Finally we note that the results obtained in this section can be modified very easily to cover the case when the perturbation term H in (S) is of the form (13) or (14) by using the variation of constants formula given in (37) and the discrete inequalities established in Theorem 3 or Theorem 2.

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ALMOST AFFINE CES SPAOVER A FIELD

BY

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1. Introduction. It is known that usual system of axioms of a vector spaces is not independent, [15, 16]. Similary, neither H. Weyl's axioms, [25], nor P. K. Raschewsky's axioms, [21], characterizing the concept of point-wise affine space, are not minimal, [17, 18]. Investigating, together with F. Radó, [19], the problem of independence of these axioms I come across a natural generalization of both vector and affine spaces, [15, 16], which I consider has a legitimate claim to be studied.

Let M be a classical affine space over a skew-field K and 1 be the identity of K. We drop the axioms $x + y = y + x$, $\forall x, y$ and $1 * x = x$, $\forall x$, for the associate vector space TM of the K-affine space M, we obtain a new structure which is called a structure of K-almost affine space and TM is called the associated K-almost linear space of M. One introduces the concepts of maximal zero-almost linear subspace, zTM and maximal unitary-almost linear subspace, uTM, in the K-almost linear space TM. One shows that zTM is a normal subspace, TM is decomposable as semi-direct sum $TM = zTM + uTM$ and it is stated the isomorphism $TM/zTM \xrightarrow{\sim} uTM$. But the subspace uTM has a structure of a K-linear space.

Then, in the K-almost affine space M one introduces the notion of foliation and proves that M has two foliations $\mathcal{F}_0$ and $\mathcal{F}_1$ which have as director spaces zTM and uTM.

We have the following three fundamental theorems:

I. Every K-almost affine space M is generated by a set $M_0 \cup M_1$, where M_0 is a leaf of $\mathcal{F}_0$ and M_1 is a leaf of $\mathcal{F}_1$.

II. The quotient K-almost affine space $M/\mathcal{F}_0$ is isomorphic to a leaf $M_1 \in \mathcal{F}_1$ and M_1 is a K-Weylean affine space.

III. If the foliation $\mathcal{F}_1$ is a normal one, then $M = M_0 \times M_1$, $M_0 \in \mathcal{F}_0$, $M_1 \in \mathcal{F}_1$.

All this construction can be given when K is an unitary ring.

I have exposed these problems at Prof. Dr. M. Barner and Prof. Dr. F. Radó's seminars at Universities of Freiburg, 1975, and Cluj-Napoca, 1976. I have benefited very much from the discussions which I had with Prof Dr. G. Ewald at University of Bochum. I will take this opportunity to thank them.

Chapter I. ALMOST AFFINE SPACES

In this chapter we define the notion of K-almost affine space and Weylean K-almost affine space.

§ 1. **First group of axioms. Torsion. Translations.** The notion of K-almost affine space is obtained from the notion of K-affine space by renouncing to the last axiom of the second group of axioms, [15]. We formulate this notion in detail.

Let M be a set whose elements $A, B, C, \ldots$ are called points. Consider the set $M \times M$. Every element $(A, B) \in M \times M$ is called an oriented segment of M whose origin is A and extremity is B. One denotes $(A, B) = \overline{AB}$.

Let ρ be an equivalence relation on the set $M \times M$, $TM = (M \times M)/\rho$ the quotient set with respect to ρ and $p: M \times M \to TM$ the natural projection. We denote $p(\overline{AB}) = \overrightarrow{AB}$ and we say that $\overrightarrow{AB}$ is the vector of the set M determinated by the oriented segment $\overline{AB}$ with respect to the equivalence ρ. We shall denote vectors by $\overrightarrow{AB}, \overrightarrow{CD}, \ldots, x, y, z, \ldots$ etc.

Obviously: $\overline{AB}\,\rho\,\overline{CD} \Leftrightarrow \overrightarrow{AB} = \overrightarrow{CD}$.

Conversely, if the set TM of vectors is given together with a surjection $p: M \times M \to TM$, then the equivalence relation ρ is well determined by $\overline{AB}\,\rho\,\overline{CD} \Leftrightarrow p(\overline{AB}) = p(\overline{CD})$.

Let K be a skew-field and $1 \in K$ its identity.

Definition 1.1. *We call a structure of K-almost afine space on the nonempty set M the following data:*

a. *An equivalence relation ρ on the set $M \times M$, which defines the quotient set $TM = (M \times M)/\rho$ of vectors of M.*

b. *A mapping*

$$*: K \times TM \to TM,\ *(\alpha, x) = \alpha * x, \quad \forall \alpha \in K, \forall x \in TM.$$

c. *The first and second group of axioms are verified (see below).*

Definition 1.2. *The pair $(M, \mathcal{A})$, where $\mathcal{A}$ is a structure of the K-almost affine space on the nonempty set M is called a K-almost affine space, in short: K-a.a.s.*

Evidently, from $M \neq \emptyset$ it follows $TM \neq \emptyset$.

First group of axioms.

I_1. $\forall A \in M, \forall x \in TM, \exists B \in M: \overrightarrow{AB} = x$.

I_2. $\forall A, B, A' \in M: \overrightarrow{AB} = \overrightarrow{A'B} \Rightarrow A = A'$.

I_3. $\forall A, B, C, A', B', C' \in M: \overrightarrow{AB} = \overrightarrow{A'B'},\ \overrightarrow{BC} = \overrightarrow{B'C'} \Rightarrow \overrightarrow{AC} = \overrightarrow{A'C'}$.

The axiom I_3 is called by S. Golab, [9], „Miron-Opaiț's axiom" or „triangle axiom".

Some immediate consequences:

Proposition 1.1. (i) $\forall A, A' \in M: \overrightarrow{AA} = \overrightarrow{A'A'}$.

(ii) $\forall A, B, A', B' \in M: \overrightarrow{AB} = \overrightarrow{A'B'} \Rightarrow \overrightarrow{BA} = \overrightarrow{B'A'}$.

(iii) $\forall A \in M, \forall x \in TM, \exists_1 B \in M: \overrightarrow{AB} = x$.

(iv) $\forall A, B, B' \in M: \overrightarrow{AB} = \overrightarrow{AB'} \Rightarrow B = B'$.

The vector $\overrightarrow{AA}$ is called the nul vector of M and is denoted by $\vec{O}$.

The vector $\overrightarrow{BA}$ is called the opposite of the vector $\overrightarrow{AB}$ and is denoted $-\overrightarrow{AB}$.

Let $(x, y) \in TM \times TM$ and $A \in M$. Then $\exists_1 B \in M: \overrightarrow{AB} = x$ and $\exists_1 C \in M: \overrightarrow{BC} = y$. The vector $\overrightarrow{AC} = z$ does not depend on the point A, (axiom I_3). It is called the sum of the vectors x and y. One writes $z = x + y$.

Definition 1.3. *The mapping $+: (x, y) \in TM \times TM \to x + y \in TM$ $\forall x, y$ is called the addition operation of vectors of M.*

We have $z = x + y$, or

$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}, \qquad x = \overrightarrow{AB}, y = \overrightarrow{BC}, \ \forall A \in M \tag{1.1}$$

Theorem 1.1. *The pair $(TM, +)$ is a group.*

Proof. $\forall x, y, z \in TM: x = \overrightarrow{AB},\ y = \overrightarrow{BC},\ z = \overrightarrow{CD},\ x + (y + z) = \overrightarrow{AB} + (\overrightarrow{BC} + \overrightarrow{CD}) = \overrightarrow{AD} = \overrightarrow{AC} + \overrightarrow{CD} = (\overrightarrow{AB} + \overrightarrow{BC}) + \overrightarrow{CD} = (x + y) + z$; $x + \vec{O} = \overrightarrow{AB} + \overrightarrow{BB} = \overrightarrow{AB} = x$; $x + -x = \overrightarrow{AB} + \overrightarrow{BA} = \overrightarrow{AA} = \vec{O}$. q.e.d.

One denotes $x + -y$ by $x - y$.

Generally the group $(TM, +)$ is not abelian.

Definition 1.4. *We call torsion of a K-a.a.s, M, the mapping $\tilde{t}: (x, y) \in TM \times TM \to -(x + y) + (y + x) \in TM$.*

Proposition 1.2. *The group $(TM, +)$ is abelian iff the torsion of M vanishes.*

Proposition 1.3. *If $x = \overrightarrow{AB} = \overrightarrow{CD'}$, $y = \overrightarrow{AC} = \overrightarrow{BD}$, then $\tilde{t}(x, y) = \overrightarrow{DD'}$.*

Proposition 1.4. $\forall A, B, A', B' \in M: \{\overrightarrow{AB} = \overrightarrow{A'B'} \Rightarrow \overrightarrow{BB'} = \overrightarrow{AA'}\} \Leftrightarrow \{\tilde{t}(x, y) = \vec{O},\ \forall x, y \in TM\}$.

Definition 1.5. *The mapping $t_x: M \to M$, for which $t_x(A) = A'$, $x = \overrightarrow{AA'}$ is called vector x translation.*

Let $\mathcal{T}$ be the set of translations. The following properties are immediatly:

Proposition 1.5. (i) *Every $t_x \in \mathcal{T}$ is a bijection.*

(ii) $\forall t_x, t_y \in \mathcal{T}: t_x(A) = A, \forall A \in M \Leftrightarrow x = \vec{O}$; $t_x(A) = A'$, $t_y(A') = A$, $\forall A \in M \Leftrightarrow x = -y$.

Proposition 1.6. *Every $t_x \in \mathcal{T}$ is well determined by a fixed point $A_0 \in M$ and by its image $A_0' = t_x(A_0)$.*

Proposition 1.7. $t_y \circ t_x = t_{x+y}$, $t_x^{-1} = t_{-x}$, $\forall t_x, t_y \in \mathcal{T}$.

Theorem 1.2. *$(\mathcal{T}, \circ)$ is a group isomorphic to the group $(M, +)$.*

Proposition 1.8. $\overline{AA'}\,\rho\,\overline{BB'} \Rightarrow \overline{t_x(A)t_x(A')}\,\rho\,\overline{t_x(B)t_x(B')}$.

Indeed, evidently $\overrightarrow{At_x(A)} = x$, $\forall A \in M$. Therefore $\overrightarrow{t_x(A)\,t_x(A')} = \overrightarrow{t_x(A)\,A} + \overrightarrow{AA'} + \overrightarrow{A't_x(A')} = -x + \overrightarrow{AA'} + x = \overrightarrow{t_x(B)\,B} + \overrightarrow{BB'} + \overrightarrow{B't_x(B')} = \overrightarrow{t_x(B)\,t_x(B')}$.

Based on this property we can consider the mapping:

$Tt_x : TM \to TM$, given by $Tt_x(\overrightarrow{AA'}) = \overrightarrow{t_x(A)\,t_x(A')}$, $\forall A \in M$, $\forall \overrightarrow{AA'} \in TM$.

Proposition 1.9. $Tt_x(y + z) = Tt_x(y) + Tt_x(z)$, $\forall y, z \in TM$.

Theorem 1.3. *$Tt_x(y) = y$, $\forall y \in TM$ if and only if M has $\tilde{t}(x, y) = \vec{O}$, $\forall x, y \in TM$.*

Proof. Let $A, A', B, B' \in M$ be four points such that $\overrightarrow{AA'} = \overrightarrow{BB'}$. We consider the translation t_x, with $x = \overrightarrow{AA'}$. Then $t_x(A) = A'$, $t_x(B) = B'$. If $Tt_x(y) = y$, $\forall y \in TM$, then $\overrightarrow{AB} = Tt_x(\overrightarrow{AB}) = \overrightarrow{t_x(A)\,t_x(B)} = \overrightarrow{A'B'}$. Therefore $\overrightarrow{AA'} = \overrightarrow{BB'} \Rightarrow \overrightarrow{AB} = \overrightarrow{A'B'}$ and the proposition 1.4 shows $\tilde{t} = 0$. Conversely, if $\tilde{t} = O$, then $\overrightarrow{AB} = \overrightarrow{A'B'} \Rightarrow \overrightarrow{AA'} = \overrightarrow{BB'}$ and $Tt_x(\overrightarrow{AB}) = \overrightarrow{t_x(A)\,t_x(B)}$. Denoting by $A' = t_x(A)$, $B' = t_x(B)$, we have $\overrightarrow{AA'} = \overrightarrow{BB'} = x$ and $\overrightarrow{AB} = \overrightarrow{A'B'}$. So that $Tt_x(\overrightarrow{AB}) = \overrightarrow{AB}$, $\forall \overrightarrow{AB} \in TM$.

§ 2. Second group of axioms. Weylean almost affine spaces. Noncontradiction and independence of axioms. In the second group of axioms the properties of compatibility of the mapping $*$ with the opperations $+$ and $\cdot$ from K and TM are postulated.

Second group of axioms.

If the opperation $+$ is given by the definition 1.3, then:

II_1. $\forall \alpha \in K$, $\forall x, y \in TM : \alpha*(x + y) = \alpha*x + \alpha*y$.

II_2. $\forall \alpha, \beta \in K$, $\forall x \in TM : (\alpha + \beta)*x = \alpha*x + \beta*x$.

II_3. $\forall \alpha, \beta \in K$, $\forall x \in TM : \alpha*(\beta*x) = (\alpha \cdot \beta)*x$.

With this, one finishes the list of axioms of a structure of K-almost affine space.

Consequences of axioms from the first and second groups.

Proposition 2.1. $\forall \alpha, \beta \in K$, $\forall x, y \in TM$:

$$\alpha*(x - y) = \alpha*x - \alpha*y \,;\quad \alpha*\vec{O} = \vec{O} \,;\quad \alpha*(-x) = -\alpha*x.$$

$$(\alpha - \beta)*x = \alpha*x - \beta + x \,;\quad O*x = \vec{O} \,;\quad (-\alpha)*x = -\alpha*x.$$

The theorem 1.1 and the second group of axioms show that $(TM, +)$ is a K-almost linear space,(15), in the sense that it contains all the axioms of an usual linear space over the field K, less the axioms $x + y = y + x$ and $1*x = x$. $(TM, +, *)$, in short TM is called the associated K-almost linear space (or tangent almost vector space) of the K-almost affine space M.

Theorem 2.1. *If the K-a.a.s, M, has the property: $\forall x \in TM/\{\vec{O}\}$: $1*x \neq \vec{O}$, then:*

(i) $1*x = x$, $\forall x \in TM$.

(ii) $x + y = y + x$, $\forall x, y \in TM$.

Proof. (i) We have $1*(x - 1*x) = 1*x - (1.1)*x = 1*x - 1*x = \vec{O}$. Therefore $x - 1*x = \vec{O}$. Then $x = 1*x$, $\forall x \in TM$.

(ii) $(1 + 1)*(x + y) = 1*(x + y) + 1*(x + y) = (x + y) + (x + y) = (1 + 1)*x + (1 + 1)*y = (1*x + 1*x) + (1*y + 1*y) = (x + x) + (y + y)$. But $(x + x) + (y + y) = (x + y) + (x + y) \Rightarrow x + y = y + x$.

Theorem 2.2. *If the K-a.a.s., M, has the property that the mapping $*: K \times TM \to TM$ is surjective, then:*

(i) $1*x$, $\forall x \in TM$.

(ii) $x + y = y + x$, $\forall x, y \in TM$.

Proof. $\forall x \in TM$, $\exists \alpha \in K$, $\exists x_1 \in TM : x = \alpha*x_1$. The $1*x = 1*(\alpha*x_1) = (1 \cdot \alpha)*x_1 = \alpha*x_1 = x$. In this condition of course $x + y = y + x$ holds.

We will show that there exists K-almost affine spaces M for which $x + y \neq y + x$, $x, y \in TM$. Indeed they are the spaces with torsion.

Definition 2.1. *We call a K-Weylean almost affine space structure on the nonempty set M the following data:*

a. *An equivalence relation ρ on the set $M \times M$, which defines the quotient set $TM = (M \times M)/\rho$ of vectors of M.*

b. *A mapping*

$$*: K \times TM \to TM, \quad *(\alpha, x) = \alpha*x, \quad \forall \alpha \in K, \ \forall x \in TM.$$

c. *The axioms I_1, I_2 of the first group, the axiom:*

$$I'_3. \ \forall A, B, A'B' \in M : \overrightarrow{AB} = \overrightarrow{A'B'} \Rightarrow \overrightarrow{AA'} = \overrightarrow{BB'}$$

and the axioms II_1, II_2, II_3 of the second group, are verified.

The axiom I'_3 is called „Raschewsky's axiom" or „the parallelogram axiom"

Theorem 2.3. *Every K-Weylean almost affine space M is a K-almost affine space.*

Proof. $\overrightarrow{AB} = \overrightarrow{A'B'}$, $\overrightarrow{BC} = \overrightarrow{B'C'}$ and the axiom I'_3 imply $\overrightarrow{AA'} = \overrightarrow{BB'}$, $\overrightarrow{BB'} = \overrightarrow{CC'}$. From $\overrightarrow{AA'} = \overrightarrow{CC'}$ and the axiom I'_3, it follows $\overrightarrow{AC} = \overrightarrow{A'C'}$. Therefore: $\forall A, B, C, A', B', C' \in M : \overrightarrow{AB} = \overrightarrow{A'B'}$, $\overrightarrow{BC} = \overrightarrow{B'C'} \Rightarrow \overrightarrow{AC} = \overrightarrow{A'C'}$ which is the axiom I_2. q.e.d.

Therefore, the triangle axiom is verified by vertue of the parallelogram axiom.

The proposition 1.2 and 1.4 show:

Theorem 2.4. *A K-almost affine space, M, is Weylean if and only if its torsion vanishes.*

Consequently:

Theorem 2.5. (i) *If the K-a.a.s., M, has the property $\forall x \in TM/\{\vec{O}\}$: $1*x \neq \vec{O}$, then it is Weylean.*

(ii). *If the mapping* $*: K \times TM \to TM$ *of the K-a.a.s., M is surjective, then it is Weylean.*

In the paper [16], we have showed that the axioms of an affine spaces over an unitary ring Γ are noncontradictory and independent.

But the axioms of a K-almost affine space form a subsystem of the system of axioms of K-affine space. It follows:

Theorem 2.6. *Let K be a given field. The axioms of the first and second group of axioms for a K-almost affine space are noncontradictory and independent.*

On the same way, using Hilbert's method of models one proves:

Theorem 2.7. *Let K be a given field. The axioms of the K-Weylean almost affine space are noncontradictory and independent.*

§ 3. **Associated almost linear space.** The associated K-almost linear space TM of M has a number of interesting properties established in the papers of R. M i r o n, F. R a d ó, N. G h e o r g h i u and D. B r â n z e i, [15, 20, 8, 2]. We give, without prove some of these.

M is called „K-almost affine trivial space" if $(TM, +)$ is the trivial group, $TM = \{\vec{O}\}$; it is called „K-zero almost affine space" if $\alpha * x = \vec{O}$ $\forall \alpha \in K$, $\forall x \in TM$; M is called „K-unitary almost affine space" if $1 * x = x$, $\forall x \in TM$.

Proposition 3.1. (i) *M is a K-almost affine trivial space iff M is formed by a single point.*

(ii) *If M is a K-unitary almost affine space then it is a K-affine space.*

Let $\emptyset \neq TM_1 \subseteq TM$ be an almost linear subspace. It is called: *almost trivial subspace if* $TM_1 = \{\vec{O}\}$; *zero — almost linear subspace if* $\forall x \in TM_1$ $\forall \alpha \in K : \alpha * x = \vec{O}$; *unitary almost linear subspace* if $1 * x = x$, $\forall x \in TM_1$.

A zero-almost subspace (unitary almost subspace) TM_1 is called *maximal* if for every zero-almost subspace (unitary almost subspace) TM_2 we have $TM_2 \subset TM_1$.

We denote by zTM the maximal zero-almost subspace and by uTM the maximal unitary almost subspace of TM.

Proposition 3.2. (i) *In the K-almost linear space TM there exists a single zTM and it is formed by all elements* $x \in TM$ *with the property* $1^* x = \vec{O}$ *and only these elements.*

(ii) *In TM there exists a single uTM and it contains all elements* $x \in TM$ *for which* $\exists y \in TM : x = 1 * y$ *and only these elements.*

An almost linear subspace TM_1 of TM is called *normal* if the subgroup $(TM_1, +)$ is a normal subgroup of the group $(TM, +)$.

Theorem 3.3. (Miron). (i) *zTM is an almost normal subspace of TM.*

(ii) *uTM is a K-linear space.*

(iii) *K-almost linear space TM is the semi-direct sum :* $TM = zTM + uTM$.

(iv) *The quotient K-almost linear space TM/zTM is a K-linear space isomorphic to uTM.*

From this theorem it results:

Proposition 3.4. *TM, zTM, uTM have the properties:*

(i) $zTM \cap uTM = \{\vec{O}\}$.

(ii) $\forall x \in TM$ *is uniquely written in the form:*

$$x = x_0 + x_1,\ x_0 \in zTM,\ x_1 \in uTM.$$

(iii) *The mapping* $P : TM \to uTM$, *defined by* $P(x) = 1 * x$ *is a projection on the maximal unitary almost subspace uTM.*

Proposition 3.5. *Let* $\{TM_i\}_{i \in I}$ *be a familly of almost linear subspaces of TM. Then* $\bigcap_{i \in I} TM_i$ *is an almost linear subspace of TM.*

Let $\mathcal{M}$ be a nonempty subset of TM. The intersection TM' of all almost linear subspaces of TM which contain $\mathcal{M}$ is caled the almost linear subspace generated by the set $\mathcal{M}$.

Proposition 3.6. *Let $\mathcal{M}$ be a nonempty subset of TM and $\mathcal{M}'$ be the set of finite sums of elements from* $\mathcal{M}, -\mathcal{M}$ *and* $K * \mathcal{M}$. *Then:*

(i). *$\mathcal{M}'$ is an almost linear subspace of TM.*

(ii). *$\mathcal{M}'$ coincides with the almost linear subspace TM' generated by $\mathcal{M}$.*

If $\mathcal{M} \subset zTM$, then TM' is a zero-almost linear subspace.

If $\mathcal{M} \subset uTM$, then TM' is an unitary almost linear subspace.

For example, TM is generated by the set $zTM \cup uTM$.

The notions of homomorphism and isomorphism of K-almost linear spaces are the usual.

Let $F : TM_1 \to TM_2$ be a homomorphism and $TM_1' \subseteq TM_1$, then $F(TM_1')$ is an almost linear subspace of TM_2. Of course Im F is an almost linear subspace of TM_2.

Proposition 3.7. *Let* $F : TM_1 \to TM_2$ *be a homomorphism of K-almost linear spaces.*

(i) *If* $TM_1' \subset zTM_1$, *then* $F(TM_1') \subset zTM_2$.

(ii) *If* $TM_1' \subset uTM_1$, *then* $F(TM_1') \subset uTM_2$.

Let $TM_2' \subset TM_2$ and $F : TM_1 \to TM_2$ a homomorphism of K-almost linear spaces. We have $F^{-1}(TM_2') \subset TM_1$.

Proposition 3.8. *Let* $F : TM_1 \to TM_2$ *be a homomorphism of K-almost linear spaces and* $TM_2' \subset TM_2$.

(i) *If TM_2' is a normal almost linear space, then* $F^{-1}(TM_2')$ *is a normal almost linear subspace of* TM_1.

(ii) *Ker F is a normal almost linear subspace.*

One proves, without dificulties, the two following theorems:

Theorem 3.4. *If* $F : TM_1 \to TM_2$ *is a homomorphism of K-almost linear spaces, then we have the isomorphism* $TM_1/\text{Ker}\, F \simeq \text{Im}\, F$.

Theorem 3.5. *Let* $F : TM_1 \to TM_2$ *be a homomorphism of the K-almost linear spaces. Then there exists a single homomorphism of the quotient K-almost linear spaces*

$$\tilde{F} : TM_1/zTM_1 \rightleftarrows uTM_1 \to TM_2/zTM_2 \rightleftarrows uTM_2$$

such that the following diagramme is commutative:

$$\begin{tikzcd} TM_1 \arrow[r, "F"] \arrow[d, "p_1"'] & TM_2 \arrow[d, "p_2"] \\ TM_1/zTM_1 \arrow[r, "\widetilde{F}"] & TM_2/zTM_2 \end{tikzcd}$$

The K-almost linear spaces as objects and the homomorphisms of the K-almost linear spaces as morphisms form a category, the composition of morphisms being the ordinary composition of mappings.

Others results concerning K-almost linear spaces.

Theorem 3.6. *A K-almost linear space TM is the direct product $TM = zTM \times uTM$ iff uTM is a normal almost linear subspace of TM.*

Theorem 3.7. (Liebeck). *For every K-almost linear space TM, for which the group $(TM, +)$ is commutative, we have the direct sum $TM = zTM \oplus uTM$.*

Now, we must see in what case one can construct a K-almost linear space TM when zTM and uTM are given.

Theorem 3.7. (Radó). *Let zTM a zero K-almost linear space and uTM a K-linear space. Then for every $0 \in$ Hom $(uTM$, Aut $(zTM))$ there exists the K-semidirect sum $zTM +_0 uTM = TM$. Its maximal zero-almost linear subspace is i_1 (zTM) and its maximal unitary linear subspace is i_2 (uTM). Up to K-isomorphisms, every K-almost linear space is obtained in this way.*

Here, $TM = zTM +{}_0uTM$ is given by the set $\{(x, y) \mid x \in zTM, y \in uTM\}$ and $(x, y) + (x', y') = (x + 0\,(y)\,(x'), y + y')$; $\alpha * (x, y) = (\alpha * x, \alpha * y)$. $i_1 : zTM \to TM$ is defined by $i_1\,(x) = (x, 0)$ and $i_2 : uTM \to TM$ by $i_2\,(y) = (0, y)$.

A consequence of this theorem:

Proposition 3.9. *There exists K-almost linear spaces TM which do not reduce to direct sum of zTM and uTM.*

Some application to K-almost affine spaces:

Proposition 3.10. (i) *The sets M, TM and $TM_{A_0} = \{\overrightarrow{A_0 A} \mid A_0 \in M, \forall A \in M\}$ are equivalent in category* Ens.

(ii) *The set TM_{A_0} is a system of generators for TM.*

Theorem 3.8. *Let TN be a K-almost linear space and M be a set with the property that TN and M are equivalent in the category* Ens. *In these conditions there exists a structure of K-almost affine space on the set M such that the associated K-almost linear space TM is isomorphic to TN.*

Indeed, if $\varphi : M \to TN$ is a bijection and $A, B, C, D \in M$, $x_1 = \varphi\,(A)$, $x_2 = \varphi\,(B)$, $x_2' = \varphi\,(C)$, $x_1' = \varphi\,(D)$, we put $\overline{AB}\,\rho\,\overline{CD} \Leftrightarrow -x_1 + x_2 = -x_1' + x_2'$. Then ρ is an equivalence. Consider the quotient set $TM = (M \times M)/\rho$ and the mapping $T\varphi : TM \to TN$, $T\varphi\,(\overline{AB}) = -x_1 + x_2$. It follows that $T\varphi$ is a bijection. So we can transfer on the set TM the structure of K-almost linear space of TN. Therefore $T\varphi$ is an isomorphism. There are verified, for M the axioms from both groups. q.e.d.

Remark. Later we shall see that this K-almost affine structure on the set M is unique up to isomorphisms of K-almost affine spaces.

Corollary 3.1. *On the set M' of a K-almost linear space TM there exists a structure of K-almost affine space such that its associated K-almost linear space TM' coincides with TM.*

Corollary 3.2. *There exists a K-almost affine space M with the property that its associated K-almost linear space TM has zTM and uTM given apriori.*

§ 4. Almost affine subspaces. Let M be a K-almost affine space and M^* be a subset of M. Consider the set of vectors.

$$TM^* = \{\overrightarrow{AB} \in TM \mid \forall A, B \in M^*\}. \tag{4.1}$$

Definition 4.1. *The subset M^* of M is called an almost affine subspace, in short a.a.s.s. if the following conditions are satisfied:*

a. *The restriction of the mapping $*$ to $K \times TM^*$ is valued in TM^*;*

b. *For the pair (M^*, TM^*) both groups of axioms are verified.*

Then TM^* is the associated almost linear space of M^* and if $M^* \neq \varnothing$, then TM^* is a K-almost linear subspace of TM.

Consider the empty set an a.a.s.s. for every almost affine space M.

Proposition 4.1. *Let M^* be a subset of the K-a.a.s. M and the set TM^* given by (4.1). M^* is an a.a.s.s. of M iff:*

(i) $\exists A_0 \in M^*, \forall x \in TM^*, \exists B \in M^* : \overrightarrow{A_0 B} = x.$

(ii) *TM^* is an almost linear subspace of TM.*

Proof. Suppose M^* is an a.a.s.s. of M and $TM^* \subset TM$. Obviously (i) and (ii) are verified. Conversely, it follows from (i) and (ii) that the second group of axioms are verified. Let $A \in M^*$ be a point and $x \in TM^*$ an arbitrary vector. From the condition (i) it results that there exists a point $A_0 \in M^* : \forall y \in TM^*, \exists B \in M^*$ such that $\overrightarrow{A_0 B} = y$. Taking $y = \overrightarrow{A_0 A} + x$ we have $\overrightarrow{A_0 B} = \overrightarrow{A_0 A} + x$ or $-\overrightarrow{A_0 A} + \overrightarrow{A_0 B} = \overrightarrow{AB} = x$ and I_1 is verified. In the same way the axioms I_2 are proved.

Proposition 4.2. *Let $A_0 \in M$ be a fixed point and TM^* be an almost linear subspace of TM. Then $M^* = \{A \in M \mid \forall \overrightarrow{A_0 A} \in TM^*\}$ is an almost affine subspace of M with TM^* as associated almost linear space.*

The proof is easy.

Theorem 4.1. *There exists a single a.a.s.s. M^* of M which contains a fixed point $A_0 \in M$ and has a given associated almost linear space $TM^* \subset TM$.*

One says that M^* from the last theorem is determinated by the point A_0 and by the *director* almost linear subspace TM^*.

Theorem 4.2. *The intersection $M_1 \cap M_2$ of two a.a.s.s. of M is an a.a.s.s. of M. If $M_1 \cap M_2 \neq \varnothing$ then $T\,(M_1 \cap M_2) = TM_1 \cap TM_2$.*

This property is valable and for a familly of a.a.s.s.

Proof. If $M_1 \cap M_2 = \varnothing$, then it is an a.a.s.s of M. If $M_1 \cap M_2 \neq \varnothing$, let $A_0 \in M_1 \cap M_2$ be a fixed point and M^* be the a.a.s.s through A_0 having $TM_1 \cap TM_2$ as director subspace. Then $\forall A \in M_1 \cap M_2 \Rightarrow \overrightarrow{A_0 A} \in TM_1 \subset TM_2 \Rightarrow$ $\Rightarrow A \in M^*$. Conversely, $\forall B \in M^* \Rightarrow \overrightarrow{A_0 B} \in TM_1 \cap TM_2 \Rightarrow \overrightarrow{A_0 B} \in$

$\in TM_1$, $\overrightarrow{A_0B} \in TM_2 \Rightarrow B \in M_1 \cap M_2$. Therefore $M_1 \cap M_2 = M^*$ and $T(M_1 \cap M_2) = TM_1 \cap TM_2$.

§ 5. **Parallelism. Foliations.** Let M_i, $i \in I$ be the a.a.s.s of K-almost affine space M.

Definition 5.1. *We say that the a.a.s.s. $M_1 \neq \emptyset$ is parallel to the almost affine subspace M_2 if $TM_1 \subset TM_2$ and we denote this $M_1 \parallel M_2$.*

Proposition 5.1. *The relation $\parallel$ is reflexive and transitive.*

Theorem 5.1. *Let $M_2 \neq \emptyset$ an a.a.s.s of M and $TM_1 \subset TM_2$ be. Then, through each point $A \in M$ it passes a unique a.a.s.s. M_1 parallel to M_2 and with TM_1 as a director subspace.*

Proposition 5.2. *If $M_1 \parallel M_2$, then one and only one of the relations $M_1 \subset M_2$ or $M_1 \cap M_2 = \emptyset$ is true.*

Indeed, if $M_1 \cap M_2 \neq \emptyset$, then $T(M_1 \cap M_2) = TM_1 \cap TM_2 = TM_1$. Taking $A_0 \in M_1 \cap M_2$, $\forall A \in M_1 : \overrightarrow{A_0A} \in TM_1 = TM_1 \cap TM_2 \Rightarrow A \in M_2$. Therefore $M_1 \subset M_2$, etc.

Definition 5.2. *A familly $\mathcal{F} = (M_i)_{i \in I}$ of a.a.s.s. of M is called a foliation of M if: a) $\forall i \in I$, $M_i \neq \emptyset$; b) $\forall A \in M$. $\exists i_0 \in I : A \in M_{i_0}$; c) $\forall i, j \in I : M_i \parallel M_j$. An element $M_i \in \mathcal{F}$ is called a leaf of the foliation $\mathcal{F}$.*

For every foliation $\mathcal{F}$ we have: $TM_i = TM_j$, $\forall i, j \in I$; $M_i \neq M_j \Rightarrow M_i \cap M_j = \emptyset$; $\forall A \in M$, $\exists M_i : A \in M_i$. Then:

Theorem 5.2. *There exists a single foliation $\mathcal{F}$ in M with the property that all its leafs have the same director subspace $TM' \subset TM$.*

A foliation $\mathcal{F} = (M_i)_{i \in I}$ for which $TM_i = \{\vec{0}\}$ is called a trivial foliation. If $TM_i = TM'$ is a normal almost linear subspace, then $\mathcal{F}$ is called a normal foliation. Let $\mathcal{F}_0$ be the foliation which has zTM as the director subspace. It is called the foliation of the maximal zero-almost affine subspaces of M. It is a normal foliation. Let $\mathcal{F}_1$ be the foliation which has uTM as the director subspace. It is called the foliation of the maximal unitary almost affine subspaces of M. These two foliations are uniquely determined in the K-almost affine space M.

One proves easely:

Proposition 5.3. *The foliation $\mathcal{F}_1$ is built up by the normal a.a.s.s. iff $\tilde{i}(x, y) = \vec{0}$, $\forall x \in u\,TM$, $\forall y \in zTM$.*

Proposition 5.4. *Let $M^0_A \in \mathcal{F}_0$ be the leaf through the point A, $M^1_A \in \mathcal{F}_1$ be the leaf through the point A. Then the leafs of the foliation $\mathcal{F}_1$ are normal a.a.s.s. iff $\forall B \in M^1_A$, $\forall C \in M^0_A$, $\exists_1 D \in M^1_C \cap M^0_B : \overrightarrow{AB} = \overrightarrow{CD} \Rightarrow \overrightarrow{AC} = \overrightarrow{BD}$.*

Theorem 5.3. *The intersection of two leafs $M_0 \in \mathcal{F}_0$ and $M_1 \in \mathcal{F}_1$ contains one point and only one.*

Proof. Since $M_0 \neq \emptyset$, $M_1 \neq \emptyset$ because $M_0 \in \mathcal{F}_0$, $M_1 \in \mathcal{F}_1$, there exists a point $A \in M_0$ and a point $B \in M_1$. By virtue of the proposition 3.4, the vector $\overrightarrow{AB}$ can uniquely be written in the form $\overrightarrow{AB} = x_0 + x_1$, $x_0 \in zTM$, $x_1 \in uTM$. There exists the point C with the property $\overrightarrow{AC} = x_0$.

We have $\overrightarrow{AB} = \overrightarrow{AC} + \overrightarrow{CB} = x_0 + x_1$. Then $\overrightarrow{CB} = x_1$ and $C \in M_0$, $C \in M_1$. Therefore $C \in M_0 \cap M_1$. But $zTM \cap uTM = \{\vec{0}\}$. It follows that C is unique. q.e.d.

Proposition 5.5. *Every leaf $M_1 \in \mathcal{F}_1$ is a K-affine space.*

§ 6. **First fundamental theorem.** Let M^* be a subset of points the K-almost affine space M. The intersection M' of all a.a.s.s. of M which contain the subset M^* is an almost affine subspace in M. This is well determined. M' is called the almost affine subspace of M generated by the set M^*. If $M^* = \emptyset$, then $M' = \emptyset$.

Let $M^* \neq \emptyset$ be, subset of M and

$$\mathcal{M}^* = \{\overrightarrow{AB} \in TM \mid \forall A, B \in M^*\} \tag{6.1}$$

the set of vectors of M^*.

Theorem 6.1. *The a.a.s.s. M' generated by the subset M^* of M, $M^* \neq \emptyset$, has the associated almost linear subspace TM' generated by the set $\mathcal{M}^*$ of vectors of M^*.*

Proof. Let $\{M_i\}_{i \in I}$ be the familly of a.a.s.s. of M which contain the subset M^* and $\{TM_i\}_{i \in I}$ be their associated almost linear subspaces. But $\forall A, B \in M^* \Rightarrow A, B \in M' = \bigcap_{i \in I} M_i$. Thus $\overrightarrow{AB} \in \mathcal{M}^* \Rightarrow \overrightarrow{AB} \in TM' = \bigcap_{i \in I} TM_i$. It follows $\mathcal{M}^* \subset TM'$. It is necessary to show that $\{TM_i\}_{i \in I}$ contains all the almost linear subspace of TM which contains $\mathcal{M}^*$. Let TM be an arbitrary almost linear subspace with the property $\mathcal{M}^* \subset TM_1$ and $A_0 \in M^*$. The a.a.s.s M_1 through A_0 having TM_1 as the director space contains the set M^*. Then $M_1 \in \{M_i\}_{i \in I}$. It follows $TM_1 \in \{TM_i\}_{i \in I}$. Therefore TM' is the almost linear subspace generated by the set $\mathcal{M}^*$ q.e.d.

Evidently, the a.a.s.s. M' generated by the set M^* is obtained as the a.a.s.s. which passes through a point $A \in M^*$ and has the almost linear subspace TM', generated by the set $\mathcal{M}^*$ as the director subspace.

A first important conclusion is given by:

Theorem 6.2. (*First fundamental theorem*). *For every K-almost affine space M, there exists a uneque foliation $\mathcal{F}_0$ of the maximal zero-almost affine subspaces and a unique foliation $\mathcal{F}_1$ of the maximal unitary almost affine subspaces. M is generated by the set of points $M_0 \cup M_1$, where $M_0 \in \mathcal{F}_0$, $M_1 \in \mathcal{F}_1$.*

Proof. TM is generated by $zTM \cup uTM$. But $zTM \cup uTM$ is contained in the set of vectors of the set $M_0 \cup M_1$. Therefore the set of vectors of $M_0 \cup M_1$ is a system of generators for the almost linear space TM. Then the almost affine subspace which contains the point $\{A_0\} = M_0 \cap M_1$ (Theorem 5.3) and has TM as the director space, coincides with the K-almost affine space M. Therefore M is generated by the set $M_0 \cup M_1$. q.e.d.

Chapter II. CATEGORY OF K-ALMOST AFFINE SPACES

The geometric structure of the K-almost affine space will be completly precised up to isomorphisms. In view of this remark we will define the notion of morphism of K-almost affine spaces and we will study the category of K-almost affine spaces. Introducing the notion of quotient almost affine space and direct product of almost affine spaces we obtain other two fundamental theorems.

§ 1. Morphisms of K-almost affine spaces. Let M_i, $i \in I$ be some K-almost affine spaces, TM_i, $i \in I$ their corresponding associated almost linear spaces and ρ_i, $i \in I$ the equivalence relations which define the vectors of M_i, $i \in I$.

Definition 1.1. *We say that a mapping $f : M_1 \to M_2$ is compatible with the equivalence relations ρ_1 and ρ_2 if $\forall A_1, A_2, B_1, B_2 \in M_1$:*

$$\overline{A_1A_2}\,\rho_1\,\overline{B_1B_2} \Rightarrow \overline{f(A_1)f(A_2)}\,\rho_2\,\overline{f(B_1)f(B_2)} \tag{1.1}$$

Evidently:

Proposition 1.1. *A mapping $f : M_1 \to M_2$ is compatible with the equivalence relations ρ_1 and ρ_2 if and only if $\forall A_1, A_2, B_1, B_2 \in M_1$:*

$$\overrightarrow{A_1A_2} = \overrightarrow{B_1B_2} \Rightarrow \overrightarrow{f(A_1)f(A_2)} = \overrightarrow{f(B_1)f(B_2)} \tag{1.2}$$

The previous proposition shows that to every mapping $f : M_1 \to M_2$ compatible with ρ_1 and ρ_2 we can associate a mapping $Tf : TM_1 \to TM_2$ defined by:

$$\forall A_1 \in M_1,\ \forall x = \overrightarrow{A_1A_2} \in TM_1 : Tf(x) = Tf(\overrightarrow{A_1A_2}) = \overrightarrow{f(A_1)f(A_2)}. \tag{1.3}$$

This mapping is well defined because from (1.2) it results that it is independent of the point A_1. In what follows, we shall say that Tf is the associated transformation (or tangent mapping) of the mapping f. We met an example of associated transformation in the case of translations, §1, I.

Definition 1.2. *A mapping $f : M_1 \to M_2$ compatible with the equivalence relations ρ_1 and ρ_2 will be called an almost affine transformation or a morphism of K-almost affine spaces if its associated transformation $Tf : TM_1 \to TM_2$ is a homomorphism of the K-almost linear spaces.*

For exemple, if M is a weylean K-almost affine space, then a translation $t_x : M \to M$ is an almost affine transformation.

Proposition 1.2. *Let $f_1, f_2 : M_1 \to M_2$ be two almost affine transformations. We have $f_1 = f_2$ iff $f_1(A) = f_2A$ for a fixed point $A \in M_1$ and $Tf_1 = Tf_2$.*

Proof. $\forall A_1 \in M_1,\ Tf_1(\overrightarrow{AA_1}) = Tf_2(\overrightarrow{AA_1}) \Rightarrow \overrightarrow{f_1(A)f_1(A_1)} =$
$= \overrightarrow{f_2(A)f_2(A_1)} \Rightarrow f_1(A_1) = f_2(A_1)$.

An almost affine transformation $f : M \to M$ with the property $f(O) = O$, for a point $O \in M$ is called a center-almost affine transformation and O will be called its center.

Proposition 1.3. *For every two center-almost affine transformation $f_1, f_2 : M \to M$, with the same center, we have $f_1 = f_2$ iff $Tf_1 = Tf_2$.*

Theorem 1.1. *Let $A \in M_1$, $A' \in M_2$ be a pair of fixed points and $F : TM_1 \to TM_2$ be a homomorphism. There exists a single almost affine transformation $f : M_1 \to M_2$ with the properties $f : A \to A'$, $Tf = F$.*

Proof. We consider the mapping $f : M_1 \to M_2$ given by $f(A) = A'$, $f(A_1) = A_1'$, where A_1' verifies relation $F(\overrightarrow{AA_1}) = \overrightarrow{A'A_1'}$. This mapping is compatible with ρ_1 and ρ_2 because $\overrightarrow{A_1A_2} = \overrightarrow{B_1B_2} \Rightarrow -\overrightarrow{AA_1} + \overrightarrow{AA_2} = -\overrightarrow{AB_1} + \overrightarrow{AB_2} \Rightarrow F(-\overrightarrow{AA_1}) + F(\overrightarrow{AA_2}) = F(-\overrightarrow{AB_1}) + F(\overrightarrow{AB_2}) \Rightarrow -\overrightarrow{A'A_1'} + \overrightarrow{A'A_2'} = -\overrightarrow{A'B_1'} + \overrightarrow{A'B_2'} \Rightarrow \overrightarrow{A_1'A_2'} = \overrightarrow{B_1'B_2'}$. Let Tf be the associated transformation of f. Thus $Tf(\overrightarrow{AA_1}) = \overrightarrow{f(A)f(A_1)} = \overrightarrow{A'A_1'} = F(\overrightarrow{AA_1})$, $\forall A_1 \in M_1$ and $Tf(\overrightarrow{A_1A_2}) = \overrightarrow{f(A_1)f(A_2)} = -\overrightarrow{A'f(A_1)} + \overrightarrow{A'f(A_2)} = F(-\overrightarrow{AA_1}) + F(\overrightarrow{AA_2}) = F(\overrightarrow{A_1A_2})$, $\forall \overrightarrow{A_1A_2} \in TM_1$. Therefore $Tf = F$. The fact that f is unique follows from proposition 1.2.

We will denote by $M_i' \subset M$, $i \in I$ an almost affine subspace and by TM_i', $i \in I$ its associated almost linear space.

Proposition 1.4. *If $f : M_1 \to M_2$ is an almost affine transformation, then:*

(i) $M_1' \subset M_1 \Rightarrow f(M_1') \subset M_2$, $T(f(M_1')) = Tf(TM_1')$.

(ii) $\mathrm{Im}\, f \subset M_2$, $T(\mathrm{Im} f) = \mathrm{Im}(Tf)$.

(iii) *M_1' being a zero-almost affine subspace in $M_1 \Rightarrow f(M_1')$ is a zero-almost affine subspace in M_2.*

(iv) *M_1' being an unitary almost affine subspace in $M_1 \Rightarrow f(M_1')$ is an unitary almost affine subspace in M_2.*

Proof. (i) Let $A_0 \in M_1'$ be a fixed point, $A_0' = f(A_0)$. The pair $(A_0', Tf(TM_1'))$ determines a unique almost affine space $\overline{M}_1'$ in M_2. We show that $f(M_1') = \overline{M}_1'$. Indeed, $\forall A_1' \in \overline{M}_1'$, $\overrightarrow{A_0'A_1'} \in Tf(TM_1') \Rightarrow \exists x \in TM_1'$, $x = \overrightarrow{A_0A_1}$, $A_1 \in M_1'$ and $Tf(x) = Tf(\overrightarrow{A_0A_1}) = \overrightarrow{f(A_0)f(A_1)} = \overrightarrow{A_0'f(A_1)} = \overrightarrow{A_0'A_1'} \Rightarrow f(A_1) = A_1'$. It follows $A_1' \in f(M_1')$ and $M_1' \subset f(M_1')$. Conversely, $\forall A_1' \in f(M_1')$, $\exists A_1 \in M_1' : A_1' = f(A_1)$. Then $\overrightarrow{f(A_0)f(A_1)} = Tf(\overrightarrow{A_0A_1}) = \overrightarrow{A_0'f(A_1)} \Rightarrow f(A_1) = A_1' \in \overline{M}_1'$ and $f(M_1') \subset \overline{M}_1'$. Therefore $f(M_1') = \overline{M}_1'$ and $f(M_1') \subset M_2$, $T(f(M_1')) = Tf(TM_1')$.

(ii) Is a consequence of (i) and (iii) and (iv) follow from the proposition 3.7.1.

Proposition 1.5. *Suppose $f : M_1 \to M_2$ is an almost affine transformation and $M_2' \subset M_2$, then:* (i) $f^{-1}(M_2') \subset M_1$, $T(f^{-1}(M_2')) = (Tf)^{-1}(TM_2')$.

(ii) *If M_2' is a normal one, then $f^{-1}(M_2')$ is normal.*

Proof. Let $A_0 \in f^{-1}(M_2')$ be a point. The pair $(A_0, (Tf)^{-1}(TM_2'))$ determins an almost affine subspace $\overline{M}_2'$ in M_1. We can establish that $\overline{M}_2' = f^{-1}(M_2')$. $\forall A_1 \in f^{-1}(M_2')$, $Tf(\overrightarrow{A_0A_1}) \in TM_2' \Rightarrow \overrightarrow{A_0A_1} \in (Tf)^{-1}(TM_2') \Rightarrow$ $\Rightarrow A_1 \in \overline{M}_2' \Rightarrow f^{-1}(M_2') \subset \overline{M}_2'$. Conversely, $\forall A_1 \in \overline{M}_2' \Rightarrow \overrightarrow{A_0A_1} \in Tf)^{-1}(TM_2') \Rightarrow$ $\Leftrightarrow Tf(\overrightarrow{A_0A_1}) = \overrightarrow{f(A_0)f(A_1)} \in TM_2'$. But $A_0 \in f^{-1}(M_1')$. It follows $f(A_1) \in M_2'$ and $\overline{M}_2' \subset f^{-1}(M_2')$. (ii) follows from (i) and form proposition 3.8, I

Theorem 1.2. *If $f: M_1 \to M_2$ is an almost affine transformation, then:*
(i) *f is injective if and only if Tf is injective.*
(ii) *f is surjective if and only if Tf is surjective.*
(iii) *f is bijective if and only if Tf is bijective.*

Proof. Let us suppose f injective. Let $x = \overrightarrow{AB} \in \operatorname{Ker}(Tf)$. Then $Tf(\overrightarrow{AB}) = \vec{O} = \overrightarrow{f(A)f(B)} \Rightarrow f(A) = f(B) \Rightarrow A = B \Rightarrow \overrightarrow{AB} = \vec{O}$. We have $\operatorname{Ker}(Tf) = \{\vec{O}\}$. Conversely, Tf injective implies $\operatorname{Ker}(Tf) = \{\vec{O}\}$. Let $A \neq B$ two arbitrary points from M. Supposing $f(A) = f(B)$ we have $Tf(\overrightarrow{AB}) = \overrightarrow{f(A)f(B)} = \vec{O} \Rightarrow \overrightarrow{AB} = \vec{O} \Rightarrow A = B$: a contradiction. On the same way one proves (ii) and (iii).

Let $f_1 : M_1 \to M_2$ and $f_2 : M_2 \to M_3$ be two almost affine transformations. Then $f_2 \circ f_1$ will denote their product mapping.

Proposition 1.6. *Let $f_1 : M_1 \to M_2, f_2 : M_2 \to M_3$ two almost affine transformations. Then:*
(i) *$f_2 \circ f_1$ is an almost affine transformation.*
(ii) $T(f_2 \circ f_1) = Tf_2 \circ Tf_1$.
(iii) $T1_{TM_1} = 1_{M_1}$.

We can consider the category of K-almost affine spaces which has as objects the K-almost affine spaces and as morphisms the almost affine transformations.

Theorem 1.3. *There exists a covariant functor T from the category of K-almost affine spaces to the category of K-almost linear spaces, with the properties: $T : M \to TM$, $T : (f : M_1 \to M_2) \to (Tf : TM_1 \to TM_2)$.*

An isomorphism in the category of K-almost affine spaces is called an isomorphism of K-almost affine spaces and it is denoted by $M_1 \rightleftarrows M_2$.

Theorem 1.4. *Every two leafs of a foliation $\mathcal{F}$ of the almost affine space M are isomorphic.*

Indeed, $\forall M_1, M_2 \in \mathcal{F} \Rightarrow TM_1 = TM_2$. Consider the mapping $f: M_1 \to M_2$ for which $f: A_1 \in M_1 \to A_2 \in M_2$, A_1 and A_2 being the fixed points and $Tf = 1_{TM_1}$. From the theorems 1.1 and 1.2 result $f: M_1 \to M_2$.

Obviously, the theorem 3.8, I, can be precised using the nation of isomorphism of K-almost affine spaces.

The binary relation $\rightleftarrows$ between K-almost affine spaces is an equivalence. The geometries of two almost affine spaces M_1, M_2 which have the property $M_1 \rightleftarrows M_2$ are considered identical.

Theorem 1.5. *The product of almost affine transformations determins a structure of group on the set of the isomorphisms from M to M.*

§ 2. Quotient K-almost affine spaces. Second fundamental theorem. Let $\mathcal{F} = (M_i)_{i \in I}$ be a foliation in the K-almost affine space M. The binary relation $\sim$ defined on M by $A \sim B \Leftrightarrow \exists\, i_0 \in I : A \in M_{i_0}, B \in M_{i_0}$ is an equivalence relation. The quotient set $M/\sim$ will be denoted by $M/\mathcal{F}$. We shall see the conditions when M induces a K-almost affine structure on the set $M/\mathcal{F} = \widetilde{M}$. Let $\pi : M \to M/\mathcal{F}$ be the canonical projection.

Theorem 2.1. *Let $\mathcal{F} = (M_i)_{i \in I}$ be a foliation in the K-almost affine space M. On the quotient set $M/\mathcal{F} = \widetilde{M}$ there exists a structure $\widetilde{\mathcal{A}}$ of K-almost affine space, uniquely determined up to an isomorphism, such that the canonical projection $\pi : M \to \widetilde{M}$ is an almost affine transformation, if and only if $\mathcal{F}$ is a normal foliation in M. When $\mathcal{F}$ is a normal foliation the pair $(\widetilde{M}, \widetilde{\mathcal{A}})$ is a K-almost affine space and its associated almost linear space $T\widetilde{M}$ is given by $T\widetilde{M} = TM/TM_i$ where TM_i is the director space of the foliation $\mathcal{F}$.*

Proof. Suppose that on the set $\widetilde{M}$ there exists a K-almost affine structure $\widetilde{\mathcal{A}}$ such that $\pi : M \to M/\mathcal{F}$ is an almost affine transformation. Let us observe that $\pi(A_1) = \pi(A_2) \Leftrightarrow \overrightarrow{A_1 A_2} \in TM_i$. Let $\Pi : TM \to T\widetilde{M}$ be the associated transformation of π, $\Pi = T\pi$. We have: $\pi(A_1) = \pi(A_2) \Leftrightarrow \Pi(\overrightarrow{A_1 A_2}) = \vec{O} \Leftrightarrow \overrightarrow{A_1 A_2} \in TM_i$. Consequently, $\operatorname{Ker} \Pi = TM_i$ so $\mathcal{F}$ is a normal foliation. But π is a surjective almost affine transformation, then $T\pi = \Pi$ is surjective, too. It follows $TM/TM_i \simeq T\widetilde{M}$. The last part of the theorem holds. The space $T\widetilde{M} = TM/TM_i$ is well determined. Hence each almost affine structure $\widetilde{\mathcal{A}}'$ on $\widetilde{M}$ with the mentioned property will be isomorphic to the almost affine structure $\widetilde{\mathcal{A}}$. Now, let us suppose that $\mathcal{F}$ is a normal foliation in M. We will show that there exists $\widetilde{\mathcal{A}}$ on $\widetilde{M}$ such that $\pi : M \to \widetilde{M}$ is an almost affine transformation. Let $p : TM \to TM/TM_i = T\widetilde{M}$ be the canonical projection. Consider the mapping $\psi : \widetilde{M} \to T\widetilde{M}$ defined in the following way. We fixe an element $\pi(O) \in \widetilde{M}$ and put $\psi : \pi(A) \in \widetilde{M} \to p(\overrightarrow{OA}) \in T\widetilde{M}$, $\forall \pi(A)$. It is obvious that $p(\overrightarrow{OA})$ depends only on $\pi(A)$ and does not depend on the point A. We have $\psi(\pi(O)) = p(\overrightarrow{OO})$. Moreover the mapping ψ is injective: $\pi(A) \neq \pi(B) \Rightarrow \overrightarrow{AB} \notin TM_i \Rightarrow p(\overrightarrow{OA}) \neq p(\overrightarrow{OB})$. This mapping is surjective since for an arbitrary $\tilde{x} = p(x) \in T\widetilde{M}$ we have $x = \overrightarrow{OA}$ and $\pi(A)$ has the property $\psi(\pi(A)) = p(\overrightarrow{OA}) = \tilde{x}$. Applying the theorem 3.8, I it follows that there exists an almost affine structure $\widetilde{\mathcal{A}}$ on the set $\widetilde{M}$ such that the associated almost linear space to $(\widetilde{M}, \widetilde{\mathcal{A}})$ is just $T\widetilde{M}$. We must prove that $\pi : M \to \widetilde{M}$ is an almost affine transformation. We have $\overrightarrow{\pi(A)\pi(B)} = -p(\overrightarrow{OA}) + p(\overrightarrow{OB}) = p(\overrightarrow{AB})$. Then $\overrightarrow{AB} = \overrightarrow{CD} \Rightarrow \overrightarrow{\pi(A)\pi(B)} = \overrightarrow{\pi(C)\pi(D)}$ and π is compatible with the relations ρ and $\tilde{\rho}$. The associated mapping $\Pi : TM \to T\widetilde{M}$ has the property $\Pi(\overrightarrow{AB}) = \overrightarrow{\pi(A)\pi(B)} = p(\overrightarrow{AB})$, $\forall \overrightarrow{AB} \in TM$. Therefore $\Pi = p$. It results that Π is a homomorphism of K-almost linear spaces and π is an almost affine transformation. q.e.d.

If $\mathcal{F}$ is a normal foliation in M, the pair $(\widetilde{M}, \widetilde{\mathcal{A}})$ from the previous theorem is called the quotient K almost affine space of M by the foliation $\mathcal{F}$ and it is denoted by $\widetilde{M} = M/\mathcal{F}$. An immediate application:

Theorem 2.2. *If $f: M_1 \to M_2$ is an almost affine transformation, then:*
(i) *$\mathcal{F} = \{f^{-1}(A') \mid \forall A' \in M_2\}$ is a normal foliation in M_1.*
(ii) *$M_1/\mathcal{F} \rightrightarrows \mathrm{Im} f$.*
In fact: (i). $f^{-1}(A')$ is an a.a.s.a of M_1 with $T(f^{-1}(A')) = \mathrm{Ker}\,(Tf)$. Therefore $\mathcal{F} = \{f^{-1}(A') \mid \forall A' \in M_2\}$ is a normal foliation. (ii). The quotient K-almost affine space $M_1/\mathcal{F}$ has $TM/\mathrm{Ker}\,(Tf) \rightrightarrows \mathrm{Im}\,(Tf)$ as the associated almost linear space. Remarking that $\mathrm{Im}\,(Tf) = T\,(\mathrm{Im} f)$ and using the theorem 1.2 we get $M_1/\mathcal{F} \to \mathrm{Im}\, f$.
We have also:
Theorem 2.3. *Let $f: M_1 \to M_2$ be an almost affine transformation and $\mathcal{F}_0^1, \mathcal{F}_0^2$ be the maximal zero-foliations in M_1 and M_2, respectively. In these conditions there exists a single almost affine transformation $\tilde{f}: M_1/\mathcal{F}_0^1 \to M_2/\mathcal{F}_0^2$ such that the following diagrame is commutative*

$$\begin{tikzcd}
M_1 \arrow[r, "f"] \arrow[d, "\pi_1"'] & M_2 \arrow[d, "\pi_2"] \\
M_1/\mathcal{F}_0^1 \arrow[r, "\tilde{f}"] & M_2/\mathcal{F}_0^2
\end{tikzcd}$$

Theorem 2.4. (*Second fundamental theorem*). *For every K-almost affine space M, in wchich $\mathcal{F}_0$ is the foliation of the maximal zero-almost affine subspaces and $\mathcal{F}_1$ is the foliation of maximal unitary almost affine subspaces, the quotient almost affine space $M/\mathcal{F}_0$ is isomorphic to a fixed leaf $M_1 \in \mathcal{F}_1$ and M_1 is a K-affine space.*

Proof. $\mathcal{F}_0$ is a normal foliation and each leaf $M_1 \in \mathcal{F}_1$ is a K-affine space (proposition 5.5, I). The quotient almost affine space $M/\mathcal{F}_0$ has the associated almost linear space $TM/zTM \rightrightarrows uTM$. But uTM is the director space of the foliation $\mathcal{F}_1$. Consider the mapping $\tilde{f}: M/\mathcal{F}_0 \to M_1$ defined on the following way: $\tilde{f}$ maps the leaf $M_0 \in \mathcal{F}_0$ into the point B given by $M_0 \cap M_1$. The point B exists and is unique. $\tilde{f}$ is an almost affine transformation and $T\tilde{f}: TM/zTM \to uTM$ is defined by $T\tilde{f}(x + zTM) = 1 * x$, $\forall x + zTM \in TM/zTM$. But $T\tilde{f}$ is an isomorphism. From the theorem 1.2 we get that $\tilde{f}$ is an almost affine isomorphism.

§ 3. **Direct product of K-almost affine spaces.** Let M_1, M_2 be two K-almost affine spaces, ρ_1 and ρ_2 be their equivalences and TM_1, TM_2 be their associated linear spaces.

Consider the set $M = M_1 \times M_2$. A point $A \in M$ has the form $A = (A_1, A_2)$. On the set $M \times M$ we have the equivalence relation $\rho = \rho_1 \times \rho_2 : \overline{AB}\, \rho\, \overline{CD} \Leftrightarrow \overline{A_i B_i}\, \rho_i\, \overline{C_i D_i}$, $(i = 1, 2)$. The vector $\overrightarrow{AB}$ from M with respect to ρ is identified with the vector $(\overrightarrow{A_1 B_1}, \overrightarrow{A_2 B_2})$ from $M_1 \times M_2$. We define the product with scalars of $K: \alpha * \overrightarrow{AB} = (\alpha * \overrightarrow{A_1 B_1}, \alpha * \overrightarrow{A_2 B_2})$ $\forall \alpha \in K$, $\forall \overrightarrow{AB} = (\overrightarrow{A_1 B_1}, \overrightarrow{A_2 B_2})$. Then M is a K-almost affine space and its associated almost linear space TM can be identified with the direct product $TM_1 \times TM_2$. M is called the direct product of M_1 and M_2 and denoted by $M = M_1 \times M_2$.

It is easy to see that the projections $\Pi_i : M \to M_i$, $(i = 1, 2)$ are almost affine surjections and $T\pi_i = p_i : TM \to TM_i$, $i = 1, 2$ are the projections $p_i(x) = p_i(x_1, x_2) = x_i$. Consider $A^0 = (A_1^0, A_2^0)$ a fixed point of M and define the mappings $i_1 : M_1 \to M$, $i_1(A_1) = (A_1, A_2^0)$, $\forall A_1 \in M_1$; $i_2 : M_2 \to M$, $i_2(A_2) = (A_1^0, A_2)$, $\forall A_2 \in M_2$. Then i_1 and i_2 are injective allmost affine transformations and $Ti_1 : TM_1 \to TM$, $Ti_2 : TM_2 \to TM$ are canonical injections. But $Ti_1(TM_1)$, $Ti_2(TM_2)$ are normal almost linear subspaces in TM. It follows that M_1 and M_2 can be identified with the normal almost affine subspaces $\mathrm{Im}\, i_1$ and $\mathrm{Im}\, i_2$ in M. $\mathrm{Im}\, i_1 \cap \mathrm{Im}\, i_2 = \{A^0\}$ and M is the direct product of its normal almost affine subspaces M_1 and M_2.

Theorem 3.1. (*Third fundamental theorem*). *The K-almost affine space M is the direct product of a fixed leaf M_0 of the foliation $\mathcal{F}_0$ of the maximal zero-almost affine subspace with a fixed leaf M_1 of the foliation $\mathcal{F}_1$ of the maximal unitary almost subspaces if and only if $\mathcal{F}_1$ is a normal foliation.*

Proof. $M = M_0 \times M_1$, $M_0 \in \mathcal{F}_0$, $M_1 \in \mathcal{F}_1$, then M_0 an M_1 are identified with the normal almost affine subspaces $i_1(M_0)$, $i_2(M_1)$. It follows that $\mathcal{F}_1$ is a normal foliation. Conversely, if $\mathcal{F}_1$ is a normal foliation then $\mathcal{F}_0$ and $\mathcal{F}_1$ are normal foliations and M_0, M_1 are normal almost affine subspaces of M. Consider $\overline{M} = M_0 \times M_1$. Change the leafs M_0 and M_1 with other two leafs of $\mathcal{F}_0$ and $\mathcal{F}_1$ one obtains a K-almost affine space isomorphic to $\overline{M}$. But $TM = zTM \times uTM$. The linear space uTM being normal it follows $TM = zTM \times uTM$. Then $\overline{M} \rightrightarrows M$.q.e.d.

The Weylean K-almost affine spaces have the property that the foliation $\mathcal{F}_1$ is normal. For this category of K-almost affine spaces theorem 3.1 is true. Of course the fundamental theorems 6.2, I and 2.4 for the Weylean K-almost affine spaces are valables.

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SUR LES SURFACES DONT DEUX NORMALES OU DROITES CANONIQUES PASSENT PAR UN POINT FIXE

PAR

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1. E. Cech s'est proposé l'étude des surfaces dont une normale ou une droite canonique passe par un point fixe, comme une généralisation à la géométrie projective différentielle d'une propriété de la sphère métrique. Il a déterminé les éléments linéaires projectives des surfaces dont l'axe de Cech passe par un point fixe [1]. Cech a attiré l'attention sur ce problème à I. Kaucky qui a obtenu, dans un mémoire très important [2], beaucoup de résultats importants. D'après Kaucky, *les surfaces dont une normale qui n'est pas la directrice de Wilczynski ou une droite canonique, passe par un point fixe, se partagent en deux espèces. Les surfaces de première espèce sont données par les solutions* $\beta = \beta(\tau)$, $\tau = u + v$, *de l'équation*

$$2(k+1)\,\beta\beta'' + (k^2-1)(\beta')^2 + 2k\beta^2\beta' - 3\beta^4 - a\beta^2 = 0 \tag{1.1}$$

où $k = -3(1+2\lambda)$, $\lambda \neq -1/2$, *a est une constante qu'on peut prendre égale à zero ou à un et* λ *est le paramètre qui fixe la normale d'une surface.* (Voir [1]) On montre en [2] que si $a = 0$, on peut, par une méthode de Cech, obtenir les équations finies de ces surfaces par des quadratures. Ces surfaces ont la propriété qu'une famille de courbes de Darboux se compose de coniques touchant une droite fixe dans un point fixe. Dans le cas $a \neq 0$, les surfaces en question ont la propriété qu'une famille de courbes de Darboux se compose de coniques qui passent par deux points fixes.

Les surfaces de deuxième espèce sont-comme l'a démontré J. Kaucky [2] - les solutions du système

$$(A_1) \qquad \beta_{uu} = c\beta\beta_v\,;\ \ \beta_{vv} = c\beta\beta_u\,,$$

$$(A_2) \qquad (k^2-1)(\beta\beta_{uv} - \beta_u\beta_v) + [\,(k+1)c^2 + kc - 3]\,\beta^4 = 0,$$

où c est une racine de l'équation quadratique

$$(B) \qquad (k+1)(k+3)\,c^2 + 4kc - 12 = 0.$$

Les systèmes $(A_1) + (A_2)$ *ont été intégrés par Kaucky dans le cas* $k = 1$ *et* $k^2 - 1 \neq 0$. *Le cas* $k + 1 = 0$, auquel correspond l'axe ou la normale de Cech a été résolu par Cech [3].