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# HOLOMORPHIC VECTOR FIELDS ON LOCALLY CONFORMAL KÄHLER MANIFOLDS

BY

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In this Note, the second order necessary and sufficient conditions of Bochner-Yano [2, 3] for a vector field on a compact Kähler manifold to be holomorphic are generalized to locally conformal Kähler manifolds. Some consequences of the generalized conditions are derived. We also characterize vector fields which are locally proportional to analytic fields. When encountered, differentiability here is always $C^\infty$.

**1. First order characterizations of holomorphic fields.** Let $M$ be a complex $n$-dimensional locally conformal Kähler (l.c.K) manifold ($n > 1$) with complex structure $J$, metric $g$, fundamental form $\Omega$ and Lee form $\omega$. This means that $M$ is covered by open neighborhoods $U_\alpha$ endowed with differentiable functions $\sigma_\alpha : U \to R$ such that

$$g'_\alpha = e^{-\sigma_\alpha}(g/_{U_\alpha}) \tag{1.1}$$

are Kähler metrics or equivalently, that

$$d\Omega = \omega \wedge \Omega, \tag{1.2}$$

where

$$\omega/_{U_\alpha} = d\sigma_\alpha . \tag{1.3}$$

For definitions, characteristic properties and examples of l.c.K. manifolds we send the reader to [4].

The useful linear connections on such a manifold $M$ are the Levi-Civita connection of $g$ which we denote by $\nabla$ and the Weyl connection of $g$, which is defined by [4]

$$\tilde{\nabla}_X Y = \nabla_X Y - (1/2)\,\omega(X)\,Y - (1/2)\,\omega(Y)\,X + (1/2)\,g(X, Y)\,B, \tag{1.4}$$

where $B = \#\,\omega$ is the contravariant field defined by $\omega$ and it will be called the *Lee vector field of M*.

It is known that the l.c.K. manifolds are characterized by the fact that $\tilde{\nabla}$ is a complex connection (i.e. $\tilde{\nabla} J = 0$) [4], whence it is easy to see that $M$ is l.c.K iff $\tilde{\nabla}_X$ preserves the complex type of the vector fields.

Locally, the Weyl connection is just the Levi-Civita connection of the Kähler metrics $g'_\alpha$ [4], whence, for instance, its Ricci tensor is the Ricci tensor of $g'_\alpha$ etc.

Finally, note that we shall use the index notation $\alpha, \bar{\alpha}, \ldots$ in the usual manner for complex manifolds (e.g. [2,3]).

Now, let

$$X = Z + \bar{Z} \qquad (Z = \zeta^{\alpha}\,(\partial/\partial z^{\alpha})) \tag{1.5}$$

be a real vector field on the l.c.K. manifold $M$. By lowering indices we get the real Pfaff form $\xi = \flat X$ and the complex (0,1) - form $\zeta = \flat Z$ with components

$$\zeta_{\bar{\lambda}} = g_{\alpha\bar{\lambda}}\,\zeta^{\alpha}. \tag{1.6}$$

Then, $X$ is a holomorphic field if the functions $\zeta^{\alpha}$ are analytic and it is classical that this happenes iff [2,3]

$$(L_X J)(Y) = [X, JY] - J\,[X, Y] = 0, \tag{1.7}$$

where $L$ denotes the Lie derivative.

If $M$ is Kähler, important characterizations of holomorphic fields $X$ exist which are expressed by the associated form $\zeta$ [2, 3] and we generalize such results for l.c.K. manifolds.

**Proposition 1.1.** *The vector field $X$ of (1.5) on the l.c.K. manifold $M$ is holomorphic iff it satisfies one of the following equivalent conditions:*

a) $\quad \tilde{\nabla}_{\bar{\alpha}}\zeta_{\bar{\beta}} = \omega_{\bar{\alpha}}\,\zeta_{\bar{\beta}},$

b) $\quad d\zeta = d''\zeta = \omega'' \wedge \zeta, \quad \nabla_{\bar{\alpha}}\zeta_{\bar{\beta}} + \nabla_{\bar{\beta}}\zeta_{\bar{\alpha}} = 0.$

*Proof.* Note first that, here and in the sequel, $d = d' + d''$, $\omega = \omega' + \omega''$ are decompositions in terms of complex type (1,0) and (0,1) respectively.

Now, to get a), we begin by the remark that holomorphy is a property of local type. Hence, it suffices to express the holomorphy of $X/_U$, which can be done by means of the Kähler metric $g'$. (We ommit thereafter the index $\alpha$ of (1.1)).

The (0,1) form of $X$ via this metric is $e^{-\sigma}\,\zeta$ and holomorphy is equivalent to [2,3]

$$\tilde{\nabla}_{\bar{\alpha}}\,(e^{-\sigma}\,\zeta_{\bar{\beta}}) = 0, \tag{1.8}$$

which by (1.3) is just a).

Finally, the equivalence of b) and a) follows by a technical computation from (1.4).

**Corollary 1.2.** *If the Lee vector field $B$ of $M$ is Killing it is also holomorphic.*

*Proof.* Indeed, for $B$, $\zeta$ is $\omega''$ and we can use the second part of Proposition 1.1.

**2. Second order holomorphy conditions.** The main role in the study of the holomorphic vector fields on compact Kähler manifolds is played by the second order conditions of B o c h n e r-Y a n o-L i c h n e r o w i c z [2]. We shall derive here such conditions in the case of the l.c.K. manifolds. These conditions are related to laplacians and we begin by stating some formulas valid for conformal changes of the metric.



**Lemma 2.1.** *Let $g_2 = f g_1$ be two conformally related $m$-dimensional Riemannian or Hermitian metrics and, as usually, denote by $\delta$ codifferentials, by $\Delta$ laplacians and by $i$ interior products. Then we have*

$$\delta_2\varphi = f^{-1}\,\delta_1\varphi - (m/2 - \deg\varphi)\,f^{-2}\,i_1\,(df)\,\varphi, \tag{2.1}$$

$$\delta''_2\,\varphi = f^{-1}\,\delta''_1\,\varphi - (m/2 - \deg\varphi)\,f^{-2}\,i_1\,(d'f)\,\varphi, \tag{2.2}$$

$$\begin{aligned}\Delta_2\,\varphi = f^{-1}\,\Delta_1\varphi\; - f^{-2}\{e\,(df)\,\delta_1\varphi + (m/2 - \deg\varphi)\,di\,(df)\,\varphi\; + \\ + (m/2 - \deg\varphi - 1)\,i_1\,(df)\,d\varphi\} + 2\,(m/2 - \deg\varphi)\,f^{-3}\,e\,(df)\,i_1\,(df)\,\varphi\end{aligned} \tag{2.3}$$

$$\begin{aligned}\Delta''_2\varphi = f^{-1}\,\Delta''_1\varphi - f^{-2}\,\{e\,d''f)\,\delta''_1\varphi + (m/2 - \deg\varphi)\,.\,d''\,i_1\,(d'f)\,\varphi\; + \\ + (m/2 - \deg\varphi - 1)\,i_1\,(d'f)\,d''\,\varphi\} + 2\,(m/2 - \deg\varphi)\,f^{-3}\,e\,(d''f)\,i_1\,(d'f)\,\varphi.\end{aligned} \tag{2.4}$$

We do not insist on the proofs which are by technical computations only and go straight to the holomorphy conditions.

**Proposition 2.2.** *If $X$ is a holomorphic vector field on the l.c.K. manifold $M^{2n}$, its covariant 1-form $\xi = \flat X$ satisfies the following necessary condition*

$$\Delta\xi + nL_B\xi - i\,(\omega)\,d\xi - \delta e\,(\omega)\,\xi + (n - 1)\,(i\,(\omega)\,\xi)\,\omega = 2\gamma\,(X), \tag{2.5}$$

*where $\gamma\,(X)$ is a 1-form of components $\gamma_{ij}\,\xi^i$, $\xi^j$ being the components of $X$ and*

$$\gamma_{ij} = \tilde{\rho}_{ij} + \frac{1}{2}(n - 1)\,\|\omega\|^2\,g_{ij} = R_{ij} + \frac{1}{2}\,(\delta\omega)\,g_{ij} + (n - 1)\left(\nabla_j\omega_i + \frac{1}{2}\,\omega_i\omega_j\right), \tag{2.6}$$

*where $\tilde{\rho}_{ij}$ is the Ricci curvature tensor of the Weyl connection and $R_{ij}$ is the Ricci tensor of the metric $g$.*

*Proof.* We use again the local character of holomorphy which allows us to look for our necessary condition by restricting the vector field to a neighbourhood $U$ endowed with the Kähler metric $g' = e^{-\sigma}\,g$ of (1.1). On $U$ the associated (0,1)-form of $X$ is $e^{-\sigma}\zeta$ and we have necessarily

$$\Delta\,(g')\,(e^{-\sigma}\zeta) = 2\tilde{\rho}_{\alpha\bar{\beta}}\,\zeta^{\alpha}\,d\bar{z}^{\beta}, \tag{2.7}$$

where $\Delta\,(g')$ is the Laplacian of the metric $g'$. Indeed, (2.7) is just the known Kählerian condition [2,3] since, as we already remarked, $\tilde{\nabla}$ is the Levi-Civita connection of $g'$.

Now, by Lemma 2.1 and after computations, (2.7) becomes

$$\Delta\zeta + nL_B\zeta - i\,(\omega)\,d\zeta - \delta e\,(\omega)\,\zeta + (n - 1)\,(i\,(\omega)\,\zeta)\,\omega = 2\gamma\,(Z). \tag{2.8}$$

Adding this relation with its complex conjugate yields (2.5) with the first expression (2.6) of $\gamma$. The second expression of $\gamma$ follows by the classical relation between the Ricci tensors of two conformally related metrics [1]. Q.e.d.

Furthermore, since $g'$ is a Kähler metric, we have $\Delta\,(g') = 2\Delta''\,(g')$ and (2.7) is equivalent to

$$\Delta''\,(g')\,(e^{-\sigma}\,\zeta) = \tilde{\rho}_{\alpha\bar{\beta}}\,\zeta^{\alpha}\,d\bar{z}^{\beta} \tag{2.9}$$

Using here Lemma 2.1 and Proposition 1.1 b) yields a new explicitation and proves

**Proposition 2.3.** *If $X$ is a holomorphic vector field on the l.c.K. manifold $M^{2n}$, its covariant (0,1)-form $\zeta$ satisfies the necessary condition*

$$\Delta''\zeta + nd''i(\omega')\zeta - \delta''e(\omega'')\zeta = \rho_{\alpha\bar{\beta}}\zeta^{\alpha}\,d\bar{z}^{\beta}. \tag{2.10}$$

Let us also derive some rather immediate consequences:

**Corollary 2.4.** *Let $M^{2n}$ be a compact non-Kähler l.c.K. manifold whose Lee vector field $B$ is holomorphic and which satisfies the inequality*

$$\tilde{\rho}(B, B) \leqslant \frac{n}{2}\|B\|^2 \operatorname{div} B. \tag{2.11}$$

*Then the Lee form $\omega$ of $M$ is harmonic and the first Betti number $b_1(M) \neq 0$.*

*Proof.* Using (2.5) for $\xi = \omega$ and taking the global scalar product by $\omega$ yields

$$(\Delta\omega, \omega) = 2\int_M\left(\tilde{\rho}(B, B) - \frac{n}{2}\|B\|^2 \operatorname{div} B\right)dv,$$

where $dv$ is the volume element of $M$. The result follows now by comparing signs.

**Corollary 2.5.** *Let $M^{2n}$ be a compact l.c.K. manifold for which the Hermitian form $\gamma$ is negative definite. Then, if $X$ is a non-zero holomorphic field on $M$, its associated 1-form $\xi$ cannot be basic.*

*Proof.* First, in this formulation, a form $\xi$ is „basic" if it satisfies

$$i(B)\xi = 0, \quad L_B\xi = 0. \tag{2.12}$$

Now, if $\xi$ is basic, we also get $i(\omega)\,d\xi = 0$ and the condition (2.5) becomes

$$\Delta\xi c - \delta(\omega)\xi = 2\gamma(X). \tag{2.13}$$

This implies

$$(\Delta\xi, \xi) = 2\int_M \gamma(X, X)\,dv,$$

and, by comparing signs, $X = 0$, which contradicts the hypothesis. Q.e.d.

**Corollary 2.6.** *Let $M^{2n}$ be a compact l.c.K. manifold on which*

$$S(X, X) = \{(\delta\omega)g_{ij} + 2(n-1)\nabla_j\omega_i\}\xi^i\xi^j \tag{2.14}$$

*is a positive definite form. Then $M$ has no non-zero holomorphic conformal infinitesimal transformation $X$ with basic form $\xi$.*

*Proof.* If $X$ is such an infinitesimal transformation, then it satisfies necessarily (2.13) and also

$$\Delta\xi - \left(1 - \frac{1}{n}\right)d\delta\xi = 2R_{ij}\,\xi^i\,dx^j, \tag{2.15}$$

which is characteristic for conformal vector fields [1]. These two conditions and (2.6) imply

$$-\frac{n-1}{n}d\delta\xi - \delta c(\omega)\xi = S(X, \cdot). \tag{2.16}$$



The conclusion follows now by taking the global scalar product with $\xi$ and comparing the signs.

In what concerns sufficiency, we are no more in the happy Kählerian case when, for compact manifolds, (2.9) or (2.10) are also sufficient. We can prove however.

**Proposition 2.7.** *Let $M^{2n}$ be a compact l.c.K. manifold and $X$ be a vector field on $M$ whose associated (0,1)-form $\zeta$ satisfies (2.10) together with the following supplementary condition*

$$\begin{aligned}(\tilde{\nabla}_{\bar{\nu}}\zeta_{\bar{\tau}} - \omega_{\bar{\nu}}\zeta_{\bar{\tau}})\,\omega^{\bar{\nu}}\,d\bar{z}^{\tau} + (n-1)(\tilde{\nabla}_{\bar{\tau}}\omega^{\bar{\lambda}})\,\zeta_{\bar{\lambda}}\,d\bar{z}^{\tau} = nd''i(\omega')\zeta - \\ - \delta''e(\omega'')\zeta - i(\omega')d''\zeta - (\delta''\zeta)\,\omega'' - (n-1)(i(\omega')\zeta)\,\omega''.\end{aligned} \tag{2.17}$$

*Then $X$ is a holomorphic vector field on $M$.*

*Proof.* One follows the standard proof of the sufficiency of (2.9) in the compact Kählerian case as exposed, e.g. in [3], i.e., one starts with the (0,1)-form

$$b = (\tilde{\nabla}_{\bar{\lambda}}\xi_{\bar{\mu}})\,\xi^{\bar{\mu}}\,d\bar{z}^{\lambda},$$

one calculates $\delta b$ and $\Delta''\zeta$ using Lemma 2.1 and one integrates $\delta b$ over $M$, which gives zero. By lengthy but technical computations one gets then the integral formula

$$(\tilde{\nabla}''\zeta, \tilde{\nabla}''\zeta) = (\Delta''\zeta - T\zeta, \zeta), \tag{2.18}$$

where $T$ is an operator defined by

$$\begin{aligned}T\zeta = \tilde{\rho}_{\alpha\bar{\beta}}\zeta^{\alpha}d\bar{z}^{\beta} + (\widetilde{\nabla_{\bar{\alpha}}\zeta_{\bar{\beta}}} - \omega_{\bar{\alpha}}\zeta_{\bar{\beta}})\,\omega^{\bar{\alpha}} - i(\omega')\,d''\zeta + \\ + \{(n-1)(\tilde{\nabla}_{\bar{\alpha}}\omega^{\bar{\beta}})\,\zeta_{\bar{\beta}} + (g^{\bar{\gamma}\beta}\tilde{\nabla}_{\beta}\zeta_{\bar{\gamma}})\,\omega_{\bar{\alpha}}\}\,d\bar{z}^{\alpha},\end{aligned} \tag{2.19}$$

and $\tilde{\nabla}''\zeta$ is a tensor whose only non-zero components are $\tilde{\nabla}_{\bar{\alpha}}\zeta_{\bar{\beta}} - \omega_{\bar{\alpha}}\zeta_{\bar{\beta}}$. Now, formula (2.18) and Proposition 1.1 a) obviously yield that

$$\Delta''\zeta = T\zeta \tag{2.20}$$

is a sufficient condition for $X$ to be a holomorphic vector field. Finally, it is straightforward that (2.10) and (2.17) imply (2.20). Q.e.d.

**3. Locally conformal holomorphic vector fields.** In this section we briefly introduce a more general class of vector fields, which make sense in the general case of a complex manifold.

**Definition 3.1.** *Let $X$ be a differentiable vector field on the complex manifold $(M, J)$. $X$ is called locally conformal holomorphic (l.c.h.) if for every $x \in M$ there is an open neighborhood $U$ endowed with a function $\varphi : U \to R$ such that $e^{-\varphi}X|_U$ is a holomorphic field on $U$. If, moreover, $X \neq 0$ at every point of $M$, it will be caled non-singular.*

Clearly, a non-singular l.c.h. vector field defines on $M$ a complex-analytic line bundle which is $C^{\infty}$-trivial and, conversely, a complex analytic line subbundle of the tangent bundle defines such a field.

**Proposition 3.2.** *A real non-singular vector field $X$ on a complex manifold $M$ is l.c.h. iff there is a closed Pfaff form $\alpha_X$ on $M$ such that for every vector field $Y$ one has*

$$(3.1) \qquad (L_X J)(Y) = \alpha_X(Y) JX - \alpha_X(JY) X.$$

*Moreover, $X$ is globally conformal holomorphic (g.c.h.) iff $\alpha_X$ is exact.*

*Proof.* Cover $M$ by neighbourhoods $U$ like in definition 3.1. For every such $U$ we have then $L_{(e^{-\varphi}X)} J = 0$ whose explicitation gives

$$(3.2) \qquad (L_X J)(Y) = d\varphi(Y) JX - d\varphi(JY) X.$$

If $U'$ is another such neighbourhood with corresponding function $\varphi'$, then $X \neq 0$ and (3.2) implies $d\varphi = d\varphi'$ on $U \cap U'$, whence we see that

$$(3.3) \qquad \alpha_X = d\varphi$$

is the requested Pfaff form.

Therefore, (3.1) is a necessary condition.

The sufficiency follows by going back-way from (3.1) to $L_{(e^{-\varphi}X)} J = 0$.

The last assertion is trivial.

$\alpha_X$ will be called the *complementary form* of the field $X$.

Now, if we suppose that $M$ is a Hermitian manifold with metric $g$, we get from (3.1)

$$(3.4) \qquad \alpha_X(Y) = (1/\|X\|^2)\{g(JX, [X, JY]) - g(X, [X, Y])\}.$$

Hence, on a Hermitian manifold, every non-singular vector field $X$ has an associated *complementary form* defined by (3.4) and $X$ is l.c.a. iff $\alpha_X$ is closed and satisfies (3.1).

In the case of an l.c.K. manifold, if we express the brackets of (3.4) by means of the Weyl connection and use the fact that the latest is almost complex, we get

$$(3.5) \qquad \alpha_X(Y) = (1/\|X\|^2) g(X, \tilde\nabla_Y X - \tilde\nabla_{JY}(JX)).$$

The same formula holds for Kähler manifolds, where $\tilde\nabla = \nabla$ = the Levi-Civita connection.

It is easy to get first order conditions for the l.c.h. property of a vector field by expressing the respective holomorphy conditions for the 1-form $e^{-\varphi}\zeta$. The same thing holds for necessary second order conditions. As for second order sufficient conditions, one can try again to find a convenient integral formula.

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# THE $d''$-CHERN CLASSES OF THE COMPLEX VECTOR BUNDLES

BY

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Let $E \to M$ be a ($C^\infty$-differentiable) complex vector bundle over the complex manifold $M$ and let $T, T^*$ be the complex tangent and cotangent bundles of $M$. As usual, denote by $\Gamma(U, E)$ the space of the $C^\infty$-differentiable sections of $E$ over the open set $U \subset M$. The local sections in $T^*$ are the complex one-forms and the following notation is used: $\Gamma(U; T^*) = A^1(U)$. The complex structure of $M$ defines a splitting of the space of one-forms into the subspaces of the one-forms of type $(1, 0)$ and type $(0, 1)$. Generally the space of the complex valued $p$-forms over $U \subset M$ is denoted by $A^p(U)$ and its subspace consisting of the forms of type $(r, p\text{-}r)$ is denoted by $A^{r, p-r}(U)$.

**Definition.** *A $d''$-connection on $E$ is a differential operator:*

$$D : \Gamma(E) \to \Gamma(T^* \otimes E); \qquad \Gamma(E) = \Gamma(M, E)$$

*which is a $d''$-derivation, i.e. for any $f \in A^0(M) = \Gamma(M \otimes \mathbf{C})$:*

$$(1) \qquad D(fu) = d''f \otimes u + f\,Du; \qquad u \in \Gamma(M, E) = \Gamma(E),$$

*where $d''$ is the operator of differentiation with respect to the complex conjugates of the (holomorphic) coordinates of $M$.*

If $s = (s_1, \ldots, s_n)$ ($n$ = fiber dimension of $E$) is a local frame of $E$ over $U \subset M$, then:

$$(2) \qquad Ds_a = \sum_{b=1}^{n} \theta_a^b \otimes s_b; \quad a = 1, \ldots, n; \quad \theta_a^b \in A^1(U).$$

The $n \times n$ — matrix $\theta = (\theta_b^a)$ with entries the complex one-forms $\theta_b^a$ is called the *$d''$-connection matrix* of $D$ with respect to the local frame $s$. Let $\tilde s = (\tilde s_1, \ldots, \tilde s_n)$ be another local frame of $E$ over $\tilde U \subset M$ and let $\tilde\theta = (\tilde\theta_b^a)$ be the connection matrix of $D$ with respect to $\tilde s$. If $A$ is the transition matrix from $s$ to $\tilde s$ on $U \cap \tilde U$ then $\theta$ and $\tilde\theta$ are related by:

$$(3) \qquad \tilde\theta = A^{-1} d''A + A^{-1}\theta A.$$

Just like in the case of the usual connections it can be introduced the curvature matrix $R$ of $D$ with respect to the local frame $s$:

$$(4) \qquad R = d''\theta + \theta \wedge \theta.$$