$$(3.1) \qquad (L_X J)(Y) = \alpha_X(Y) JX - \alpha_X(JY) X.$$

Moreover, X is globally conformal holomorphic (g.c.h.) iff α_X is exact.

Proof. Cover M by neighbourhoods U like in definition 3.1. For every such U we have then $L_{(e^{-\varphi}X)} J = 0$ whose explicitation gives

$$(3.2) \qquad (L_X J)(Y) = d\varphi(Y) JX - d\varphi(JY) X.$$

If U' is another such neighbourhood with corresponding function φ', then $X \neq 0$ and (3.2) implies $d\varphi = d\varphi'$ on $U \cap U'$, whence we see that

$$(3.3) \qquad \alpha_X = d\varphi$$

is the requested Pfaff form.

Therefore, (3.1) is a necessary condition.

The sufficiency follows by going back-way from (3.1) to $L_{(e^{-\varphi}X)} J = 0$.

The last assertion is trivial.

α_X will be called the *complementary form* of the field X.

Now, if we suppose that M is a Hermitian manifold with metric g, we get from (3.1)

$$(3.4) \qquad \alpha_X(Y) = (1/\|X\|^2)\{g(JX, [X, JY]) - g(X, [X, Y])\}.$$

Hence, on a Hermitian manifold, every non-singular vector field X has an associated *complementary form* defined by (3.4) and X is l.c.a. iff α_X is closed and satisfies (3.1).

In the case of an l.c.K. manifold, if we express the brackets of (3.4) by means of the Weyl connection and use the fact that the latest is almost complex, we get

$$(3.5) \qquad \alpha_X(Y) = (1/\|X\|^2)\, g(X, \tilde{\nabla}_Y X - \tilde{\nabla}_{JY}(JX)).$$

The same formula holds for Kähler manifolds, where $\tilde{\nabla} = \nabla =$ the Levi-Civita connection.

It is easy to get first order conditions for the l.c.h. property of a vector field by expressing the respective holomorphy conditions for the 1-form $e^{-\varphi}\zeta$. The same thing holds for necessary second order conditions. As for second order sufficient conditions, one can try again to find a convenient integral formula.

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THE d''-CHERN CLASSES OF THE COMPLEX VECTOR BUNDLES

BY

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Let $E \to M$ be a (C^∞-differentiable) complex vector bundle over the complex manifold M and let T, T^* be the complex tangent and cotangent bundles of M. As usual, denote by $\Gamma(U, E)$ the space of the C^∞-differentiable sections of E over the open set $U \subset M$. The local sections in T^* are the complex one-forms and the following notation is used: $\Gamma(U; T^*) = A^1(U)$. The complex structure of M defines a splitting of the space of one-forms into the subspaces of the one-forms of type $(1, 0)$ and type $(0, 1)$. Generally the space of the complex valued p-forms over $U \subset M$ is denoted by $A^p(U)$ and its subspace consisting of the forms of type $(r, p{-}r)$ is denoted by $A^{r, p-r}(U)$.

Definition. *A d''-connection on E is a differential operator:*

$$D : \Gamma(E) \to \Gamma(T^* \otimes E); \qquad \Gamma(E) = \Gamma(M, E)$$

which is a d''-derivation, i.e. for any $f \in A^0(M) = \Gamma(M \otimes \mathbf{C})$:

$$(1) \qquad D(fu) = d''f \otimes u + f\,Du; \qquad u \in \Gamma(M, E) = \Gamma(E),$$

where d'' is the operator of differentiation with respect to the complex conjugates of the (holomorphic) coordinates of M.

If $s = (s_1, \ldots, s_n)$ (n = fiber dimension of E) is a local frame of E over $U \subset M$, then:

$$(2) \qquad Ds_a = \sum_{b=1}^{n} \theta_a^b \otimes s_b; \qquad a = 1, \ldots, n; \qquad \theta_a^b \in A^1(U).$$

The $n \times n$ — matrix $\theta = (\theta_b^a)$ with entries the complex one-forms θ_b^a is called the *d''-connection matrix* of D with respect to the local frame s. Let $\tilde{s} = (\tilde{s}_1, \ldots, \tilde{s}_n)$ be another local frame of E over $\tilde{U} \subset M$ and let $\tilde{\theta} = (\tilde{\theta}_b^a)$ be the connection matrix of D with respect to $\tilde{s}$. If A is the transition matrix from s to $\tilde{s}$ on $U \cap \tilde{U}$ then θ and $\tilde{\theta}$ are related by:

$$(3) \qquad \tilde{\theta} = A^{-1} d''A + A^{-1} \theta A.$$

Just like in the case of the usual connections it can be introduced the curvature matrix R of D with respect to the local frame s:

$$(4) \qquad R = d''\theta + \theta \wedge \theta.$$

Then the change rule for the curvature matrices is:

$$\tilde{R} = A^{-1} R A, \tag{5}$$

where $\tilde{R}$ is the curvature matrix of D with respect to $\tilde{s}$.

The curvature matrix satisfies the Bianchi identity:

$$d'' R = R \wedge \theta - \theta \wedge R. \tag{6}$$

Proposition 1. *There exists a d''-connection D such that its connection matrices are of type $(0,1)$. Then the curvature matrices of D are of type $(0,2)$.*

Proof. If ∇ is a usual connection on E then its *part of type* $(0,1)$: $\nabla^{0,1} = D$ is constructed as follows. Let u be a (local) section of E and X a complex vector tangent to M in a point from the domain of u. Then $X = X^{1,0} + X^{0,1}$ where $X^{1,0}$, $X^{0,1}$ are the parts of type $(1,0)$ and $(0,1)$ of X. Then:

$$\nabla^{0,1} u(X) = \nabla u\,(X^{0,1}).$$

It is easily verified that $\nabla^{0,1}$ is a d''-connection satisfying the conditions of the proposition.

Remark. Every d''-connection on E can be obtained as the $(0,1)$-type part of a usual connection. Let D be a d''-connection and let ∇' be a usual connection. If, as above $X = X^{1,0} + X^{0,1} \in T$ and u is a local section of E then:

$$\nabla u\,(X) = \nabla' u\,(X^{1,0}) + Du\,(X^{0,1})$$

defines a connection on E such that its part of type $(0,1)$ is D.

Let $I_n = I\,(GL\,(n, \mathbf{C}))$ be the algebra of the complex valued polynomials defined on the Lie algebra $gl\,(n, \mathbf{C})$ of $GL\,(n, \mathbf{C})$, invariant under the action Ad of $GL\,(n, \mathbf{C})$ on $gl\,(n, \mathbf{C})$. If $\Phi_p \in I_n$, $m \in gl\,(n, \mathbf{C})$, $g \in GL\,(n, \mathbf{C})$ then:

$$\Phi_p(m) = \Phi_p(\mathrm{Ad}\, g\,(m)) = \Phi_p(gmg^{-1}).$$

Let $\Phi_p \in I_n$ be an invariant polynominal of degree p and let $\Phi_p\,(R)$ be the local $2p$-form obtained by replacing the matrix of $gl\,(n, \mathbf{C})$ by the curvature matrix R of a d''-connection D on E with respect to a local frame s of E. Of course, the products are replaced by exterior products.

Theorem 2. *For every invariant polynominal $\Phi_p \in I_n$ of degree p the $2p$-form $\Phi_p\,(R)$ is independent of the frame s and defines a d''-closed form of type $(0,2p)$, globally defined on M. The Dolbeault cohomology class of this form is independent of the d''-connection D.*

Proof. The proof is similar to that of the classical theorem.

Definition. *The (total) d''-Chern class of the complex vector bundle E is the Dolbeault cohomology class $c\,(E)$ represented by the following d''-closed differential form:*

$$c(E, D) = \det\left(I + \frac{i}{2\pi} R\right). \tag{7}$$

If $\mathcal{O}$ is the sheaf of germs of holomorphic functions on M then

$$c(E) \in H^*(M, \mathcal{O})$$

Theorem 3. *If E is holomorphic or anti-holomorphic then its d''-Chern class vanishes.*

Proof. If E is holomorphic there exists a connection on E such that its connection matrix with respect to the holomorphic local frames of E is of type $(1,0)$. Then the connection matrix of its part of type $(0,1)$ vanishes and the curvature matrix vanishes too. The anti-holomorphic bundles are holomorphic bundles with respect to the conjugate complex structure of the base manifold.

Thus the d''-Chern class of a complex vector bundle is an obstruction to the finding a holomorphic (anti-holomorphic) vector bundle equivalent to the given one.

Theorem 4. *The d''-Chern class of a complex vector bundle $E \to M$ is the image of the usual Chern class of E by $j : H^*\,(M, \mathbf{C}) \to H^*\,(M, \mathcal{O})$ induced by the inclusion $\mathbf{C} \subset \mathcal{O}$.*

Proof. First remark that the d''-Chern class of E can be constructed with the help of the part of type $(0,2)$ of the connection matrix of a usual connection on E. Then, it is an exercise in tracing the de Rham and Dolbeault isomorphism maps to show that the image by $j : H^*(M, \mathbf{C}) \to H^*(M, \mathcal{O})$ of a class $c \in H^p\,(M, \mathbf{C})$ represented in de Rham cohomology by the p-form ω can be represented in Dolbeault cohomology by the part of type $(0,p)$ of ω.

We shall now consider an example of complex vector bundle with non-trivial d''-Chern class. The set of the equivalence classes of complex line bundles over a complex manifold M is isomorphic to the integer cohomology group $H^2\,(M, \mathbf{Z})$ and the elements of $H^2\,(M, \mathbf{Z})$ are just the Chern classes of these complex line bundles. The images of these classes in $H^2\,(M, \mathbf{C})$ are the usual Chern classes as they are constructed with the help of their representative in de Rham cohomology. Consider the complex torus $\mathbf{C}T^2$ obtained from $\mathbf{C}^2$ by factorization with respect to the group generated over $\mathbf{Z}$ by $(1,0), (i,0), (0,1), (0,i) \in \mathbf{C}^2$. Then $\mathbf{C}T^2$ can be identified with the 4-dimensional real torus $T^4 = (S^1)^4$. We have:

$$H^2(\mathbf{C}T^2, \mathbf{Z}) = H^2(T^4, \mathbf{Z}) = \mathbf{Z}^6.$$

The usual Hermitian metric of $\mathbf{C}^2$ can be factorized to give a flat Kaehler metric on $\mathbf{C}T^2$. Consider the form:

$$\omega = dz^1 \wedge dz^2 + d\bar{z}^1 \wedge d\bar{z}^2$$

defined on $\mathbf{C}T^2$. This is a harmonic form, hence it defines a non-trivial cohomology class in $H^2\,(\mathbf{C}T^2, \mathbf{C})$. The image of this class in $H^2\,(\mathbf{C}T^2, \mathcal{O})$ is represented by the two-form $d\bar{z}^1 \wedge d\bar{z}^2$ which is also harmonic (the d''-Laplacian on Kaehler manifolds coincides, up to a constant factor, with the usual Laplacian). Hence the form $d\bar{z}^1 \wedge d\bar{z}^2$ defines a non-trivial cohomology class in $H^2\,(\mathbf{C}T^2, \mathcal{O})$. Remark that ω is a real form hence it defines an element in $H^2\,(\mathbf{C}T^2, \mathbf{R})$. Moreover, this element is in the image of $H^2\,(\mathbf{C}T^2, \mathbf{Z})$ in $H^2\,(\mathbf{C}T^2, \mathbf{R})$ by the natural inclusion $\mathbf{Z} \subset \mathbf{R}$, since the periods of ω on the generators of $H_2\,(\mathbf{C}T^2, \mathbf{Z})$ are integers. Hence ω represents the Chern class of a complex line bundle on M and the d''-Chern class of this complex line bundle is not trivial.

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SPIN STRUCTURES ON HILBERT MANIFOLDS

BY

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1. Introduction. Let H be a separable, real Hilbert space. We denote by $L(H)$ the algebra of bounded linear operators on H, by $O(H)$ the orthogonal operators on H and by I the identity operator on H. Let $GL_p(H)$ be the group of those invertible operators on H which can be written as $I+A$, where A is in a „perturbation class" $P(H)$ [2, p. 46] of $L(H)$. The group $O(H)_p = O(H) \cap GL_p(H)$ is doubly connected and we denote by $SO(H)_p$ the connected component of I.

Let $F(H)$ be the ideal of finite rank operators on H. We denote by $F(H)_p$ the closure of the ideal $F(H)$ in the p-norm defined by $\| X \|_p = (\operatorname{trace} (X^*X)^{\frac{p}{2}})^{\frac{1}{p}}$ for $1 \leq p < \infty$ and by the usual norm, for $p = \infty$. The ideals $F(H)_p$ are „perturbation classes" for $L(H)$. For $p = 1$ (resp. $p = 2$) we obtain the ideal of nuclear operators (resp. the ideal of Hilbert-Schmid operators) and in this case the groups $O(H)_p$, $SO(H)_p$ will be denoted by $O(H)_1$, $SO(H)_1$ (resp. $O(H)_2$, $SO(H)_2$). It follows, from general principles, that the universal covering of $SO(H)_p$ is a Banach-Lie group and that the covering map is 2-sheeted. An explicit construction of the universal covering group Spin $(H)_1$ of $SO(H)_1$ has been given by P. de la Harpe [3]. Later, R. J. Plymen and R. F. Streater [9] gave an explicit construction of the universal covering group Spin $(H)_2$ of $SO(H)_2$. Both groups Spin $(H)_1$ and Spin $(H)_2$ will be called spinor groups and will be denoted by Spin (H).

We denote by $SO(H)$ both the groups $SO(H)_1$ and $SO(H)_2$ and by $\rho : \mathrm{Spin}(H) \to SO(H)$ the corresponding covering maps.

In the following, we define the spin structures using the groups Spin (H) and we give some properties of these structures. Some results about the Riemannian P-structures are given, too.

2. Definitions of the spin structures. All bundles, manifolds and morphisms considered in the following will be assumed of class C^∞. Let M be a connected and paracompact manifold, modelled on a Banach space and let $\xi : E \to M$ be a vector bundle over M with fibre H.

Definition 2.1. *A Riemannian P-structure on the vector bundle ξ is a reduction of its structural group to the group $O(H)_p$.*

Remark 2.1. The existence of the Riemannian P-structures is a direct consequence of the fact that $GL(H)$ is contractible (Kuiper's theorem).

Definition 2.2. *A Riemannian P-structure on the vector bundle ξ is said*