$

for all $n\in N$. Then all solutions of (S^*) approach zero as $n\to\infty$.

Theorem 9. *Let the solution $y(n)$ of (U) be uniformly slowly growing relative to its resolvent $r(n)$. Suppose that the functions f, h, and k in (S^*) satisfy the hypotheses (30), (31), and (32) of Theorem 6, where $a(n)$, $b(n)$, and $c(n)$ are real-valued nonnegative functions defined on N such that*

$$\prod_{t=0}^{n-1} [1+Ma(m)e^{-\alpha}]\Big[1+\sum_{m=0}^{n-1} Mb(m)e^{-\alpha}\,.\prod_{t=0}^{m-1} [1+Ma(m)e^{-\alpha}+ \tag{40}$$

$$+Mb(m)e^{-\alpha}+c(m)e^{-\alpha}]\prod_{\tau=0}^{m} [1+Ma(\tau)e^{-\alpha}]\Big]<\infty,$$

for all $n\in N$. Then all solutions of (S^*) are slowly growing.

The details of the proofs of Theorems 7—9 follow by the similar arguments as in the proofs of Theorems 4—6 by using the variation of constants formula (37) and the discrete inequality given in Theorem 1. We omit the details.

Finally we note that the results obtained in this section can be modified very easily to cover the case when the perturbation term H in (S) is of the form (13) or (14) by using the variation of constants formula given in (37) and the discrete inequalities established in Theorem 3 or Theorem 2.

REFERENCES

1. Bykov J. A. and Linenko V. G. — *On stability of the solutions of difference equations (Russian)*, Differential'nye Uravnenija, 9 (1973), 349—354.
2. Coffman C. V. — *Asymptotic behavior of solutions of ordinary difference equations*, Trans. Amer. Math. Soc. 110 (1964), 22—51.
3. Corduneanu C. — *Integral equations and stability of feedback sytems*, Academic Press, New York and London, 1973.
4. Goldberg S.—*Introduction to Difference Equations*, John Wiley, New York, 1958.
5. Grossman S. I. and Miller R. K. — *Perturbation theory for Volterra integrodifferential systems*, J. Differential Equations, 8 (1970), 457—566.
6. Grossman S. I. and Miller R. K. — *Nonlinear Volterra integrodifferential system with L^1-kernels*, J. Differential Equations 13 (1973), 551—566.
7. Luca N. and Talpalaru P. — *Stability and asymptotic bahavior of a class of discrete systems*, Ann. Math Pura. Appl. 112 (1977), 351—382.
8. Masolockaja L. V. — *Stability of difference equations (Russian)*, Differential'nye Uravnenija 34 (1967), 1976—1983.
9. Pachpatte B.G. — *On the discrete generalizations of Gronwall's inequality*, J. Indian Math. Soc. 37 (1973), 147—156.
10. Pachpatte B. G. — *Finite difference inequalities and their applications*, Proc. Nat. Acad. Sci., India 43 (A) (1973), 348—356.
11. Pachpatte B. G. — *On discrete inequalities related to Gronwall's inequality*, Proc. Indian Acad Sci. 85 (A) 1977), 26—40.
12. Pachpatte B. G. — *Finite difference inequalities and an extension of Lyapunov's method*, Michigan Math. J. 18 (1971), 385—391.
13. Pachpatte B. G. — *On perturbed stochastic discrete systems*, Bull. Austral. Math. Soc. 11 (1974), 385—393.
14. Pachpatte B. G. — *A note on some fundamental discrete inequalities of the Gronwall-Bellman type*, Bull. Inst. Math.Acad. Sinica 5 (1977), 121—128.
15. Pachpatte B. G. — *Stability and asymptotic behavior of perturbed Volterra integral equations*, J. Math. Physical Sci. 10 (1976), 519—533.
16. Petrovanu D. —*Equations Hammerstein integrales et discrètes*, Ann. Mat. Pura Appl. 70 (1966), 227—254.

Received 7.X.1977

Department of Mathematics and Statistics
Marathwada University Aurangabad
(Maharashtra), India

Analele ştiinţifice ale Universităţii „Al. I. Cuza" Iaşi
Tomul XXIV, f. 2, s. I a, 1978

ALMOST AFFINE CES SPAOVER A FIELD

BY

RADU MIRON

1. Introduction. It is known that usual system of axioms of a vector spaces is not independent, [15, 16]. Similary, neither H. Weyl's axioms, [25], nor P. K. Raschewsky's axioms, [21], characterizing the concept of point-wise affine space, are not minimal, [17, 18]. Investigating, together with F. Radó, [19], the problem of independence of these axioms I come across a natural generalization of both vector and affine spaces, [15, 16], which I consider has a legitimate claim to be studied.

Let M be a classical affine space over a skew-field K and 1 be the identity of K. We drop the axioms $x + y = y + x$, $\forall x, y$ and $1 * x = x$, $\forall x$, for the associate vector space TM of the K-affine space M, we obtain a new structure which is called a structure of K-almost affine space and TM is called the associated K-almost linear space of M. One introduces the concepts of maximal zero-almost linear subspace, zTM and maximal unitary-almost linear subspace, uTM, in the K-almost linear space TM. One shows that zTM is a normal subspace, TM is decomposable as semi-direct sum $TM = zTM + uTM$ and it is stated the isomorphism $TM/zTM \cong uTM$. But the subspace uTM has a structure of a K-linear space.

Then, in the K-almost affine space M one introduces the notion of foliation and proves that M has two foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ which have as director spaces zTM and uTM.

We have the following three fundamental theorems:

I. Every K-almost affine space M is generated by a set $M_0 \cup M_1$, where M_0 is a leaf of $\mathcal{F}_0$ and M_1 is a leaf of $\mathcal{F}_1$.

II. The quotient K-almost affine space $M/\mathcal{F}_0$ is isomorphic to a leaf $M_1 \in \mathcal{F}_1$ and M_1 is a K-Weylean affine space.

III. If the foliation $\mathcal{F}_1$ is a normal one, then $M = M_0 \times M_1$, $M_0 \in \mathcal{F}_0$, $M_1 \in \mathcal{F}_1$.

All this construction can be given when K is an unitary ring.

I have exposed these problems at Prof. Dr. M. Barner and Prof. Dr. F. Radó's seminars at Universities of Freiburg, 1975, and Cluj-Napoca, 1976. I have benefited very much from the discussions which I had with Prof Dr. G. Ewald at University of Bochum. I will take this opportunity to thank them.