Hence $\bar{S} \subset S$, which contradicts $\bar{S} = S$. So the order of S is an odd integer.

From the above proposition we find that a cyclic group of even order is not generalized simple. Regarding the cyclic group of even order we have the following proposition.

Proposition 9. *Every cyclic group of even order has two generalized semi ideals.*

Proof. Let S be a cyclic group of even order n generated by g. Then $S = \{1, g, \ldots, g^{n-1}\}$. Now $S_1 = \{1, g^2, g^4, \ldots, g^{n-2}\}$ and $S_2 = \{g, g^3, \ldots, g^{n-1}\}$ are two generalized semi-ideals of S. Thus S contains two generalized semi-ideals. Next we shall show that these are the only generalized semi-ideals of S. If possible let A be any other generalized semi-ideal of S. Then either $1 \in A$ or $1 \notin A$. If $1 \in A$, then A contains all the elements of S_1, and no element of S_2. Because if any one of the elements of S_2 belongs to A then A coincides with S which contradicts that A is a proper generalized semi ideal of S. So if $1 \in A$ then $A = S_1$. If $1 \in A$ then A does contain any element of S_1 and contains all the elements of S_2. So $A = S_2$, Hence the proposition.

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ON WARD QUASIGROUPS

BY

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The purpose of the present note is to solve a problem suggested in [2] which is concerned with the structures of quasigroups and groups in a same set.

Definition 1. *A subtractive groupoid is a groupoid* $(G, 0)$ *which satisfies the axioms*:

$$(S\,1)\ a \circ (b \circ c) = c \circ (b \circ a), \text{ for every } a, b, c, \text{ in } G,$$

$$(S\,2)\ a \circ (b \circ b) = a, \text{ for every } a, b, \text{ in } G.$$

Examples. The set Z of integers with $x \circ y = x - y$. The set Q^* of rationals different of zero with $x \circ y = x/y$.

Theorem 1. *If x and y are elements of a subtractive groupoid $(G, \circ)$ then*

$$x \circ x = y \circ y.$$

Proof. $x \circ x = x \circ (x \circ (y \circ y)) = (y \circ y) \circ (x \circ x) = y \circ y.$

Theorem 2. *If $(G, \circ)$ is a subtractive groupoid, the element $i = x \circ x$, for every $x \in G$, is a right identity in G.*

Proof. By $(S\,2)$, $a \circ (x \circ x) = a$.

Theorem 3. *If a, b, c, d are elements of the subtractive groupoid $(G, \circ)$, then*

$$(a \circ b) \circ (c \circ d) = (a \circ c) \circ (b \circ d).$$

Proof.

$$(a \circ b) \circ (c \circ d) = d \circ (c \circ (a \circ b)) = d \circ (b \circ (a \circ c)) = \\ = (a \circ c) \circ (b \circ d).$$

Theorem 4. *If i is the right identity of the subtractive groupoid $(G, \circ)$ then*

(i) $i \circ (a \circ b) = b \circ a$, *for every* a, b *in* G,

(ii) $(a \circ b) \circ c = (a \circ c) \circ b$, *for every* a, b, c *in* G,

(iii) $(a \circ b) \circ c = a \circ (c \circ (i \circ b))$, *for every* a, b, c *in* G.

Proof.

(i) $i \circ (a \circ b) = b \circ (a \circ i) = b \circ a,$

(ii) $(a \circ b) \circ c = i \circ (c \circ (a \circ b)) = i \circ (b \circ (a \circ c)) = (a \circ c) \circ b,$

(iii) $a \circ (c \circ (i \circ b)) = a \circ (b \circ (i \circ c)) = (i \circ c) \circ (b \circ a) = \\ = (i \circ (b \circ a)) \circ c = (a \circ b) \circ c.$

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Theorem 5. *If* $(g, \circ)$ *is a subtractive groupoid, then the two cancellation laws*

$$a \circ x = b \circ x \quad \text{implies} \quad a = b$$

and

$$x \circ a = x \circ b \quad \text{implies} \quad a = b$$

hold in G.

Proof. If $\quad a \circ x = b \circ x$, then

$$(a \circ x) \circ (i \circ x) = (b \circ x) \circ (i \circ x)$$

$$(a \circ i) \circ (x \circ x) = (b \circ i) \circ (x \circ x)$$

$$a \circ i = b \circ i$$

$$a = b.$$

In a similar manner we can prove the second part of the theorem.

Corollary. Every subtractive groupoid is a quasigroup.

Definition 2. *A Ward quasigroup is a quasigroup* $(G, \circ)$ *which satisfies the axioms:*

(W 1) *There exists an element* $i \in G$ *such that*

$$a \circ a = i, \text{ for all } a \in G.$$

(W 2) *For every* a, b, c *in* G,

$$(a \circ b) \circ c = a \circ (c \circ (i \circ b)),$$

where i *is the element whose existence was postulated in* (W 1).

Examples. Any subtractive quasigroup is Ward quasigroup but the converse is not true. The quasigroup given by the Cayley table

$\circ$	i	a	b	c	d	e
i	i	c	b	a	d	e
a	a	i	e	c	b	d
b	b	e	i	d	a	c
c	c	a	d	i	e	b
d	d	b	c	e	i	a
e	e	d	a	b	c	i

is Ward but $a \circ (b \circ d) = i \neq a = d \circ (b \circ a)$.

Theorem 6. *Let* $(G, \circ)$ *be a Ward quasigroup. The element* i *whose existence was postulated in* (W 1) *is such that*

(i) $\quad a \circ i = a,$

(ii) $\quad i \circ (i \circ a) = a,$

(iii) $\quad i \circ (a \circ b) = b \circ a,$

for every a, b, c in G.

Proof.

(i) $\quad i \circ a = (i \circ i) \circ a = i \circ (a \circ (i \circ i)) = i \circ (a \circ i),$

hence

$$a = a \circ i.$$

(ii) $\quad i \circ a = (i \circ a) \circ i = i \circ (i \circ (i \circ a)),$

hence

$$a = i \circ (i \circ a).$$

(iii) $\quad c \circ (a \circ b) = c \circ (a \circ (i \circ (i \circ b))) = (c \circ (i \circ b)) \circ a$

and in particular

$$i \circ (a \circ b) = (i \circ (i \circ b)) \circ a = b \circ a.$$

Theorem 7. *If* $(G, \$)$ *is a group, then* $(G, \circ)$ *is a Ward quasigroup, where operation „*$\circ$*" is defined by* $x \circ y = x \$ y^{-1}$. *If* $(G, \$)$ *is abelian then* $(G, \circ)$ *is subtractive.*

Proof.

(i) If i is the identity of $(G, \$)$, then

$$a \circ a = a \$ a^{-1} = i$$

for all $a \in G$.

(ii) $\quad a \circ (c \circ (i \circ b)) = a \$ (c \$ (i \$ b^{-1})^{-1})^{-1} = a \$ (c \$ b)^{-1} =$
$\quad = a \$ (b^{-1} \$ c^{-1}) = (a \$ b^{-1}) \$ c^{-1} = (a \circ b) \circ c.$

(iii) If $(G, \$)$ is abelian then

$$a \circ (b \circ c) = a \$ (b \$ c^{-1})^{-1} = a \$ (c \$ b^{-1}) = c \$ (a \$ b^{-1}) = c \circ (b \circ a).$$

Theorem 8. *If* $(G, \circ)$ *is a Ward quasigroup and* i *is its right identity, then* G *is a group relative to the operation defined by* $x \$ y = x \circ (i \circ y)$. *The right identity* i *is the identity of* $(G, \$)$ *and the inverse of* x *is* $i \circ x$. *If* $(G, \circ)$ *is subtractive then* $(G, \$)$ *is abelian.*

Proof.

(i) $\quad a \$ i = a \circ (i \circ i) = a \circ i = a = i \circ (i \circ a) = i \$ a.$

(ii) $\quad (a \$ b) \$ c = (a \circ (i \circ b)) \circ (i \circ c) = a \circ ((i \circ c) \circ b) =$
$\quad = a \circ (i \circ (b \circ (i \circ c))) = a \$ (b \$ c).$

(iii) $\quad a \$ (i \circ a) = a \circ (i \circ (i \circ a)) = a \circ a = i$
$\quad (i \circ a) \$ a = (i \circ a) \circ (i \circ a) = i.$

(iv) If $(G, \circ)$ is a subtractive quasigroup then

$$a \$ b = a \circ (i \circ b) = b \circ (i \circ a) = b \$ a.$$

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